4-5-2023

Tilted densities

Given $X \sim$ pdf $f(x)$

$$M(t) = E(e^{tX}) = \int e^{tx} f(x)\,dx \qquad t \in \mathbb{R}$$

tilted density $f_t(x)$

$$f_t(x) = \frac{e^{tx} f(x)}{M(t)}$$

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① $X \sim Exp(\lambda)$ $\quad f(x) = \lambda e^{-\lambda x}, \; x > 0.$

$$f_t(x) = C e^{tx} e^{-\lambda x} = C e^{-(\lambda - t)x} \qquad t < \lambda.$$

$$\parallel$$

$$\sim Exp(\lambda - t) \qquad (\lambda - t) e^{-(\lambda-t)x}$$

$$E_{f_t}(X) = \frac{1}{\lambda - t} \qquad \uparrow_t$$

$$F_t(x) = 1 - e^{-(\lambda-t)x} \qquad \downarrow_t \qquad 1 - F_t(x) \uparrow_t \; \forall x$$

$$P_{f_t}(X > x) \uparrow_t$$

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② $X \sim \text{Bernoulli}(p)$

$$f(x) = \begin{cases} p, & x = 1 \\ 1-p, & x = 0 \end{cases} = p^x (1-p)^{1-x}, \quad x = 0, 1$$

$$M(t) = E(e^{tX}) = pe^t + (1-p)$$

$$f_t(x) = \frac{e^{tx} p^x (1-p)^{1-x}}{pe^t + 1 - p} =$$

$$= \frac{(pe^t)^x \cdot (1-p)^{1-x}}{pe^t + (1-p)} = q^x (1-q)^{1-x} \quad x = 0, 1$$

$$\boxed{q = \frac{pe^t}{pe^t + 1 - p}} \qquad X_{f_t} \sim \text{Ber}\left(\frac{pe^t}{pe^t + 1 - p}\right)$$

③ $X \sim \mathcal{N}(\mu, \sigma^2)$

$$X|_{f_t} \sim \mathcal{N}(\mu + \sigma^2 t, \sigma^2)$$

Παραδείγματα

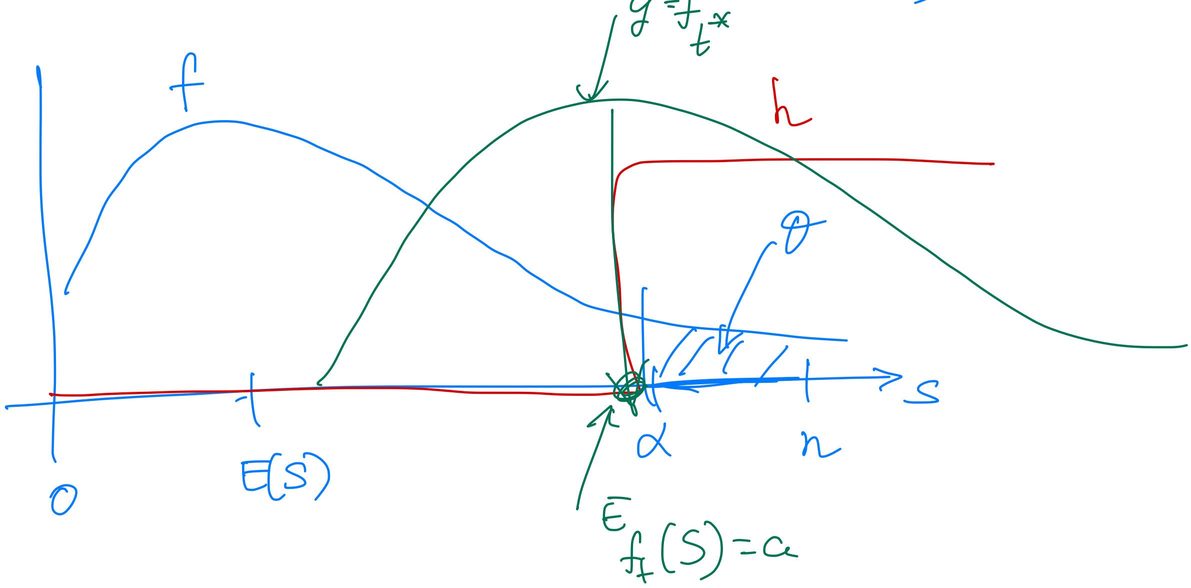
① $X_1, \dots, X_n$ ανεξ.,

$X_j \sim Ber(p_j)$

$S = X_1 + \dots + X_n$

$\underbrace{\theta = P(S \geq \alpha)}$, όπου $\alpha > E(S) = \sum p_j$

$\theta \approx 0$ $\theta = E(\underbrace{1(S > \alpha)}_{h(S)})$



$\theta = E_f(1(S > \alpha))$ $f(x_1, \dots, x_n) = f_1(x) \dots f_n(x_n)$

1) Monte Carlo
N σενάρια
$\begin{cases} \Delta\eta\mu.\ X_1, \dots, X_n \\ \quad \downarrow \\ Ber(p_1) \dots Ber(p_n) \\ h(S) = 1(S > \alpha) \end{cases}$

2) Tilted densities των $f_1, \dots, f_n$ με ίδιο t:

$$f_t(x_1, \dots, x_n) = f_{t,1}(x_1) \cdots f_{t,n}(x_n)$$

$$\theta = E_{f_t}\left[1(S \geq \alpha) \cdot \frac{f(x_1, \dots, x_n)}{f_t(x_1, \dots, x_n)} \right]$$

$$= E_{f_t}\left[\underbrace{1(S \geq \alpha) \cdot \prod_{j=1}^{n} \frac{f_j(x_j)}{f_{jt}(x_j)}}_{\hat{\theta}} \right]$$

$$f_{jt}(x_j) = \frac{e^{t x_j} f_j(x_j)}{M_j(t)} \qquad M_j(t) = E_{f_j}\left(e^{tX_j}\right)$$

$$\frac{f_j(x_j)}{f_{jt}(x_j)} = e^{-tX_j} M_j(t)$$

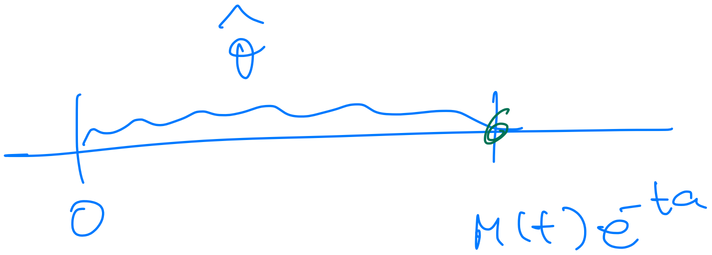
$$\hat{\theta} = 1(S \geq \alpha) \cdot e^{-t\overbrace{(X_1 + \dots + X_j)}^{S}} \underbrace{\prod_{j=1}^{n} M_j(t)}_{M(t)}$$

$$\hat{\theta} = 1(S \geq \alpha) \cdot \underline{M(t)}\, e^{-tS}$$

$$M(t) = \prod_{j=1}^{n} \left(p_j e^t + 1 - p_j\right)$$

$$1(S \ge \alpha)\, e^{-tS} \le e^{-ta} \quad \forall S.$$

$$\Rightarrow \boxed{\hat{\theta} \le M(t)\, e^{-ta}}$$



Επιλέγουμε t^* : $M(t)\, e^{-ta} \leftarrow \min.$

$$\left(M(t)\, e^{-ta}\right)' = M'(t)\, e^{-ta} - a M(t)\, e^{-ta} =$$

$$= \left[M'(t) - a M(t)\right] e^{-ta} = 0$$

$$\Rightarrow M'(t) - a M(t) = 0,$$

Όμως $M(t) = E(e^{tS})$

$$M'(t) = E_f(S e^{tS})$$

$$\Rightarrow E_f\left[S e^{tS} - a e^{tS}\right] = 0 \Rightarrow$$

$$\Rightarrow E_f\left[(S-a)\, e^{tS}\right] = 0 \Rightarrow$$

$$\Rightarrow \quad E_{f_t}(S-a) = 0 \quad \Rightarrow$$

$$t^* \quad \boxed{E_{f_{t^*}}(S) = a} \leftarrow \text{numerically.}$$

$$f_{j,t} \quad \text{Bernoulli}\left(p_j e^t \Big/ (p_j e^t + 1 - p_j)\right)$$

$$E_{f_{t^*}}(S) = \sum_{i=1}^{n} \frac{p_i e^t}{p_i e^t + (1-p_i)} = a \quad (\text{solve in } t)$$

$$\sum_{i=1}^{n} \left[1 - \frac{1-p_i}{p_i e^t + (1-p_i)}\right] = a \quad \Rightarrow$$

$$\Rightarrow \sum \frac{1-p_i}{p_i e^t + (1-p_i)} = n - a$$

Monte Carlo $\quad \hat{\theta} = 1(S > a)$

IS $\quad \hat{\theta} : 1(S > a) \cdot M(t) e^{-tS}$

$$M(t) = \prod_{j=1}^{n} \left(p_j e^t + (1-p_j)\right)$$