15-5-2023

Παράδειγμα Metropolis-Hastings

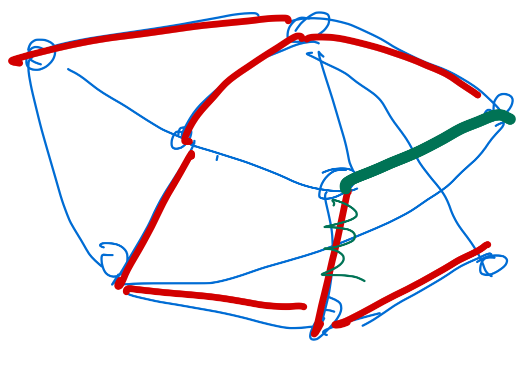
Σύνολο L $\quad |L| < \infty$. $|L|$ πολύ μεγάλο

π.χ. ① $L = \{ (x_1, \dots, x_n) \in \mathbb{Z}^n , \ x_j \geq 0 , \ x_1 + \dots + x_n = K \}$

$$\pi(x) = \begin{cases} C , & x \in L \\ 0 , & x \notin L \end{cases}$$

$$C = \frac{1}{|L|}$$

② Έστω γράφημα G

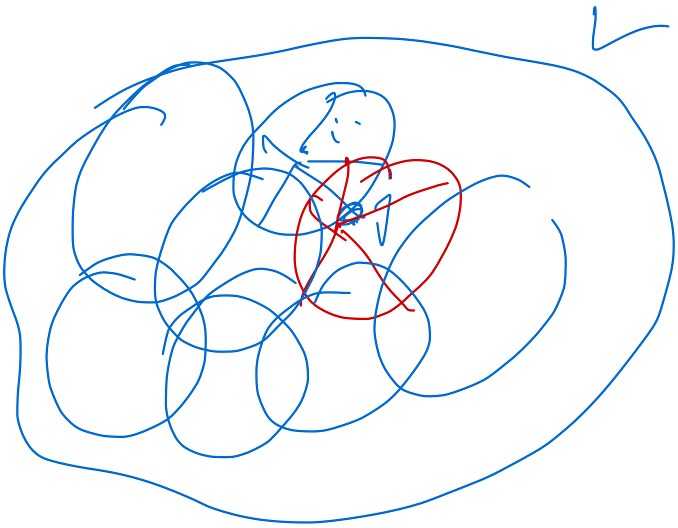


$L = \{$ υπογραφήματα του G που είναι δέντρα $\}$

$|L| < \infty$

Metropolis - Hastings

MC στο $S = L$



$\forall\, i \in L$: $N(i) = \{$γειτονικά στοιχεία του $i\}$

π.χ. στο παρ. 1 :

Αν $\quad x = (1, 5, 2, 7) \qquad K = 15$

$y \in N(x)$: $\quad y = (1, 2, 5, 7) \rightarrow \times$

ή $\quad y = (0, 6, 2, 7)$
$\quad = (2, 4, 2, 7)$
$\quad = (1, 4, 3, 7)$ ✓

$$q_{ij} = \frac{1}{|N(i)|}, \quad j \in N(i)$$

$$\pi_i = C = C \cdot 1, \quad b_i = 1 \quad \forall i \in L$$

$$a_{ij} = \min\left\{ \frac{b_j q_{ji}}{b_i q_{ij}}, 1 \right\} = \min\left\{ \frac{\frac{1}{|N(j)|}}{\frac{1}{|N(i)|}}, 1 \right\}$$

$$= \min\left\{ \frac{|N(i)|}{|N(j)|}, 1 \right\}$$

Gibbs Sampler

$X = (X_1, \dots, X_n)$ διακριτή

$$f(x) = P(X=x) = P(X_1 = x_1, \dots, X_n = x_n)$$
$$= f(x_1, \dots, x_n)$$

$$f(x) = C\, g(x) \qquad \left[C = \frac{1}{\sum_x g(x)} \right]$$

π.χ. $n=2$

$f(x,y)$,

$$f_X(x) = \sum_y f(x,y) = C \overbrace{\sum_y g(x,y)}^{g_X(y)}$$

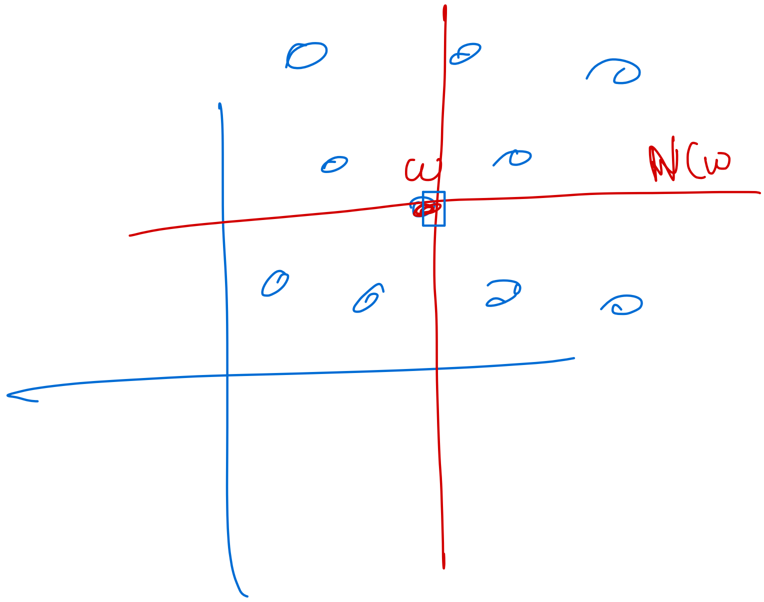
$$f_Y(y) = C \sum_x g(x,y)$$

$$f_{X|Y}(x|y) = P(X = x | Y = y) = \frac{f(x,y)}{f_Y(y)} = \frac{g(x,y)}{g_Y(y)}$$

Υποθέτουμε ότι υπάρχουν γεννήτορες από $f_{X|Y}$ κ' $f_{Y|X}$

Γενικά από $f(x_j \mid x_{-j})$ $\qquad x_{-j} = (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n)$

$$S = \{ \underbrace{(x,y)}_{w} \}$$



$$q_{w_1, w_2} = \dots$$

Επιλέγουμε κατεύθυνση $\begin{cases} \to x & 1/2 \\ \to y & 1/2 \end{cases}$

$$\left.\begin{array}{l} \text{Αν επιλ. } x \text{ : δημιουργούμε μια παρατήρηση } x' \\ \qquad \text{από } f(x'|y) \\ \\ \text{Όμοια αν επιλ. } y. \end{array}\right\} Q$$

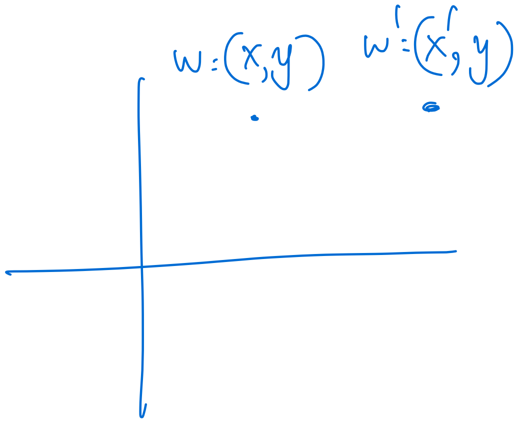
$$a_{w,w'} = \min\left\{ \frac{g(w')\, q_{w'w}}{g(w)\, q_{ww'}} , 1 \right\}$$

$$\text{Αν } w = (x,y) \Rightarrow w' = \begin{cases} (x,y') & \text{μ.π. } \frac{1}{2} f_{Y|X}(y'|x) \\ (x',y) & \text{μ.π. } \frac{1}{2} \cdot f_{X|Y}(x'|y) \end{cases}$$

Όμως $f_{Y|X}(y'|x) = \dfrac{f(x,y')}{f_X(x)} = \dfrac{g(x,y')}{g_X(x)}$

$$f_{X|Y}(x'|y) = \frac{f(x',y)}{f_Y(y)} = \frac{g(x',y)}{g_Y(y)}$$

Αν $w' = (x', y)$



$$q_{ww'} = \frac{1}{2}\,\frac{g(x',y)}{g_Y(y)},$$

$$q_{w'w} = \frac{1}{2}\,\frac{g(x,y)}{g_Y(y)}$$

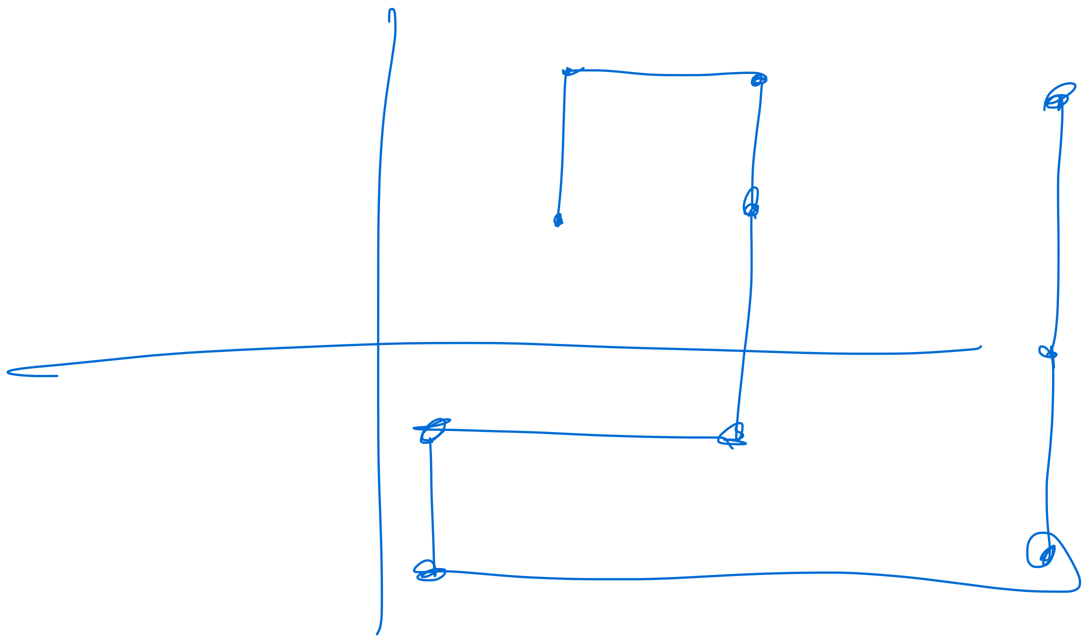
Επομένως $a_{ww'} = \min\left\{ \dfrac{g(x',y)\cdot\frac{1}{2}\frac{g(x,y)}{g_Y(y)}}{g(x,y)\cdot\frac{1}{2}\frac{g(x',y)}{g_Y(y)}},\ 1 \right\}$

$\boxed{a_{ww'} = 1}$ όμοια για $w' = (x, y')$

Γενικά $(x_1, \dots, x_n)$

1. Επιλέγουμε κατεύθυνση $j = 1, \dots, n$ τυχαία
2. Δημ. μια παρατήρηση από

$$f(x_j \mid x_{-j})$$



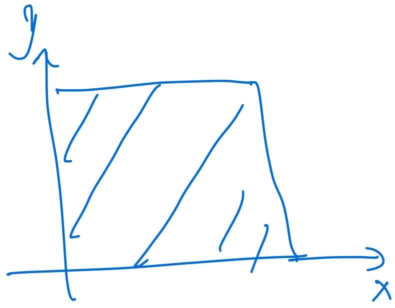
Παράδειγμα 1

$$(X,Y) \qquad f(x,y) = K(x+y) \qquad (x,y) \in [0,1]^2$$

Εφαρμογή Gibbs sampler

$$f_X(x) = \int_{y=0}^{1} K(x+y)\,dy = Kx + K\cdot\frac{1}{2} = K\left(x+\frac{1}{2}\right)$$

$$f_Y(y) = K\left(y+\frac{1}{2}\right)$$



$$f_{X|Y}(x|y) = \frac{K(x+y)}{K(y+1/2)} = \frac{x+y}{y+1/2}, \quad x\in[0,1].$$

$$f_{Y|X}(y|x) = \frac{x+y}{x+\frac{1}{2}}, \quad y\in[0,1].$$

Γεννήτριες από $f_{X|Y}$, $f_{Y|X}$ μέσω αντιστροφής

$$f_{X|Y}(x,y) = \frac{x+y}{y+\frac{1}{2}} \Rightarrow F(x,y) = \int_0^x \frac{u+y}{y+\frac{1}{2}}\,du =$$

$$= \frac{\frac{x^2}{2} + xy}{y+1/2} \qquad 0 \le x \le 1$$

$$F(X) = u \Rightarrow \frac{X^2}{2} + Xy = u\left(y + \frac{1}{2}\right) \Rightarrow$$

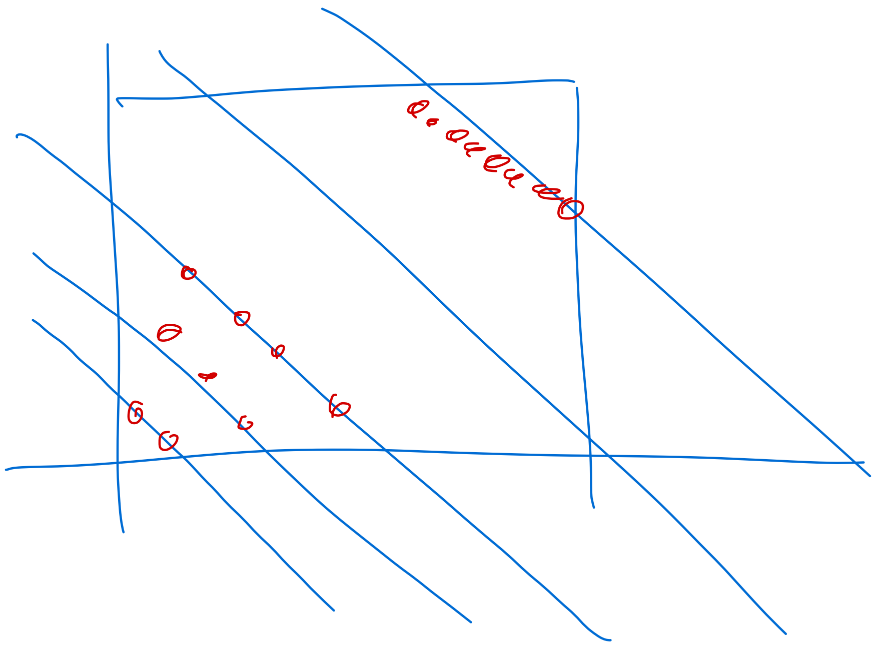
$$\Rightarrow X^2 + 2Xy - 2u\left(y + \frac{1}{2}\right) = 0$$

$$\Delta = 4y^2 + 8\left(y + \frac{1}{2}\right)u > 0$$

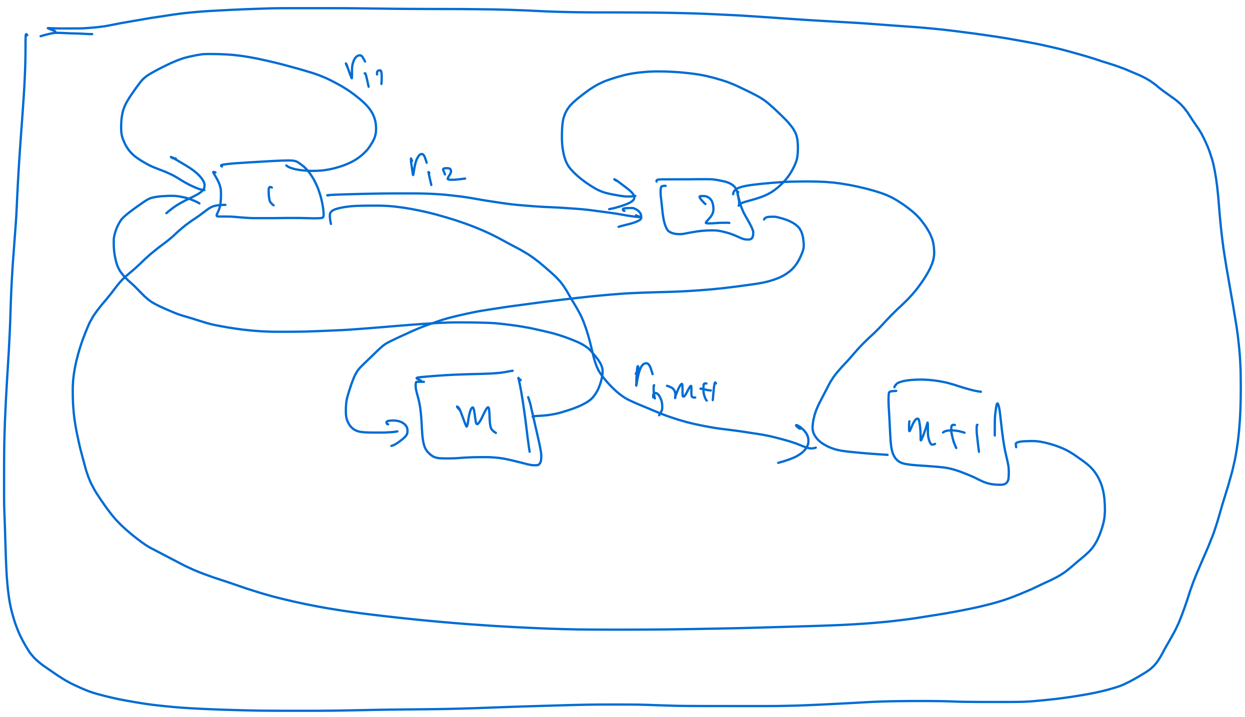
$$\boxed{X = \frac{-2y + \sqrt{4y^2 + 8u(y + 1/2)}}{2}}$$

Γεννήτρια από $f(y|x)$

$$\boxed{Y = \frac{-2x + \sqrt{4x^2 + 8u\left(x + \frac{1}{2}\right)}}{2}}$$



Παράδειγμα 2 Closed Jackson queueing network.



r customers

Ουρά j $\quad c_j$ servers $\quad S \sim Exp(\mu_j)$

Κατάσταση $\quad x = (x_1, \dots, x_{m+1})$

$x_j \geq 0$, $x_j \in \mathbb{Z}$, $\quad x_1 + \dots + x_{m+1} = r$

Ισοδύναμα : $x = (x_1, \dots, x_m)$, $\quad x_1 + \dots + x_m \leq r$

$$S = \{ (x_1, \dots, x_m),\ x_j \geq 0,\ x_j \in \mathbb{Z},\ x_1 + \dots + x_m \leq r \}$$

Στάσιμη κατανομή μορφή γινομένου (product form)

$$p(x_1, \dots, x_m) = \begin{cases} C \prod_{j=1}^{m} p_j(x_j), & \sum x_j \leq r \\ 0 & \text{διαφ.} \end{cases}$$

όπου $P_j(x_j)$: γνωστή μονοδιάστατη κατανομή (αποκ. γεωμετρική)

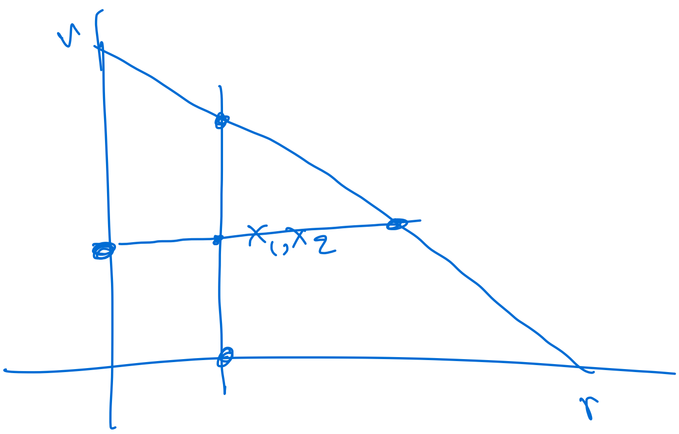
$$C = \frac{1}{\sum_{x\in S}\prod_{j=1}^{n} P_j(x_j)} \qquad ??$$

Gibbs sampler

$$P_j(x_j \mid x_{-j}) = P(X = x_j \mid X_k = x_k,\ k \neq j)$$

Έστω $x = (x_1, \dots, x_m)$ Έστω $\sum_{j=1}^{m} x_j = c$ $(\leq r)$

m=2



Δεδ. $\underbrace{x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_m} \Rightarrow \boxed{0 \leq x_j' \leq r - \sum_{k\neq j} x_k}$

$$P(X = x_j' \mid x_k,\ k \neq j) = \frac{P(x_1, \dots, x_j', \dots, x_m)}{\sum_j P(x_1, \dots, x_j', x_m)}, \quad 0 \leq x_j' \leq r - \sum$$

$$= \frac{C\,\cancel{P_1(x_1)}\cdots P_j(x_j')\cdots \cancel{P_m(x_m)}}{C\sum_{x_j'} \cancel{P_1(x_1)}\cdots P_j(x_j')\cdots \cancel{P_m(x_m)}}$$

$$= \frac{P_j(x_j')}{\sum_{x_j'} P_j(x_j')} = K \cdot P(x_j') \qquad 0 \le x_j' \le r - \sum_{k \ne j} x_k$$