Welcome to Fourier Analysis and Lebesgue Integration

Summary until March 31

http://eclass.uoa.gr/courses/MATH121/

Summer semester 2019-2020

Complex-valued functions on the unit circle

Denote by $\mathbb{T}$ the unit circle

$$\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\} = \{e^{i\theta} : \theta \in \mathbb{R}\}.$$

If $\phi : \mathbb{T} \to \mathbb{C}$, define $f : \mathbb{R} \to \mathbb{C}$ by

$$f(\theta) = \phi(e^{i\theta}).$$

The function f is 2π-periodic.
Conversely, if $f : \mathbb{R} \to \mathbb{C}$ is 2π-periodic, then the function $\phi : \mathbb{T} \to \mathbb{C}$ given by $\phi(e^{i\theta}) = f(\theta)$ is well defined.
Thus we have a $1 - 1$ correspondence between functions $\phi : \mathbb{T} \to \mathbb{C}$ and 2π-periodic functions $f : \mathbb{R} \to \mathbb{C}$.
We say ϕ is integrable if f is integrable in some interval of length 2π (hence in all such intervals), we say ϕ is continuous if f is continuous, we say ϕ is differentiable if f is differentiable, we say ϕ is continuously differentiable if f is continuously differentiable and so on.
In what follows we shall make no distinction between ϕ and f.

Fourier Series

Remark (Trigonometric Polynomial)

$$\textit{If} \quad f(x) = \sum_{k=-N}^{N} c_k \exp ikx$$

then,

$$c_m = \frac{1}{2\pi}\int_0^{2\pi} f(x)\exp(-imx)dx, \quad -N \le m \le N.$$

because if $k \in \mathbb{Z}$,

$$\frac{1}{2\pi}\int_0^{2\pi} \exp(ikx)dx = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}$$

Fourier Series

Generalisation: Given a 2π-periodic function $f : \mathbb{R} \to \mathbb{C}$, we define

$$a_n = a_n(f) = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx dx, \quad (n = 0, 1, 2, \ldots)$$

$$b_m = b_m(f) = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin mx dx, \quad (m = 1, 2, \ldots)$$

$$\hat{f}(k) = \frac{1}{2\pi} \int_0^{2\pi} f(x) \exp(-ikx) dx, \quad (k \in \mathbb{Z})$$

It suffices that the integrals exist.

Definition: The Fourier series $S(f)$ of f:

$$S(f, x) := \frac{a_o}{2} + \sum_{k=1}^{\infty} a_k \cos kx + \sum_{k=1}^{\infty} b_k \sin kx$$

$$= \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{ikx} \quad \text{(complex form)}$$

(For now, we are not concerned with convergence or divergence of these series.)

Fourier series

Remark

• *The Fourier series of a trigonometric polynomial* p *is the trig. polynomial itself:* $S_n(p) = p$ *when* $n \geq \deg p$*, hence* $S(p) = p$*.*

• *If a trigonometric series* $s(x) = \sum_k c_k e^{ikx}$ *converge uniformly, then the Fourier coefficients* $\hat{s}(k)$ *of* s *are the* c_k*, hence the Fourier series of* s *is* s*.*

• *It is not however always true that every convergent trigonometric series is the Fourier series of some function (see later).*

Fourier series

Proposition (Linearity!)

If f and g are integrable on $[0, 2\pi]$ and $\lambda \in \mathbb{C}$,

$$a_n(f + \lambda g) = a_n(f) + \lambda a_n(g),$$

$$b_n(f + \lambda g) = b_n(f) + \lambda b_n(g) \quad (n, m \in \mathbb{N})$$

equivalently $$(\widehat{f + \lambda g})(k) = \hat{f}(k) + \lambda \hat{g}(k) \quad (k \in \mathbb{Z})$$

therefore $$S_n(f + \lambda g) = S_n(f) + \lambda S_n(g) \quad (n \in \mathbb{N}).$$

The Uniqueness Theorem

Theorem

If f and g is continuous and 2π-periodic functions with $\hat{g}(k) = \hat{f}(k)$ for each $k \in \mathbb{Z}$ (equivalently $a_n(f) = a_n(g)$ and $b_n(f) = b_n(g)$ for each $n \in \mathbb{N}$), then $f = g$.

Continuity was used only at the point t_0 :

Theorem

If f and g are integrable on $[-\pi, \pi]$ and $\hat{g}(k) = \hat{f}(k)$ for each $k \in \mathbb{Z}$ (equivalently $a_n(f) = a_n(g)$ and $b_n(f) = b_n(g)$ for each $n \in \mathbb{N}$), then $f(t_0) = g(t_0)$ at each point t_0 where $f - g$ is continuous.

Simple cases of convergence

Proposition

If f continuous, 2π-periodic and $\sum |\hat{f}(k)| < \infty$ (equivalently $\sum(|a_k(f)| + |b_k(f)| < \infty$) then $(S_N(f))$ converges uniformly to f.

Proposition

If f continuous, 2π-periodic and its derivative f' exists and is integrable,

$$\widehat{f'}(k) = ik\hat{f}(k) \quad (k \in \mathbb{Z}).$$

Simple cases of convergence

Proposition

If $f : \mathbb{R} \to \mathbb{C}$ is continuous, 2π-periodic and $\sum |k\hat{f}(k)| < \infty$, then f is continuously differentiable and the series $\sum ik\hat{f}(k) \exp ikx$ converges to f' uniformly.

Proposition

If f, f' and f'' are continuous and 2π-periodic, the series $\sum \hat{f}(k) \exp ikx$ converges uniformly to f.

Fejér' s Theorem

Let $f : \mathbb{R} \to \mathbb{C}$ be continuous and 2π-periodic.
Reminder: $S_n(f,t) = \sum_{|k|\leq n} \hat{f}(k)e^{ikt}$.
The sequence $(S_n(f))$ is not always always convergent (not even pointwise). However,

Theorem (Fejér)

If $f : \mathbb{R} \to \mathbb{C}$ is a continuous and 2π-periodic function, then the sequence $(\sigma_n(f))$ where

$$\sigma_m(f) = \frac{1}{m+1}\sum_{n=0}^{m} S_n(f) \quad (m \in \mathbb{N})$$

converges to f uniformly.

Two kernels: Dirichlet against Fejér

$$\text{Dirichlet:}\quad D_n(x) = \sum_{k=-n}^{k=n} \exp(ikx) = \begin{cases} \frac{\sin\left(\frac{2n+1}{2}x\right)}{\sin(x/2)}, & x \neq 0, \\ 2n+1, & x = 0 \end{cases} \tag{d}$$

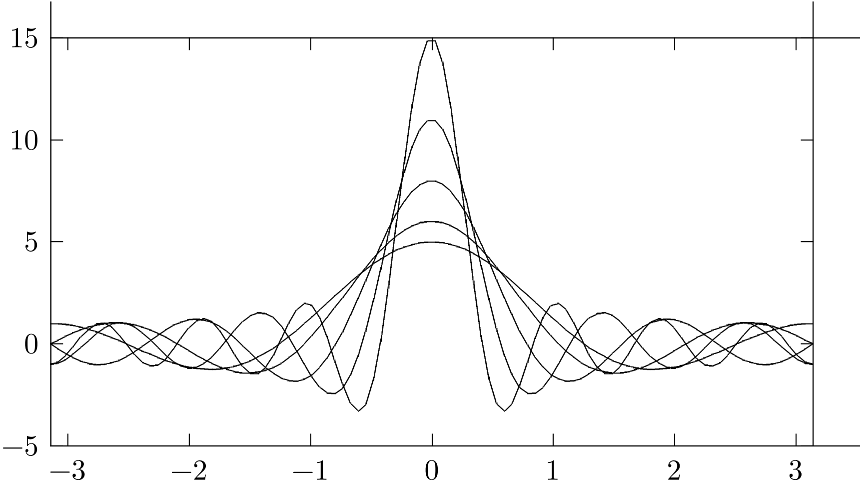
$$\text{Fejér:}\quad K_m(x) = \frac{1}{m+1}\sum_{n=0}^{m}\left(\sum_{k=-n}^{n}\exp(ikx)\right)$$

$$= \begin{cases} \frac{1}{m+1}\left(\frac{\sin\left(\frac{m+1}{2}x\right)}{\sin(x/2)}\right)^2, & x \neq 0, \\ m+1, & x = 0 \end{cases} \tag{k}$$

$$K_m = \sum_{k=-m}^{m}\left(1 - \frac{|k|}{m+1}\right) e_k \,.$$

The Dirichlet kernel

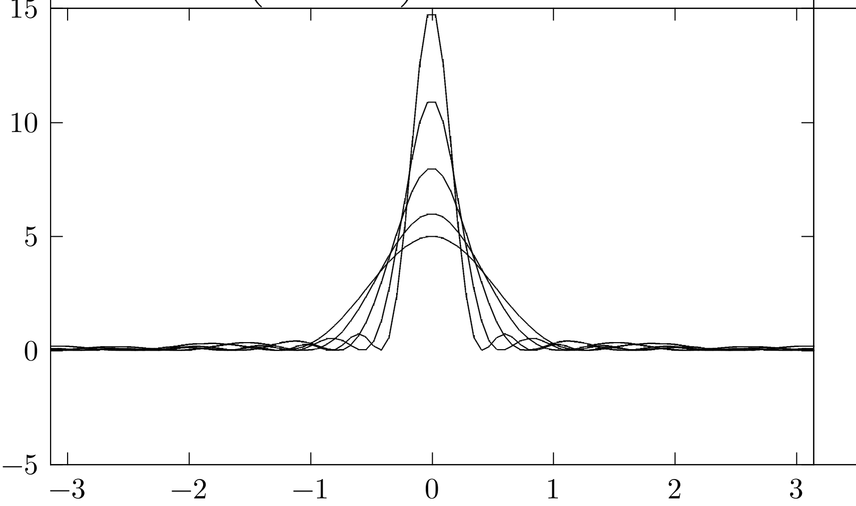
$$D_m(x) = \frac{\sin\left(\frac{2m+1}{2}x\right)}{\sin(x/2)}, \; x \neq 0, \quad D_m(0) = 2m+1.$$



$m = 4, 5, 7, 10, 14.$

The Fejér kernel

$$K_m(x) = \frac{1}{m+1}\left(\frac{\sin(\frac{m+1}{2}x)}{\sin(x/2)}\right)^2, \; x \neq 0, \quad K_m(0) = m+1.$$



$m = 4, 5, 7, 10, 14.$

Properties of Fejér's kernel K_m

Remark

The Fejér kernel has the following properties:

(α) *There exists* M *so that* $\|K_m\|_1 \leq M$ *for each* m.

(β) *If* $\delta \in (0, \pi)$ *and* $E_\delta = [-\pi, -\delta] \cup [\delta, \pi]$, *then* $\lim_m \int_{E_\delta} |K_m| = 0$.

(γ) $\frac{1}{2\pi} \int_{-\pi}^{\pi} K_m(x)dx = 1$ *for every* m.

First consequences of Fejér's Theorem

- Uniqueness. If f, g are continuous, 2π-periodic and $\hat{f}(k) = \hat{g}(k)$ for all $k \in \mathbb{Z}$, then $f = g$.

Second Proof. We have $\sigma_n(f) = \sigma_n(g)$ for each $n \in \mathbb{N}$, hence $f = \lim_n \sigma_n(f) = \lim_n \sigma_n(g) = g$ by Fejér.

- Proposition [Fejér] Let $f : \mathbb{R} \to \mathbb{C}$ be Riemann integrable in $[-\pi, \pi]$ and 2π-periodic. If f is continuous at some $t \in [-\pi, \pi]$, then $\sigma_n(f, t) \to f(t)$. [The proof is a variation of the previous one: now δ will depend on t, and convergence is shown at t.]

[Remark: More generally, if the one-sided limits $f(t_+)$ and $f(t_-)$ exist, then $\sigma_n(f, t) \to \frac{f(t_+)+f(t_-)}{2}$. (Proof omitted).]

- Corollary Under the conditions of the Proposition, if $(S_n(f, t_0))$ converges, then it must converge to $f(t_0)$.

- Remark For every f, Riemann integrable in $[-\pi, \pi]$ and 2π-periodic, we have $\|\sigma_n(f)\|_\infty \le \|f\|_\infty$.

Mean square convergence

Proposition (Best mean square approximation)

Let $f : [-\pi, \pi] \to \mathbb{C}$ *be a Riemann-integrable function and* $n \in \mathbb{N}$.
Then for every trigonometric polynomial p *of degree* $\deg(p) \leq n$ *we have*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f-p|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f-S_n(f)|^2 + \frac{1}{2\pi}\int_{-\pi}^{\pi} |S_n(f)-p|^2. \quad (1)$$

Therefore the inequality

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f-p|^2 \geq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f-S_n(f)|^2 \quad (2)$$

holds, and we have equality we have if and only if $p = S_n$.
In other words, S_n *is the unique trigonometric polynomial which minimizes the integral* $\frac{1}{2\pi}\int_{-\pi}^{\pi} |f-p|^2$ *among all choices of trigonometric polynomials* p *of degree at most* n.

In particular, if $m \leq n$ then $\quad \|f - S_m(f)\|_2 \geq \|f - S_n(f)\|_2$.

Mean square convergence

If f, g are two (Riemann) integrable functions defined on $[-\pi, \pi]$ we define

$$\|f - g\|_2 := \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(t) - g(t)|^2 dt \right)^{1/2}$$

and $$\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\overline{g(t)} dt \,.$$

Observe that $\|\cdot\|_2$ satisfies

$$\|f - g\|_2 \le \|f - g\|_\infty := \sup\{|f(t) - g(t)| : t \in [-\pi, \pi]\}$$

and that $\|f\|_2 = \langle f, f \rangle^{1/2}$.

Remark $\hat{f}(k) = \langle f, e_k \rangle \,, \ k \in \mathbb{Z}.$

Mean square convergence

Corollary

The map $(f,g) \to \langle f,g\rangle$ *is an inner product and the map* $(f,g) \to d_2(f,g) := \|f-g\|_2$ *is a metric on the linear space* $C([-\pi,\pi])$. [1] *That is, they satisfy*

$$
\begin{array}{rcl}
 & \langle f,g\rangle \in \mathbb{C} & d_2(f,g) \in \mathbb{R}_+ \\
(i) & \langle f+\lambda g,h\rangle = \langle f,h\rangle + \lambda\langle g,h\rangle & (a)\ d_2(f,g) = d_2(g,f) \\
(ii) & \langle g,f\rangle = \overline{\langle f,g\rangle} & (b)\ d_2(f,g) \le d_2(f,h) + d_2(h,g) \\
(iii) & \langle f,f\rangle \ge 0 & (c)\ d_2(f,g) = 0 \iff f = g. \\
(iv) & \langle f,f\rangle = 0 \iff f = 0. &
\end{array}
$$

[1] However it is not a metric on the space of integrable functions.

Mean square convergence

Although the sequence $(S_n(f))$ for a continuous f may fail to converge, even pointwise, it does converge to f with respect to the metric d_2:

Theorem

If $f : [-\pi, \pi] \to \mathbb{C}$ is are continuous and 2π-periodic, then

$$S_n(f) \xrightarrow{\|\cdot\|_2} f$$

that is

$$\lim_n \frac{1}{2\pi} \int_{-\pi}^{\pi} |S_n(f) - f|^2 = 0.$$

Proposition (Bessel's Inequality)

Let $f : [-\pi, \pi] \to \mathbb{C}$ be integrable. Then

$$\sum_{k=-\infty}^{+\infty} |\hat{f}(k)|^2 \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f|^2.$$

Mean square convergence

Theorem (Riemann - Lebesgue)

If $f : [-\pi, \pi] \to \mathbb{C}$ *is an integrable function, then*

$$\lim_{k\to+\infty} \widehat{f}(k) = \lim_{k\to\infty} \widehat{f}(-k) = 0$$

$$\textit{equivalently} \quad \lim_{n\to+\infty} a_n(f) = \lim_{n\to\infty} b_n(f) = 0.$$

Corollary (Parseval's equality)

If $f : [-\pi, \pi] \to \mathbb{C}$ *is a continuous function, then*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f|^2 = \sum_{k=-\infty}^{\infty} |\widehat{f}(k)|^2.$$

Note Let us state once again that the results of this Section will be generalised and strengthened, if one uses the Lebesgue integral instead of the Riemann integral.

Abel summability and the Poisson kernel

If $f : [-\pi, \pi] \to \mathbb{C}$ is an integrable function, for each $0 \leq r < 1$, the series

$$A_r(f)(t) = f_r(t) := \sum_{k \in \mathbb{Z}} r^{|k|} \hat{f}(k) e^{ikt}, \qquad t \in [-\pi, \pi]$$

converges absolutely and uniformly, hence defines a continuous function $f_r : [-\pi, \pi] \to \mathbb{C}$. We find

$$f_r(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(s) P_r(t-s) ds$$

$$\text{where} \quad P_r(t) := \sum_{n \in \mathbb{Z}} r^{|n|} e^{int} = 1 + 2 \sum_{n=1}^{\infty} r^n \cos nt$$

The Poisson kernel

$$P_r(t) = \frac{1-r^2}{1-2r\cos t + r^2}\,, \qquad 0 \le r < 1$$
$$\widehat{P_r}(k) = r^{|k|}\,, \qquad k \in \mathbb{Z}.$$

Proposition

(α) *For each* $r \in [0,1)$*, the function* $P_r : [-\pi, \pi] \to \mathbb{R}$ *is continuoust and non-negative.*

(β) *If* $\delta \in (0, \pi/2)$ *and* $E_\delta := [-\pi, -\delta] \cup [\delta, \pi]$*, we have*
$$\lim_{r \nearrow 1} \int_{E_\delta} P_r(x)dx = 0.$$

(γ) $\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} P_r(x)dx = 1$ *for every* $r \in [0,1)$.

Abel summability and the Poisson kernel

Theorem

If f is Riemann integrable and 2π-periodic, then at every point t of continuity of f we have $\lim_{r \nearrow 1} f_r(t) = f(t)$.
If f is continuous*, then* $\lim_{r \nearrow 1} f_r(x) = f(x)$ uniformly*, that is* $\lim_{r \nearrow 1} \|f_r - f\|_\infty = 0$.

Remark

Note that although the functions f_r are (in general) not trigonometric polynomials, they are continuous (in fact differentiable - why?) functions given by absolutely and uniformly convergent Fourier series.