2-3-2023

Γεννήτριες από συνεχείς κατανομές.

1) Αντίστροφος Μετασχηματισμός

Πρόταση Έστω $X \sim F(x)$ X: συνεχής

και έστω $Y \sim F(X)$.

Τότε $Y \sim \mathcal{U}(0,1)$ $(F: \mathbb{R} \to [0,1])$

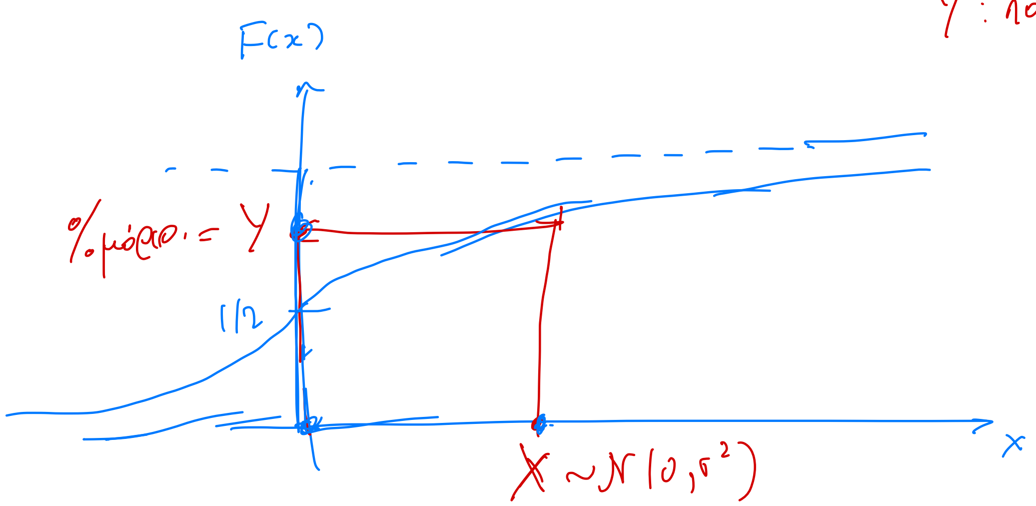
Απόδειξη $P(Y \le y) = P(F(X) \le y) =$

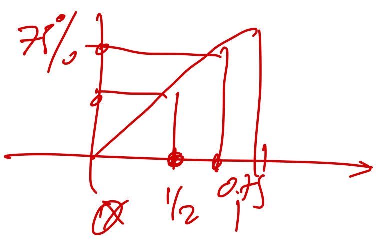
$= P(X \le F^{-1}(y))$ $\left[F(x) \le y \Leftrightarrow x \le F^{-1}(y) \right]$

$= F(F^{-1}(y)) = y, \quad y \in [0,1]$

$\Rightarrow Y \sim \mathcal{U}(0,1)$

Y: ποσοστημόριο της X

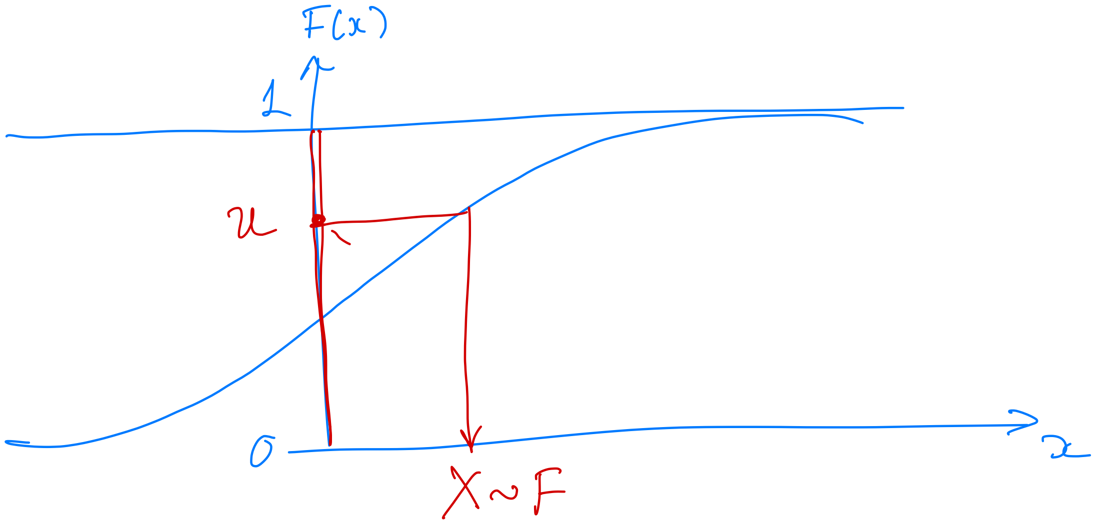




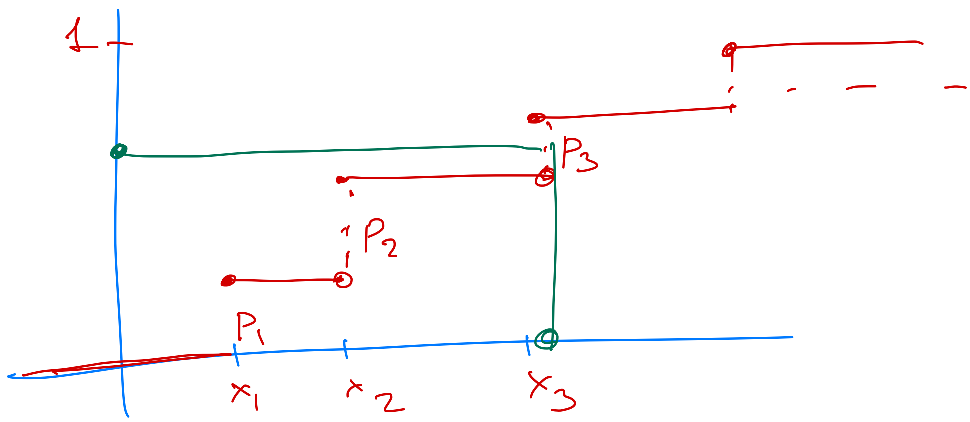
Γεννήτρια

Πόρισμα Αν $U \sim \mathcal{U}(0,1)$ κ'

$$X = F^{-1}(U) \Rightarrow \boxed{X \sim F}$$



(Διακριτή κατανομή)



Παραδείγματα

① $X \sim \mathcal{U}(a,b) \Rightarrow F(x) = \dfrac{x-a}{b-a}, \quad x \in [a,b]$

$$F(X) = \frac{X-a}{b-a} = U \Rightarrow \boxed{X = a + U(b-a)}$$

② $X \sim Exp(\lambda) \Rightarrow F(x) = 1 - e^{-\lambda x}, \quad x \geq 0.$

$$\Rightarrow F(x) = U \Rightarrow 1 - e^{-\lambda X} = U \Rightarrow \boxed{X = -\frac{1}{\lambda}\ln(\underbrace{1-U}_{\mathcal{U}(0,1)})}$$

$1 - U \sim \mathcal{U}(0,1)$

Επομ. $\boxed{X = -\frac{1}{\lambda}\ln U} \sim Exp(\lambda)$

③ $X \sim Gamma(n, \lambda) \quad n \in \mathbb{N}$

$\sim Erlang(n, \lambda)$

$$X = \sum_{j=1}^{n} X_j, \quad X_j \text{ iid} \sim Exp(\lambda)$$

$U_1, U_2, \ldots, U_n$ iid $\mathcal{U}(0,1)$

$$X_j = -\frac{1}{\lambda}\ln(U_j), \quad j = 1, \ldots, n$$

$$X = \sum_{j=1}^{n} X_j = -\frac{1}{\lambda}\sum_{j=1}^{n}\ln(U_j) = -\frac{1}{\lambda}\ln\left(\prod_{j=1}^{n} U_j\right)$$

(4) $X \sim F(x) = x^n \quad , \quad 0 < x < 1$

$F(X) = U \Rightarrow X = U^{1/n}$

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(5) $X_1, \dots, X_n \sim F$ (iid) (F : αντιστρέψιμη)

$M = \max(X_1, \dots, X_n)$

(a) $U_1, \dots, U_n$ iid $\mathcal{U}(0,1)$

$X_1 = F^{-1}(U_1), \; X_2 = F^{-1}(U_2), \dots, X_n = F^{-1}(U_n)$

$M = \max(X_1, \dots, X_n)$

(b) $U \sim \mathcal{U}(0,1)$

$F_M(y) = P(M \le y) = P(X_1 \le y, \dots, X_n \le y) = (F(y))^n$

$F_M(M) = U \Rightarrow (F(M))^n = U \Rightarrow F(M) = U^{1/n}$

$\Rightarrow \boxed{M = F^{-1}(U^{1/n})}$

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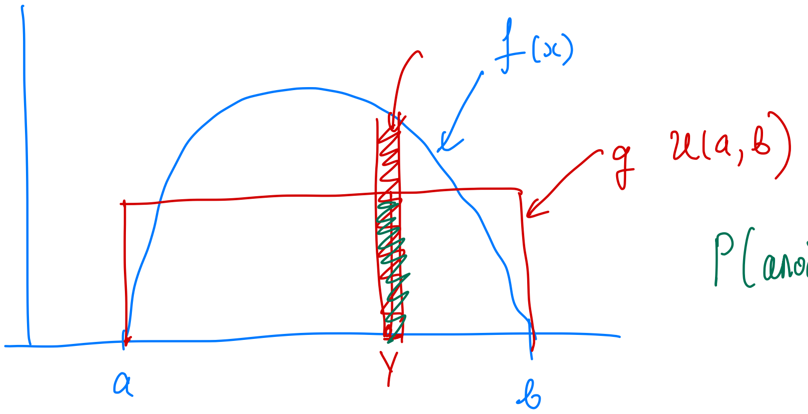
② Μέθοδος Απόρριψης (Accept/Reject)

Έστω $X \sim f(x)$ pdf

κ' $Y \sim g(y)$ pdf ; ∃ γεννήτρια της Y

$g(x) > 0$ όταν $f(x) > 0$



$P(\text{αποδ. } Y=y) \sim \dfrac{f(y)}{g(y)}$

Έστω $c : \dfrac{f(y)}{g(y)} \le c \;\; \forall y \iff c \ge \sup_y \left\{ \dfrac{f(y)}{g(y)} \right\}$

Αλγόριθμος Αποδοχής-Απόρριψης

① Δημιουργία $Y = y \sim g$

② " $U \sim U(0,1)$

③ Αν $U \le \dfrac{f(y)}{c\,g(y)} \Rightarrow$ accept, $X = y$

διαφορετικά απορρ. Y, επιστροφή στο ①

Ισοδύναμα $Y = y$ γίνεται δεκτό με πιθ. $\dfrac{f(y)}{c\,g(y)}$

Θεώρημα

1) $X \sim f$

2) $N =$ αρ. βημάτων μέχρι την πρώτη αποδοχή της Y, $N \sim Geom(\frac{1}{c})$

$\Rightarrow \boxed{E(N) = c}$

Παράδειγμα 1

$f(x) = 20x(1-x)^3, \quad 0 < x < 1$

Έστω $g(x) = 1, \quad 0 \le x \le 1 \quad (Y \sim U(0,1))$

$$\frac{f(x)}{g(x)} = 20x(1-x)^3 \Rightarrow \max_{x\in[0,1]} \frac{f(x)}{g(x)}$$

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = 20\left[(1-x)^3 - 3x(1-x)^2\right] =$$

$$= 2(1-x)^2(1-4x) = 0 \Rightarrow x = \frac{1}{4} \leftarrow \max$$

$$\max_{x[0,1]} \frac{f(x)}{g(x)} = f\left(\frac{1}{4}\right) = \boxed{\frac{135}{64} = c} \approx 2.1$$

$$\frac{f(x)}{cg(x)} = \frac{20x(1-x)^3}{\frac{135}{64}} = \frac{256}{27} x(1-x)^3$$

Αλγόριθμος

1. $Y \sim U(0,1)$ έστω $Y = y$
2. $U \sim U(0,1)$
3. Αν $U \le \frac{256}{27} y(1-y)^3 \Rightarrow X = y$

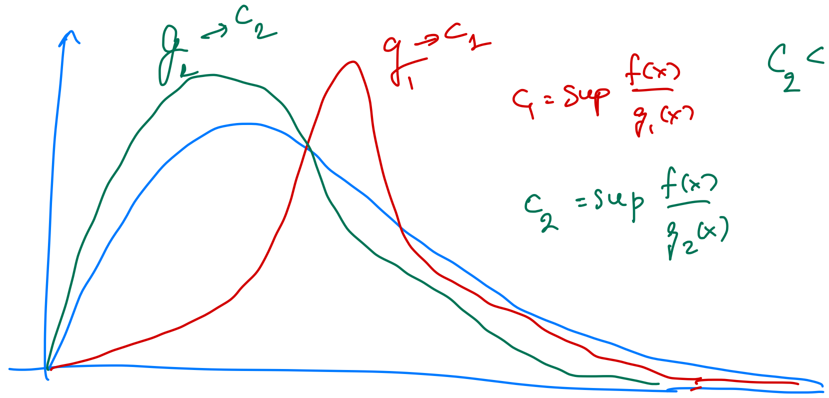
   διαφ. επιστρ. στο 1

Παραδ 2

$X \sim \Gamma(\alpha, \lambda)$ $\qquad x \in (0, \infty)$

$$X \sim \Gamma(\alpha,\lambda) \Rightarrow f(x) = \frac{\lambda^{\alpha} x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)} = K x^{\alpha-1} e^{-\lambda x}$$

$$K = \frac{\lambda^{\alpha}}{\Gamma(\alpha)}$$



$c_1 = \sup \frac{f(x)}{g_1(x)}$ $\qquad c_2 < c_1$

$c_2 = \sup \frac{f(x)}{g_2(x)}$

$g(x)$: $\exists$ γεννήτρια (εύκολη)

Έστω ότι επιλέγουμε τη $g \sim Exp(\mu)$, (μ : κατάλληλα επιλεγμένο)

$g(x) = \mu e^{-\mu x}$

Έστω $\quad C(\mu) = \sup_{x>0} \frac{f(x)}{\mu e^{-\mu x}}$

$$\mu^* : \quad C(\mu^*) = \min_{\mu>0} C(\mu)$$

$$C(\mu) : \quad \frac{f(x)}{\mu e^{-\mu x}} = \frac{K x^{\alpha-1} e^{-\lambda x}}{\mu e^{-\mu x}} = \frac{K}{\mu} \cdot \underbrace{x^{\alpha-1} e^{(\mu-\lambda)x}}_{\uparrow \sup_x}$$

1) $\alpha < 1 \Rightarrow \lim_{x \to 0} x^{\alpha-1} e^{(\mu-\lambda)x} = \infty \quad \forall \mu,$

$$\Rightarrow \sup_{x>0} x^{\alpha-1} e^{(\mu-\lambda)x} = +\infty \Rightarrow C(\mu) = \infty \quad \forall \mu.$$

Επομένως δε μπορεί να χρησιμοποιηθεί η $g(\mu) \sim Exp(\mu)$ για κανένα μ.

2) $\alpha > 1$: $\lim_{x \to 0} x^{\alpha-1} e^{(\mu-\lambda)x} = 0$

$$\lim_{x \to \infty} x^{\alpha-1} e^{(\mu-\lambda)x} = \begin{cases} 0 & \text{όταν } \mu < \lambda \\ +\infty & \text{όταν } \mu > \lambda . \end{cases}$$

Επομένως πρέπει $\mu < \lambda$ $\quad c(\mu) = \sup_{x \geq 0} \frac{f(x)}{g(x)}$

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{d}{dx} x^{\alpha-1} e^{(\mu-\lambda)x} = (\alpha-1) x^{\alpha-2} e^{(\mu-\lambda)x} + (\mu-\lambda) x^{\alpha-1} e^{(\mu-\lambda)x} = 0$$

$$\Rightarrow \quad x = \frac{\alpha-1}{\lambda-\mu} \Rightarrow$$

$$\Rightarrow \quad c(\mu) = \frac{f\left(\frac{\alpha-1}{\lambda-\mu}\right)}{g\left(\frac{\alpha-1}{\lambda-\mu}\right)} = \cdots = \frac{K}{\mu}\left(\frac{\alpha-1}{\lambda-\mu}\right)^{\alpha-1} \cdot e^{1-\alpha}$$

$$\min_{0 \leq \mu \leq \lambda} c(\mu) : \quad \frac{d}{d\mu} c(\mu) = 0 \Rightarrow \cdots \Rightarrow \boxed{\mu = \frac{\lambda}{\alpha}}$$

$$\frac{1}{\mu} = \frac{\alpha}{\lambda}$$

$$E_g(Y) = E_f(X)$$