10-5-2023

Ασκήσεις Κεφ. 9

(2) $$\theta = \int_0^1 \int_0^1 e^{x^2+y^2}\, dy\, dx$$

[Αντιθετικές μεταβλητές. Είναι χρήσιμη η μέθοδος εδώ;]

(a) $\theta = E\left[G(U_1, U_2)\right]$,

$G(u_1, u_2) = e^{u_1^2 + u_2^2}$, U_1, U_2 iid $U(0,1)$

Ζεύγη (U_{1j}, U_{2j}) $j = 1, \dots, n$

$G_j = e^{U_{1j}^2 + U_{2j}^2}$, $j = 1, \dots, n$

$$\hat{\theta} = \frac{1}{n} \sum_{j=1}^{n} G_j$$

(b) Antithetic Variables

Ζεύγη (U_{1j}, U_{2j}), $j = 1, \dots, n/2$

$(U'_{1j}, U'_{2j}) = 1 - U_{1j}, 1 - U_{2j}$, $j = 1, \dots, n/2$

$$G_j = e^{u_{1j}^2 + u_{2j}^2}$$

$$G_j' = e^{u_{1j}'^2 + u_{2j}'^2}$$

$$\hat{\theta} = \frac{1}{n/2} \sum_{j=1}^{n/2} (G_j + G_j')$$

(γ) Είναι χρήσιμη η μέθοδος;

Πρέπει $Cov(G_j, G_j') < 0$.

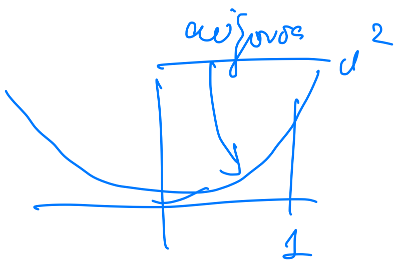
<u>Λήμμα</u> αν $H(U_1, \dots, U_k)$ μονότονη ως προς όλα τα ορίσματα

τότε $Cov\big(H(U_1,\dots,U_k),\ H(1-U_1,\dots,1-U_k)\big) < 0$.

$$G = H(U_1, U_2) = e^{U_1^2 + U_2^2}$$

εδώ $U_i \in (0,1)$

$e^{u_i^2} \uparrow \quad u \in (0,1)$



ισχύει το Λήμμα $\Rightarrow$ $\underline{Cov(G_j, G_j') < 0}$ ✓

(άσκηση προγραμματίστε τις δύο μεθόδους και δείτε κατά πόσο μειώνεται η $Var(\hat{\theta})$.)

(3) $X_1, \ldots, X_5$ iid $\sim \mathcal{E}xp(1)$

$$\theta = P\left(\sum_{j=1}^{5} j X_j \geq \alpha\right)$$

(a) Αντιθετικές μετ.

(b) Χρήσιμη η μέθοδος;

(a) $\theta = E\left[1\left(\sum_{j=1}^{5} j X_j \geq \alpha\right)\right]$ $\quad X_j \sim$ iid $\mathcal{E}xp(1)$

raw simulation

For $k = 1, \ldots, N$

$X_k = (X_{k1}, \ldots, X_{k5})$ $\quad \sim \mathcal{E}xp(1)$ (μπορούμε $X_k =$ rexp(5,1))

$Z_k = 1\left(\sum_{j=1}^{5} j X_{kj} \geq \alpha\right)$ $\quad k = 1, \ldots, N$

$\hat{\theta} = \bar{Z}_N$

Antithetic $\quad X = -\log U \;,\; U \sim \mathcal{U}(0,1)$.

$k = 1, \ldots, N/2 \;:\; X_{kj} = -\log(U_{kj})$ $\quad j = 1, \ldots, 5$, $k = 1, \ldots, n/2$

$X'_{kj} = -\log(1 - U_{kj})$ $\quad j = 1, \ldots, 5$, $k = 1, \ldots, n/2$

$$Z_k = 1\left(-\sum_{j=1}^{5} j \log(U_{kj}) \geq \alpha\right)$$

$$= 1\left(\sum_{j=1}^{5} j \log(U_{kj}) \leq -\alpha\right) = H(U_1, \ldots, U_5)$$

$$Z'_k = 1\left(\sum_{j=1}^{5} j \log(1 - U_{kj}) \leq -\alpha\right)$$

$$H(u_1,\dots,u_5) = 1\left(\sum_{j=1}^{5} j\log(u_j) \le -\alpha\right)$$

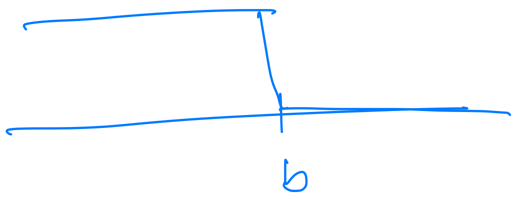
ως προς u_1: $\log u_1 + \sum_{j=2}^{5} j\log u_j \le -\alpha \Rightarrow$

$$\Rightarrow \log u_1 \le -\alpha - \sum_{j=2}^{5} j\log u_j$$

$$u_1 \le b(u_2,\dots,u_5)$$

$$H(u_1,\dots,u_5) = \begin{cases} 1, & u_1 \le b \\ 0, & u_1 > b \end{cases}$$ φθίνουσα

φθίνουσα ως προς $u_1,\dots,u_5$



$$\Longrightarrow \quad Cov(Z_k, Z_k') < 0 \quad \forall k.$$

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6 Πως μπορούμε να παράξουμε την αντιθετική της $X \sim Exp(1)$, χωρίς U ?

χρησιμ. X = rexp(1 1)

$$X \rightarrow X' = G(X) \ ?$$

$X = -\log U, \quad U \sim U(0,1)$

$\Rightarrow U = e^{-X}$

$\Rightarrow 1 - U = 1 - e^{-X}$

αντίσ.
$F(X) = U \Rightarrow 1 - e^{-X} = U$
$\Rightarrow X = -\log(1-U)$
Επίσης $X = -\log U$ ✓

$$\Rightarrow X' = -\log(1-u) = -\log(1-e^{-x})$$

$$\Rightarrow \boxed{X' = -\log(1-e^{-X})}$$

1, 12 $\quad \theta = \int_0^1 e^{x^2} dx$ (αναθετικές μεθ).

Μεταβλητή ελέγχου (?)

$$\theta = E(e^{U^2}), \quad U \sim U(0,1).$$

$$\theta = E(X), \quad X = e^{U^2}, \quad U \sim U(0,1).$$

Y : $E(Y) = \mu_Y$ γνωστή

$Cov(X,Y) \neq 0$.

Αν $Y = U \Rightarrow \mu_Y = \frac{1}{2}$

$Cov(e^{U^2}, U) \neq 0$.

$$X' = e^{U^2} + c\left(U - \frac{1}{2}\right)$$

$$c^* = -\frac{Cov(U, e^{U^2})}{Var(U)} = -12\, Cov(U, e^{U^2})$$

$$U_1, \ldots, U_n \quad \begin{array}{l} \nearrow x_1, \ldots, x_n \\ \searrow u_1, \ldots, u_n \end{array} \Bigg\} \quad \hat{c} = -12\, \widehat{Cov}(X, U)$$

$$\Rightarrow \quad X' = X + \hat{c}\left(U - 1/2\right) \quad \ldots$$

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$$(X, Y)$$

$$Y \sim N(1, 1)$$

$$X \mid Y = y \sim N(y, 4)$$

$$\theta = P(X > 1)$$

ⓐ <u>Raw simulation</u>

$\forall\ j = 1, \ldots, N$

$$\begin{cases} Y \sim N(1,1) & (\pi.\chi.\ \ Y = \text{rnorm}(1, 1, 1)) \\ X \underset{Y=y}{\mid} \sim N(y, 4) & (\pi.\chi.\ \ X = \text{rnorm}(1, Y, 2)) \\ W_j = 1(X > 1) & \end{cases}$$

$$\hat{\theta} = \overline{W}_N$$

ⓑ Variance Reduction with Conditioning:

$$\theta = E(X)$$

$$= E\left(E(X \mid Y)\right) = E\left(m(Y)\right)$$

όπου $m(Y)$ γνωστή συνάρτηση (υπολογίζεται απευθείας)

Έστω $\theta = E(1(X>1)) = E_Y[\underbrace{E(1(X>1)|Y)}]$

$$m(y) = E(1(X>1)|Y=y) = P(X>1|Y=y)$$

$$= P\left(\frac{X-y}{2} > \frac{1-y}{2} \,\middle|\, Y=y\right) =$$

$$= P\left(Z > \frac{1-y}{2}\right) \qquad Z \sim N(0,1)$$

$$\Rightarrow \quad m(y) = 1 - \Phi\left(\frac{1-y}{2}\right) = \Phi\left(\frac{y-1}{2}\right)$$

Π.χ. $m(y) = \text{pnorm}\left(\frac{y-1}{2}, 0, 1\right)$

$$\boxed{\tilde{\theta} = E_Y\left[\Phi\left(\frac{Y-1}{2}\right)\right], \quad Y \sim N(1,1)}$$

(β) Σε αυτό επιλέγω αντιθετικές μεταβλητές

$Y \sim N(1,1)$ από ?

$$Y \sim N(1,1) \Rightarrow Y = 1 + Z \qquad Z \sim N(0,1)$$

Z μέσω αντιστρ. μετ/σμού : $\Phi(Z) = U \Rightarrow Z = \Phi^{-1}(U)$

$$Y = 1 + \Phi^{-1}(U)$$

[στο R: $Y = 1 + \text{qnorm}(U, 0, 1)$]

$\tilde{\theta} = E(R)$, $\quad R = \Phi\left(\frac{Y-1}{2}\right)$, $\quad Y \sim N(1,1)$

$$R = \Phi\left(\frac{1+\Phi^{-1}(u)-1}{2}\right) = \underbrace{\Phi\left(\frac{\Phi^{-1}(u)}{2}\right)}_{H(u)}$$

$$R' = \Phi\left(\frac{\Phi^{-1}(1-u)}{2}\right)$$

$$\frac{\Phi^{-1}(u)}{2} \uparrow, \quad \Phi\left(\frac{\Phi^{-1}(u)}{2}\right) \uparrow_u$$

(γ) Έστω $\theta = E\left(\Phi\left(\frac{Y-1}{2}\right)\right) = E(R)$

επιλέγω control variate

Control variate $Y \Rightarrow E(Y) = 1$

(π.χ. άλλη επιλογή $\tilde{Y} = Y^2$, $E(Y^2) = \underbrace{Var(Y) + (E(Y))^2}_{=2}$)

ομοίως. $h(Y)$: $E(h(Y))$ γνωστή.

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Ασφ. εταιρεία

N = αρ. ατυχημάτων σε 1 εβδομάδα.

U: ητρικό παράγοντας $\sim \mathcal{U}(0,1)$

$$N \mid U=u \sim \text{Poisson}\left(\frac{15}{0.5+u}\right)$$

$$\theta = P(N>k) = E(X), \quad X = \mathbb{1}(N>k)$$

(a) raw simulation ~~(simple)~~ $\forall j$ $U_j \to N_j \to X_j = \mathbb{1}(N_j>k)$

$$\hat{\theta} = \bar{X}_n$$

$U \sim \text{runif}(0,1)$, $U=u$

$$N = \text{rpoisson}\left(1, \frac{15}{0.5+u}\right)$$

(b) Συνδ. conditioning + control variable.

$$\theta = E(X) = E_u\left[E(X|U)\right]$$

$$E(X|U=u) = E\left[\mathbb{1}(N>k) \mid U=u\right] =$$

$$= P(N>k \mid U=u) = \overbrace{1-\text{ppoisson}\left(k, \frac{15}{1/2+u}\right)}^{m(u)}$$

$$= 1-\sum_{j=0}^{k} e^{-\lambda}\frac{\lambda^j}{j!}, \quad \lambda = \frac{15}{1/2+u}$$

$$\theta = E_u\left[m(u)\right]$$

Αντιθ. μετ. $u_1, \dots, u_{n/2}$

$1-u_1, \dots, 1-u_{n/2} = u'_1, \dots, u'_{n/2}$

$X_j = m(u_j)$

$X'_j = m(u'_j)$

$$\tilde{\theta} = \frac{1}{n/2} \sum_{j=1}^{n/2} (X_j + X'_j)$$

$m(u) = 1 - F_u(k)$, F : Poisson$(\lambda(u))$

$u_1 > u_2$ $\quad N_1 \sim \text{Poisson}(\lambda(u_1))$

$N_2 \sim \text{Poisson}(\lambda(u_2))$

$$\lambda(u) = \frac{15}{1/2 + u} = \frac{30}{1+2u}$$

Θέλ νδο. $\underbrace{P(N_1 > k)}_{m(u_1)} \overset{?}{\gtrless} \underbrace{P(N_2 > k)}_{m(u_2)}$

$$N_1 \underset{st}{\gtrless} N_2$$

$\lambda(u_1) < \lambda(u_2)$

$E(N_1) < E(N_2)$

$\Rightarrow ?$ $N_1 \leq_{st} N_2$?

Στοχ. σύγκριση

$X_1 \sim F_1$

$X_2 \sim F_2$

$X_1 \leq X_2$ μ.π. 1 $\quad$ (π.χ. $X_1 \sim U(0,1)$, $X_2 \sim U(7,10)$)

Στοχαστική διάταξη

$X_1 \leq_{st} X_2 \Leftrightarrow P(X_1 > \alpha) \leq P(X_2 > \alpha) \quad \forall \alpha \in \mathbb{R}$

$1 - F_1(\alpha) \leq 1 - F_2(\alpha)$

$$X_1 \leq_{st} X_2 \Leftrightarrow F_1(\alpha) \geq F_2(\alpha) \quad \forall \alpha \in \mathbb{R}.$$

π.χ. $X_1 \sim Exp(\lambda_1)$, $X_2 \sim Exp(\lambda_2)$ $\quad \lambda_1 < \lambda_2$ $\left(\frac{1}{\lambda_1} > \frac{1}{\lambda_2}, E(X_1) > E(X_2)\right)$

$$F_1(x) = 1 - e^{-\lambda_1 x} \overset{(\lambda_1)}{<} 1 - e^{-\lambda_2 x} \quad \forall x \Rightarrow$$

$X_1 \geq_{st} X_2$

Λήμμα

$$\boxed{X_1 \leq_{st} X_2 \Leftrightarrow E(h(X_1)) \leq E(h(X_2)) \quad \forall h: \text{αύξουσα}}$$

Poisson (?)