# Waring's Problem

W. J. Ellison

*The American Mathematical Monthly*, Vol. 78, No. 1 (Jan., 1971), 10-36.

Stable URL:
http://links.jstor.org/sici?sici=0002-9890%28197101%2978%3A1%3C10%3AWP%3E2.0.CO%3B2-Q

*The American Mathematical Monthly* is currently published by Mathematical Association of America.

Choose a title for your paper that is short (not an abstract of the paper) but informative. Remember that each page of your article has a running head, which, hopefully, is most of your title.

BAD: Remark on a Theorem of Hilbert
BAD: A generalization of Hilbert's Norm Theorem to Abelian Extensions of Skew Fields of Characteristic Zero
GOOD: Hilbert's Norm Theorem for Skew Fields.

Avoid mathematical symbols in your title.

## 8. References

1. A manual for authors of mathematical papers, revised edition, Amer. Math. Soc., 1966.
2. Abbreviations of the names of scientific periodicals, Math. Reviews (reprints of the 1965 edition are sold by the Amer. Math. Soc.).
3. William Strunk, Jr. and E. B. White, The Elements of Style, Macmillan Paperbacks 107, 1962, $.95.
4. A Manual of Style, 12th edition, Univ. of Chicago Press, 1969.
5. P. G. Perrin, Writer's Guide and Index to English, 4th edition, Scott Foresman, 1965.
6. H. W. Fowler, Modern English Usage, 2nd edition corrected, Oxford Univ. Press, 1968.
7. The preparation and typing of mathematical manuscripts, Bell Telephone Labs, 1963. (Available from: Mathematical Manuscripts, Bell Telephone Labs, Inc., 463 West Street, New York, N. Y. 10014.)
8. Mathematics in Type, William Byrd Press, 1954.
9. Arthur Phillips, Setting Mathematics: A Guide to Printers Interested in the Art, John Wright and Sons, 1956.
10. T. W. Chaundy, P. R. Barrett, and C. Batey, The Printing of Mathematics, Oxford Univ. Press, 1954.
11. Rita de Clercq Zubli and Cynthia B. Wong, Guide to Technical Typing, D. H. Marks Publishing Company, Braintree, Mass., 1969, $7.95.

# WARING'S PROBLEM

W. J. ELLISON, University of Michigan

**1. Introduction.** Edward Waring in his book *Meditationes Algebraicae* (1770 edition, pages 203–204) makes the following statement: "*Omnis integer numerus vel est cubus; vel e duobus, tribus,* 4, 5, 6, 7, 8, *vel novem cubus compositus: est etiam quadratoquadratus; vel e duobus, tribus &c. usque ad novemdecim compositus &sic deinceps.*" In the 1782 edition, page 349, he adds guardedly "... *consimilia etiam affirmari possunt* (*exceptis excipiendis*) *de eodem numero quantitatum earundem dimensionum.*"

It has become traditional to interpret these assertions as: "Can every posi-

W. J. Ellison took his B.A. at Cambridge University and is continuing work there for a Ph.D. under Prof. J. W. S. Cassels. He spent 1969–70 at the University of Michigan with D. J. Lewis. Evidently he is a promising young number theorist. *Editor.*

tive integer be expressed as a sum of at most $g(k)$ $k$th powers of positive integers, where $g(k)$ depends only on $k$, not on the number being represented?" and to call the resulting problem "Waring's problem." There seems little doubt that Waring had only limited numerical evidence in favor of his assertions and no shadow of a proof.

The case $k=2$ had been stated by Fermat in 1640 and was attacked unsuccessfully by Euler for a very long time. It was finally proved by Lagrange in 1770, who showed that each positive integer could be expressed as a sum of at most four squares of positive integers. During the next 139 years, special cases of the problem were solved for $k=3, 4, 5, 6, 7, 8, 10$. It was in 1909 that Hilbert solved the problem in the affirmative for all $k$. His proof was extremely complicated in its detailed arguments. The key result was a proof of the following lemma.

LEMMA 1.1. *For each pair of positive integers $k$ and $n$ there are: an integer $M=(2k+1)\cdots(2k+n-1)/(n-1)!$, positive rationals $\lambda_0, \cdots, \lambda_M$, and integers $\alpha_{11}, \cdots, \alpha_{1M}, \alpha_{21}, \cdots, \alpha_{nM}$ such that*

$$(x_1^2 + \cdots + x_n^2)^k = \sum_{i=0}^{M} \lambda_i(\alpha_{1i}x_1 + \cdots + \alpha_{ni}x_n)^{2k}.$$

The reader may wish to test his ingenuity by proving the lemma in the following special cases: $n=2$, $k=2$; and $n=3$, $k=3$. In section 5 of this survey we shall give a complete, short, elementary version of Hilbert's solution of Waring's problem.

Once one knows that the answer to Waring's problem is "yes!" it is natural to ask "How big is $g(k)$?" We can easily see that $g(k) \geqq [(3/2)^k]+2^k-2$ because the integer $n=2^k[(3/2)^k]-1$ is less than $3^k$, and its minimal expression as a sum of $k$th powers is a sum of $([(3/2)^k]-1)$ $k$th powers of 2 and $(2^k-1)$ $k$th powers of 1. We shall see later that $[(3/2)^k]+2^k-2$ is probably the correct value of $g(k)$ for all $k$. Obtaining an upper bound looks (and is) very much harder.

Hilbert's proof, as it stood in 1909, was not very amenable to giving an explicit upper bound for $g(k)$. Stridsberg [13] however, gave an explicit proof of Lemma 1.1 and the way was open for obtaining an explicit upper bound for $g(k)$. Strangely enough nobody took it until 1953 when Rieger [11] worked out the details and showed that

$$g(k) \leqq (2k+1)^{260(k+3)^{(3k+8)}}.$$

In the intervening period Hardy and Littlewood published a series of papers during the 1920's in which they used a powerful new analytic technique to resolve Waring's problem and to show that $g(k)=O(k2^k+1)$. This upper bound, though large compared to the trivial lower bound for $g(k)$, heralded the beginning of an era in the theory of numbers.

From the work of Hardy and Littlewood it became apparent that a more fundamental number than $g(k)$ was $G(k)$, which is defined to be the least positive integer such that all sufficiently large integers can be represented as a sum of at most $G(k)$ $k$th powers of positive integers. That is, there are infinitely many integers which actually need $G(k)$ $k$th powers. For example, each integer of the form $8n+7$ really does need 4 squares in its representation as a sum of squares. For the squares modulo 8 are congruent to 0, 1 or 4, so if $n\equiv 7 \pmod 8$, then $n$ cannot be written as a sum of three squares; hence $G(2)\geqq 4$. As $G(2) \leqq g(2)=4$ we have $4=g(2)=G(4)$. As a further illustration, it was proved by Dickson [35] that $g(3)=9$, but 23 and 239 are the only integers which actually need 9 cubes; each integer greater than 239 can be written as a sum of at most 8 cubes. But only a finite number of integers really need 8 cubes, from some point on 7 cubes suffice. The tables seem to indicate that the point is 8042. The precise value of $G(3)$ is not known; the best result that I know is $4\leqq G(3)\leqq 7$.

Hardy and Littlewood did much more than obtain an upper bound for $G(k)$; they obtained an asymptotic formula for the number of integral solutions of the equation $x_1^k+\cdots+x_s^k=N$, $x_1\geqq 0,\cdots,x_s\geqq 0$. In section 3 I shall show how the following theorem is proved.

THEOREM 1.2. *If* $s\geqq 2^k+1$, *then the number* $r_{k,s}(N)$ *of solutions in integers of* $x_1^k+\cdots+x_s^k=N$, $x_1\geqq 0\cdots,x_s\geqq 0$, *satisfies*

$$r_{k,s}(N)=\frac{\Gamma(1+(1/k))^s}{\Gamma(s/k)}N^{(s/k)-1}\mathfrak{S}(N)+o(N^{(s/k)-1}) \quad as\ N\to\infty,$$

*where* $\mathfrak{S}(N)\geqq c>0$ *is a certain arithmetical function of* $N$.

Vinogradov made great technical improvements to the Hardy-Littlewood method, and he was able to show that the conclusion of Theorem 1.2 holds if $s>c_1k^2\log k$, where $c_1$ is a positive real number. For large values of $k$ this is a much weaker condition than $s\geqq 2^k+1$. Vinogradov also obtained upper bounds for $G(k)$. In [62] he proved $G(k)\leqq 6k\log k+k\log 216$. No substantial improvements on this estimate have been made, though in recent years the numerical factors have been reduced slightly.

Before going on to describe the Hardy-Littlewood-Vinogradov method in detail, I shall discuss Linnik's solution [17] of Waring's problem. Though his proof is strictly arithmetical, it was clearly very much influenced by the analytic method. Linnik's proof uses methods from the general theory of sequences. For a beautiful introduction to this fascinating branch of number theory one cannot do better than to read Halberstam and Roth's book *Sequences*.

**2. Linnik's method.** If $\mathfrak{A}$ is a sequence of positive integers the **Schnirelmann density** $\sigma(\mathfrak{A})$ is defined by $\sigma(\mathfrak{A})=$ greatest lower bound of $A(x)/x$ for $x\geqq 1$, where $A(x)$ is the counting function of $\mathfrak{A}$, that is $A(x)$ is the number of elements of $\mathfrak{A}$ less than or equal to $x$. Clearly $0\leqq\sigma(\mathfrak{A})\leqq 1$. If $\sigma(\mathfrak{A})>0$ then $1\in\mathfrak{A}$. If $\sigma(\mathfrak{A})=1$ then $\mathfrak{A}=\boldsymbol{Z}^+$, the set of positive integers.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be two sequences of positive integers, then $\mathfrak{A}\oplus\mathfrak{B}$ is to consist of all positive integers which are either $a$ or $b$ or $a+b$, where $a\in\mathfrak{A}$ and $b\in\mathfrak{B}$, each counted only once. For example, if $\mathfrak{A}=\{1, 2, 5, 8\}$ and $\mathfrak{B}=\{3, 7, 10, 11\}$, then $\mathfrak{A}\oplus\mathfrak{B}=\{1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 18, 19\}$.

THEOREM 2.1. *For any sequences $\mathfrak{A}$ and $\mathfrak{B}$,*

$$\sigma(\mathfrak{A} \oplus \mathfrak{B}) \geqq \sigma(\mathfrak{A}) + \sigma(\mathfrak{B}) - \sigma(\mathfrak{A})\sigma(\mathfrak{B}).$$

*Proof.* Without any loss of generality, we can assume $\sigma(\mathfrak{A})>0$; thus $1\in\mathfrak{A}$. Let $1, a_2, \cdots, a_r\leqq x$ be the part of $\mathfrak{A}$ up to $x$. Whenever $a_i+1<a_{i+1}$ we note that $\mathfrak{A}\oplus\mathfrak{B}$ contains $a_i+b$ for all $b\in\mathfrak{B}$ satisfying $1\leqq b\leqq a_{i+1}-a_i-1$. Similarly if $a_r<x$, then $\mathfrak{A}\oplus\mathfrak{B}$ contains $a_r+b$ for all $b\in\mathfrak{B}$ satisfying $1\leqq b\leqq x-a_r$. Hence if $C(x)$ and $B(x)$ are the counting functions for $\mathfrak{A}\oplus\mathfrak{B}$ and $\mathfrak{B}$ respectively, we have the following inequality:

$$C(x) \geqq r + \sum_{i=1}^{r-1} B(a_{i+1} - a_i - 1) + B(x - a_r).$$

If we write $\beta=\sigma(\mathfrak{B})$ and $\alpha=\sigma(\mathfrak{A})$, we have $B(y)\geqq\beta y$ and $A(y)\geqq\alpha y$ for $y\geqq 0$. Thus

$$C(x) \geqq r + \sum_{i=1}^{r-1} \beta(a_{i+1} - a_i - 1) + \beta(x - a_r) \geqq (1 - \beta)r + \beta x.$$

Now $r=A(x)\geqq\alpha x$, hence we have the inequality

$$\frac{C(x)}{x} \geqq (1 - \beta)\alpha + \beta.$$

Therefore

$$\sigma(\mathfrak{A} \oplus \mathfrak{B}) \geqq \sigma(\mathfrak{A}) + \sigma(\mathfrak{B}) - \sigma(\mathfrak{A})\sigma(\mathfrak{B}).$$

In fact $\sigma(\mathfrak{A}+\mathfrak{B})\geqq\min(1, \sigma(\mathfrak{A})+\sigma\mathfrak{B}))$, but this is much harder to prove. The reader should attempt a proof of it though, as a challenge.

THEOREM 2.2. *If $\sigma(\mathfrak{A})+\sigma(\mathfrak{B})>1$, then $\mathfrak{A}\oplus\mathfrak{B}$ consists of all the positive integers.*

*Proof.* Suppose $n\notin\mathfrak{A}+\mathfrak{B}$, so in particular $n\notin\mathfrak{A}$. Consider the integers $n-a_i$, where $a_i\in\mathfrak{A}$ and $1\leqq a_i\leqq n-1$. The number of such integers is $A(n)\geqq\alpha n>n-\beta n\geqq n-B(n)$. The integers $n-a_i$ lie between 1 and $n-1$ inclusive. The number of elements of $\mathfrak{B}$ in this range is $B(n-1)$. So the total number of integers of the form $n-a_i$ and the number of members of $\mathfrak{B}$ in the range 1 to $(n-1)$ is

$$A(n) + B(n - 1) > n - B(n) + B(n - 1) \geqq n - 1.$$

Thus at least one member of the set of integers $\{n-a_i\}$ must belong to $\mathfrak{B}$, so we have $n=a_i+b\in\mathfrak{A}\oplus\mathfrak{B}$, a contradiction.

A sequence $\mathfrak{A}$ is called a **basis of order** $h$ if $\mathfrak{A}\oplus \cdots \oplus\mathfrak{A}$, taken $h$ times, consists of all the positive integers. That is, each positive integer can be expressed in the form $a_{i_1}+ \cdots +a_{i_k}$ with $k\leqq h$, and the $a$'s all belonging to $\mathfrak{A}$.

THEOREM 2.3. *If $\sigma(\mathfrak{A})>0$, then $\mathfrak{A}$ is a basis.*

*Proof.* Define $\mathfrak{A}_2=\mathfrak{A}\oplus\mathfrak{A}$ and $\mathfrak{A}_{r+1}=\mathfrak{A}_r\oplus\mathfrak{A}$ for $r\geqq 2$. By Theorem 2.1, we have $\sigma(\mathfrak{A}_2)\geqq 2\alpha-\alpha^2=1-(1-\alpha)^2$; by induction we easily prove that $\sigma(\mathfrak{A}_r)\geqq 1-(1-\alpha)^r$. We choose $r$ so that $(1-\alpha)^r<\frac{1}{2}$, hence $\sigma(\mathfrak{A}_r)>\frac{1}{2}$. Then by Theorem 2.2 we see $\mathfrak{A}_{2r}=\mathfrak{A}_r\oplus\mathfrak{A}_r$ consists of all the positive integers.

We shall take as our sequence $\mathfrak{A}^{(k)}=\{n^k:n=1,2,3,\cdots\}$. Now $\sigma(\mathfrak{A}^{(k)})=0$, but if we can show that for some integer $s$ the sequence $\mathfrak{A}_s^{(k)}=\mathfrak{A}^{(k)}\oplus\cdots\oplus\mathfrak{A}^{(k)}$, taken $s$ times, has positive density, then by Theorem 2.3 we can find an integer $r$ such that $\mathfrak{A}_{2rs}^k$ is the sequence of positive integers. This will imply that $g(k)\leqq 2rs$.

Let us now see how Linnik proved $\sigma(\mathfrak{A}_s^{(k)})>0$ for some $s$. Denote by $r_t(N)$ the number of integral solutions of the equation

$$x_1^k+\cdots+x_t^k=N \quad \text{with } x_i\geqq 0,\cdots,x_t\geqq 0,$$

and by $R_t(N)$ the number of integral solutions of the inequalities

$$0\leqq x_1^k+\cdots+x_t^k\leqq N \quad \text{with } x_1\geqq 0,\cdots,x_t\geqq 0.$$

Thus, $R_t(N)=r_t(0)+\cdots+r_t(N)$ and Linnik's fundamental result is the following lemma.

LEMMA 2.4. *There exists $s=s(k)$ such that $r_s(n)\leqq cN^{(s/k)-1}$ for $0\leqq n\leqq N$, where $c>0$ depends only on $k$.*

Unfortunately, I do not know a proof of this lemma that is not long and complicated. However, one can prove quite easily that on average the value of $r_s(n)$ is $O(n^{(s/k)-1})$. The hard part is to prove that if $t$ is sufficiently large, then $r_t(n)$ does not differ from the average value. As an exercise show that $\sum_{n=1}^N r_t(n)=R_t(N)\leqq c(k,t)N^{t/k}$. (Hint: How is the volume of the $t$ dimensional solid defined by $x_1^k+\cdots+x_t^k\leqq N$, $x_1\geqq 0,\cdots,x_t\geqq 0$ related to the number of integral solutions of the inequalities and to the sum $\sum_{n=1}^N r_t(n)$?)

It is now straightforward to show that $\sigma(\mathfrak{A}_s^{(k)})>0$, where $s$ is as in Lemma 2.4. By Theorem 2.3 this is sufficient to solve Waring's problem. We shall suppose $\sigma(\mathfrak{A}_s^{(k)})=0$ and deduce a contradiction. This assumption implies that for any $\epsilon>0$ there are an infinity of $N$ such that $A_s^{(k)}(N)<\epsilon N$, where $A_s^{(k)}(X)$ is the counting function of the sequence $\mathfrak{A}_s^{(k)}$. Now we have:

$$R_s(N)=r_s(0)+\cdots+r_s(N)\leqq 1+cN^{(s/k)-1}A_s^{(k)}(N),$$

since each $r_s(n)\leqq cN^{(s/k)-1}$ and the number of nonzero terms is $A_s^{(k)}(N)$. Thus $R_s(N)\leqq 1+cN^{(s/k)-1}\epsilon N\leqq 2c\epsilon N^{s/k}$ if $N$ is large enough, $2c\epsilon N^{s/k}<(N/s)^{s/k}$ if $\epsilon$ is small enough. Thus we have $R_s(N)<(N/s)^{s/k}$ if $N$ is large enough. But it is trivial that $R_s(N)\geqq(N/s)^{s/k}$ for all $N$, this is a contradiction. Consequently, $\sigma(\mathfrak{A}_s^{(k)})>0$ and this solves Waring's problem.

Linnik did not give an explicit estimate for $g(k)$. This was done later by Rieger [**20**], who proved that $g(k)\leqq 2^{2\cdot 16^k(k+1)!}$.

Schnirelmann took the sequence $\mathfrak{P}=\{p, \text{ primes}\}$, which also has $\sigma(\mathfrak{P})=0$, and proved the analogue of Lemma 2.4. From this he was able to deduce, as above, that each integer can be expressed as a sum of a bounded number of primes. (An estimate for the number of primes needed is $\leqq 2\times 10^{10}$.) This was the first step towards a proof of Goldbach's conjecture. The last step has not been taken yet!

As a generalization of Waring's problem, Rieger [21] proved that if $\mathfrak{A}$ is a monotone sequence of positive integers with $\sigma(\mathfrak{A})>0$, then the sequence $\mathfrak{A}^{(k)}=\{a^k, a\in\mathfrak{A}\}$, $k\in \boldsymbol{Z}^+$, forms a basis for $\boldsymbol{Z}^+$. The classical Waring problem is the special case $\mathfrak{A}=\boldsymbol{Z}^+$.

**3. The Hardy-Littlewood-Vinogradov method.** It is not possible in a short space to give a detailed account, with full proofs, of the method. I propose therefore to give the skeleton structure, indicating how all the bits fit together and refer the reader to Davenport [1] for the proofs. Many of the proofs are straightforward manipulations and applications of standard analytic and number-theoretic arguments. Enthusiastic readers are urged to attempt proofs of all the lemmas for themselves, before looking at Davenport to see how it should be done.

For convenience we introduce the notation $e(\theta)=e^{2\pi i\theta}$. Consider the function $f(\alpha)=\sum_{x=1}^{P} e(x^k\alpha)$, where $\alpha$ is any real number. Then

$$\{f(\alpha)\}^s=\sum_{N=1}^{sP^k} R_P(N)e(N\alpha),$$

where $R_P(N)$ is the number of integral solutions of the equation $x_1^k+\cdots+x_s^k=N$, with $0\leqq x_i\leqq P$ and $1\leqq i\leqq s$.

If $P\geqq [N^{1/k}]$, then $R_P(N)=R(N)$, the number of integral solutions of the equation $x_1^k+\cdots+x_s^k=N$ with $x_1\geqq 0,\cdots, x_s\geqq 0$. By elementary calculus we have:

$$R_P(N)=\int_0^1\{f(\alpha)\}^s e(-N\alpha)d\alpha. \tag{3.1}$$

The idea of the method is to show that if $s$, depending only on $k$, is chosen sufficiently large, then the integral is positive for all $N\geqq N_0(k)$. This means that the equation $x_1^k+\cdots+x_s^k=N$ has at least one solution for all $N\geqq N_0$. Since all integers less than $N_0$ can be written as a sum of at most $N_0$ 1's we shall have solved Waring's problem. To show that $R_P(N)$ is positive if $s$ is large enough, we shall obtain an asymptotic formula for $R_P(N)$ of the shape

$$R_P(N)=\frac{\Gamma(1+(1/k))^s}{\Gamma(s/k)}P^{s-k}\mathfrak{S}(N)+o(P^{s-k})\quad\text{provided}\quad s\geqq 2^k+1.$$

Thus, if we take $P=[N^{1/k}]$, then $R_P(N)=R(N)$; and if $N$ is sufficiently large, then the first term is always larger than the error term, so $R(N)>0$.

Unfortunately the integral (3.1) is extremely complicated, and we can do nothing with it as it stands. What we are going to do is to divide the range of integration into two disjoint sets $M$ and $m$, called the *major* and *minor sets* respectively. In the major set we shall make a series of approximations to $f(\alpha)$. It will turn out that the approximating functions are comparatively easy to handle, and we can explicitly evaluate the integrals which arise. On the minor set we shall show that the integral

$$\int_m \{f(\alpha)\}^s e(-N\alpha)d\alpha$$

is comparatively small, and it can be absorbed into the error term of the asymptotic formula.

The details are as follows. Consider the rational numbers $a/q$ with $(a, q)=1$, $1 \leqq a < q$, and $q < P^{1/2}$; there are only a finite number of them. Take intervals $M_{a,q}$ centered on $a/q$ of the form

$$\left|\alpha - \frac{a}{q}\right| = \beta \leqq \frac{1}{2kqP^{k-1}}.$$

The set $M = \bigcup M_{a,q}$ is called **the major set**. The **minor set** is its complement $m = [0, 1]\backslash M$. So we have

$$R_P(N) = \int_M \{f(\alpha)\}^s e(-N\alpha)d\alpha + \int_m \{f(\alpha)\}^s e(-N\alpha)d\alpha. \tag{3.2}$$

In each of the intervals $M_{a,q}$ we find a "good" function which approximates to $f(\alpha)$. These are given by our next lemma.

LEMMA 3.1. *If* $\alpha \in M_{a,q}$, *then* $f(\alpha) = (1/q)S_{a,q}I(\beta) + O(q)$ *where* $S_{a,q} = \sum_{x=1}^{q} e(ax^k/q)$, $I(\beta) = \int_0^P e(\beta x^k)dx$.

The proof of this lemma is a straightforward substitution; write $\alpha = a/q + \beta$ and use a partial summation argument.

We can now estimate the first integral in (3.2), for we have

$$\int_M \{f(\alpha)\}^s e(-N\alpha)d\alpha = \sum_{a,q} \int_{M_{a,q}} \{f(\alpha)\}^s e(-N\alpha)d\alpha,$$

and using our approximations for $f(\alpha)$ in each of the $M_{a,q}$ we obtain the following result.

LEMMA 3.2. *If* $s \geqq 4k$ *then* $\int_M \{f(\alpha)\}^s e(-N\alpha)d\alpha = J\mathfrak{S}(N) + O(P^{s-k-1})$, *where* $J = \int_{-\infty}^{\infty} I(\beta)^s e(-N\beta)d\beta$ *and*

$$\mathfrak{S}(N) = \sum_{q=1}^{\infty} \sum_{\substack{a=1\\(a,q)=1}}^{q} (q^{-1}S_{a,q})^s e\left(-N\frac{a}{q}\right).$$

The proof is quite routine; the estimation of the error term accounts for the condition $s \geqq 4k$ and requires a little care.

The integral $J$ can be evaluated by good, hard, nineteenth century integral calculus (it requires Fourier's integral theorem and a careful justification for the interchange of limits of integration). When one does this the result is as follows:

LEMMA 3.3.

$$J = \frac{\Gamma(1 + (1/k))^s}{\Gamma(s/k)} P^{s-k}, \qquad \text{if } P \geqq [N^{1/k}].$$

To summarize what we have done so far; we have shown that if $s \geqq 4k$, then

$$\int_M \{f(\alpha)\}^s e(-N\alpha)d\alpha = \frac{\Gamma(1 + (1/k))^s}{\Gamma(s/k)} P^{s-k}\mathfrak{S}(N) + o(P^{s-k-1}), \qquad \text{as } P \to \infty.$$

There are two further steps to take. First, we must show that

$$\int_m \{f(\alpha)\}^s e(-N\alpha)d\alpha = o(P^{s-k}) \qquad \text{as } P \to \infty$$

if $s$ is large enough. This will ensure that the integral over the minor set can be absorbed into the error term. Second, we must show that $\mathfrak{S}(N) \geqq c > 0$, where $c$ depends only on $k$. This will ensure that the expression we have actually written down as a main term *is* the leading term in an asymptotic expansion for $R(N)$.

The estimation of the integral on the minor set is the hardest part of the whole argument. The key results are due to Weyl and Hua. They are our next two lemmas.

LEMMA 3.4 (Weyl). *For* $\alpha \in m$, $|f(\alpha)| = O(P^{1-(1/2^k)+\epsilon})$.

LEMMA 3.5 (Hua). *For* $\nu \in Z^+$, $\int_0^1 |f(\alpha)|^{2^\nu} d\alpha = O(P^{2^\nu - \nu + \epsilon})$.

These hold for any $\epsilon > 0$, the implied '$O$' constants depend only on $\epsilon$, $k$, $\nu$. The proofs are by induction on $k$ and $\nu$ respectively and they are by no means easy. Readers should try and prove the cases when $k = 1$ and $\nu = 1$ respectively.

We can now easily deduce our next result.

LEMMA 3.6. *If* $s \geqq 2^k + 1$, *then* $\int_m \{f(\alpha)\}^s e(-N\alpha)d\alpha = O(P^{s-k-\delta})$ *where* $\delta > 0$.

*Proof.*

$$\left|\int_m \{f(\alpha)\}^s e(-N\alpha)d\alpha\right| \leqq \int_m |f(\alpha)|^s d\alpha \leqq \max_{\alpha \in m} |f(\alpha)|^{s-2k} \int_0^1 |f(\alpha)|^{2k} d\alpha$$
$$= O(P^{(s-2^k)(1-(1/2^k)+\epsilon)} P^{2^k - k + \epsilon}) = O(P^{s-k-\delta}),$$

where $\delta = (s/2^k) - \epsilon(s - 2^k + 1) > 0$, if $\epsilon$ is sufficiently small.

Now it only remains to show that $\mathfrak{S}(N) \geqq c > 0$ for all $N$. We recall that

$$\mathfrak{S}(N) = \sum_{q=1}^{\infty} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} (q^{-1}S_{a,q})^s e(-Na/q),$$

where $S_{a,q} = \sum_{x=1}^{q} e(ax^k/q)$. We simplify this rather complicated expression by defining

$$A(q) = \sum_{a=1,\,(a,q)=1}^{q} (q^{-1}S_{a,q})^s e(-Na/q)$$

and

$$\chi(p) = 1 + \sum_{\nu=1}^{\infty} A(p^\nu),$$

where $p$ is a prime number. It is now easy to check that

$$\mathfrak{S}(N) = \prod_p \chi(p),$$

the product being over all the primes. One now proves the following lemma.

LEMMA 3.7. (i) $\mathfrak{S}(N)$ *is an absolutely convergent series.* (ii) *For any positive* $\eta$ *there is a prime number* $p_0$, *depending only on* $\eta$, *such that*

$$1 - \eta < \prod_{p > p_0} \chi(p) < 1 + \eta.$$

The truth of part (ii) follows easily from part (i), but the proof of part (i) is difficult and we omit it. If we prove that $\chi(p) \geqq c(p) > 0$ for each prime number $p$, then we can quickly show that $\mathfrak{S}(N) \geqq c > 0$ for all $N$, since

$$\mathfrak{S}(N) = \prod_{p \geqq p_0} \chi(p) \prod_{p < p_0} \chi(p)$$

$$\geqq (1 - \eta) \prod_{p < p_0} c(p) = c > 0.$$

To show that $\chi(p) > c(p)$ for all prime numbers $p$, we define $M(q)$ to be the number of solutions of the congruence $x_1^k + \cdots + x_s^k \equiv N \pmod q$ and then note the following lemma.

LEMMA 3.8.

$$\chi(p) = \lim_{\nu \to \infty} \left\{ \frac{M(p^\nu)}{p^{\nu(s-1)}} \right\}.$$

The proof is straightforward; one merely notes that

$$M(q) = \frac{1}{q} \sum_{t=1}^{q} \sum_{x_1=0}^{q} \cdots \sum_{x_s=0}^{q} e((t/q)(x_1^k + \cdots + x_s^k)),$$

and one then verifies that

$$\frac{M(p^\nu)}{p^{\nu(s-1)}} = \sum_{\mu=0}^{\nu} A(p^\mu) \to \chi(p) \qquad \text{as } \nu \to \infty.$$

LEMMA 3.9. *If for any prime $p$ the congruence $x_1^k + \cdots + x_s^k \equiv N(p^\gamma)$ is soluble with not all the $x_i \equiv 0 \pmod p$, then $\chi(p) \geqq c(p) > 0$. Here $\gamma$ is defined as follows: if $p^\tau \| k$, then $\gamma = \tau + 1$ if $p > 2$, $\gamma = \tau + 2$ if $p = 2$.*

From one solution of the congruence $x_1^k + \cdots + x_s^k \equiv N \pmod{p^\gamma}$ it is possible to derive many solutions of the congruence $x_1^k + \cdots + x_s^k \equiv N \pmod{p^\nu}$ for $\nu > \gamma$. In particular one can show $M(p^\nu) \geqq p^{(\nu-\gamma)(s-1)}$, so by Lemma 3.8 we have $c(p) \geqq p^{-\gamma(s-1)} > 0$.

The final link in the chain is the next lemma which tells us when the congruence $x_1^k + \cdots + x_s^k \equiv N \pmod{p^\gamma}$ does have solutions with not all the $x_i \equiv 0 \pmod p$.

LEMMA 3.10. *The congruence $x_1^k + \cdots + x_s^k \equiv N \pmod{p^\gamma}$ is soluble with $x_1, \cdots, x_s$ not all congruent to $0 \pmod p$ provided $s \geqq 2k$ if $k$ is odd or $s \geqq 4k$ if $k$ is even.*

The proof of this lemma is just an exercise in the theory of congruences.

Thus, the final link in the chain of arguments leading to a proof of Theorem 1.2 is completed. The reader is once again urged to read Professor Davenport's book so that all the details of the argument can be filled in.

The result which we now go on to discuss is Vinogradov's estimate for $G(k)$. The theorem we shall prove is the following.

THEOREM 3.11. $G(k) \leqq k(6 \log k + 4 + \log 216)$.

In order to get this estimate Vinogradov solves the following, apparently harder, problem. Let $r(N)$ denote the number of solutions of the equation

$$x_1^k + \cdots + x_{4k}^k + u + u_1 + vy^k = N,$$

where $1 \leqq x_i \leqq P$, $0 \leqq u$, $u_1 \leqq \frac{1}{4}P^k$, $0 \leqq y \leqq P^{1/2k}$, $0 \leqq v \leqq \frac{1}{4}P^{k-1/2}$, and $u$, $u_1$, $v$ can all be expressed as the sum of at most $l$ $k$th powers of positive integers. We shall choose $l$ as a function of $k$ later. If we can show that $r(N) > 0$ for all $N > N_0(k)$, then we shall have shown that $G(k) \leqq 4k + 3l$.

Just as before we have an expression for $r(N)$ as an integral:

$$r(N) = \int_0^1 \{f(\alpha)\}^{4k}\{U(\alpha)\}^2\{V(\alpha)\} e(-N\alpha)\, d\alpha,$$

where

$$f(\alpha) = \sum_{x=1}^{P} e(\alpha x^k), \qquad U(\alpha) = \sum_u e(\alpha u), \quad \text{and} \quad V(\alpha) = \sum_v e(\alpha v).$$

We divide the interval [0, 1] into the major and minor sets $M$ and $m$ and so obtain:

$$r(N) = \int_m \{f(\alpha)\}^{4k}\{U(\alpha)\}^2 V(\alpha)e(-N\alpha)d\alpha + \int_m \{f(\alpha)\}^{4k}\{U(\alpha)\}^2 V(\alpha)e(-N\alpha)d\alpha,$$

$$= I_M + I_m, \quad \text{say.}$$

The basic idea is to show that if $l$ is chosen appropriately, then $r(N)>0$ for all sufficiently large $N$. We do this by showing that $\lim_{N\to\infty} |I_m|/I_M = 0$, for this means that from a certain point on $I_M > |I_m|$ and so $r(N) = I_M + I_m > 0$.

Of course one cannot evaluate the integrals $I_M$ and $I_m$ directly. What we are going to do is obtain a lower bound $B_M$ for $I_M$ and an upper bound $b_m$ for $|I_m|$, and then prove that $b_m/B_M \to 0$ as $N\to\infty$ if $l$ is chosen appropriately. This is sufficient because

$$0 \leqq \frac{|I_m|}{I_M} \leqq \frac{b_m}{B_M}.$$

LEMMA 3.12. *$I_M > c_1 P^{3k} U(0)^2 V(0)$, where $c_1$ is a positive real number depending only on $k$.*

*Proof.* By direct substitution we see that

$$I_M = \sum_{u,u_1,v,y} \int_M \{f(\alpha)\}^{4k} e(-N_1\alpha)d\alpha,$$

where $N_1 = N - u - u_1 - vy^k$ and $\frac{1}{4}P^k \leqq N_1 \leqq P^k$. From Lemma 3.2 we know that

$$\int_M \{f(\alpha)\}^{4k} e(-N_1\alpha)d\alpha = J(N_1)\mathfrak{S}(N_1) + O(P^{3k-\delta}).$$

Now

$$J(N_1) = \frac{\Gamma(1+(1/k))^{4k}}{\Gamma(4)} N_1^{(4k-1)/k} \geqq c_2 P^{3k},$$

where $c_2 = c_2(k) > 0$, and by Lemmas 3.7–3.10 $\mathfrak{S}(N_1) \geqq c_3 > 0$, with $c_3$ depending only on $k$. Hence

$$I_M \geqq c_4 P^{3k} \sum_{u,u_1,v,y} = c_4 P^{3k} U(0)^2 V(0).$$

We now investigate the integral over the minor set:

$$|I_m| = \left| \int_m \{f(\alpha)\}^{4k} U(\alpha)^2 V(\alpha) e(-N\alpha)d\alpha \right| \leqq \int_m |f(\alpha)|^{4k} |U(\alpha)|^2 |V(\alpha)| d\alpha.$$

Now $|f(\alpha)| = |\sum_{x=1}^P e(\alpha x^k)| \leqq \sum_{x=1}^P |e(\alpha x^k)| = P$, so we can give an upper bound for the last expression:

$$\int_m |f(\alpha)|^{4k}|U(\alpha)|^2|V(\alpha)|\,d\alpha \leqq P^{4k}\max_{\alpha\in m}|V(\alpha)|\int_m |U(\alpha)|^2 d\alpha$$
$$\leqq P^{4k}\max_{\alpha\in m}|V(\alpha)|\,U(0)^2.$$

To investigate $\max_{\alpha\in m}|V(\alpha)|$ we need the following two lemmas:

LEMMA 3.13. *Let $D_l(X)$ denote the number of distinct numbers up to $X$ that can be represented as sums of $l$ positive $k$th powers. Then $D_l(X)>c_lX^{1-\delta}$, where* $\delta=(1-(1/k))^l$, $c_l>0$ depends only on $l$.

*Proof.* We use induction on $l$. For $l=1$, $D_1(x)=[x^{1/k}]>c_1x^{1/k}$. Consider the numbers $x^k+z$, where

$$(\tfrac{1}{4}x)^{1/k} < x < (\tfrac{1}{2}x)^{1/k} \quad\text{and}\quad 0 < z < \tfrac{1}{2}x^{1-1/k},$$

and $z$ is a sum of $l-1$ positive integral $k$th powers. The integers $x^k+z$ are sums of $l$ positive $k$th powers, and they are all distinct. Hence

$$D_l(x) \geqq x^{1/k}D_{l-1}(\tfrac{1}{2}x^{1-(1/k)}) \geqq c_lx^{1-\delta},$$

by the induction hypothesis on $D_{l-1}(\frac{1}{2}x^{1-(1/k)})$.

LEMMA 3.14. *Suppose $x$ runs through $X_0$ distinct integer values in an interval of length $X$, and $Y$ runs through $Y_0$ distinct integer values in a set of length $Y$. Suppose further that $\alpha=(a/q)+O(1/q^2)$*, where $(a,q)=1$. *Then*

$$\left|\sum_x\sum_y e(\alpha xy)\right|^2 \leqq cX_0Y_0\frac{(\log q)}{q}(q+X)(q+Y).$$

This lemma was proved by Vinogradov and applied by him in several important investigations in number theory, such as his proof that each sufficiently large odd number can be expressed as a sum of at most three primes. However, the lemma does occur in the book *Inequalities* by Hardy, Littlewood and Polya, page 205. They seem to have missed its applications in number theory.

We now return to our investigation of $\max_{\alpha\in m}|V(\alpha)|$. Since $\alpha\in m$ we have $|\alpha-(a/q)|\leqq 1/(2kqP^{k-1})$ with $P^{1/2}<q<2kP^{k-1}$, in particular $|\alpha-)a/q)|<1/q^2$. We apply Lemma 3.14 to $V(\alpha)$ with

$$X_0 = D_l(\tfrac{1}{2}P^{k-1/2}),\qquad Y_0 = P^{1/2k},\qquad X = \tfrac{1}{4}P^{k-1/2},\ \text{and}\quad Y = P^{1/2}$$

to get

$$|V(\alpha)|^2 = \left|\sum_y\sum_v e(\alpha y^kv)\right|^2 < c_1X_0P^{1/2k}\frac{\log q}{q}(q+\tfrac{1}{4}P^{k-1/2})(q+P^{1/2})$$
$$\leqq c_2X_0P^{1/2k}\frac{\log q}{q}\cdot\tfrac{1}{4}P^{k-1/2}q \leqq c_3X_0P^{\mu+\epsilon},$$

where $\mu=k+1/2k-1/2$ and $\epsilon>0$, since $\log q<c_4P^\epsilon$ for any $\epsilon>0$. Therefore

$$\max_{\alpha\in m} |V(\alpha)| \leqq c_5 X_0^{1/2} P^{(\mu/2)+\epsilon},$$

$$|I_m| \leqq c_5 X_0^{1/2}\cdot U(0)\cdot P^{(\mu/2)+\epsilon+4k}.$$

Thus

$$\frac{|I_m|}{I_M} \leqq \frac{c_6 X_0^{1/2} U(0) P^{4k+(\mu/2)+\epsilon}}{P^{3k}U^2(0)V(0)}.$$

Now $V(0)\geqq c_1 X_0 P^{1/2k}$ and $U(0)=D_l(\frac{1}{4}\cdot P^k)$, so we have

$$\frac{|I_m|}{I_M} \leqq \frac{c_8 P^{k+(\mu/2)+\epsilon-1/2k}}{X_0^{1/2} D_l(\frac{1}{4}P^k)}. \tag{3.3}$$

Now $X_0=D_l(\frac{1}{2}P^{k-1/2})\geqq c_l(\frac{1}{2}P^{k-2})^{1-\delta}$ and $D_l(\frac{1}{4}P^k)\geqq c_l(\frac{1}{4}P^k)^{1-\delta}$. On substituting these lower bounds into (3.3) we obtain

$$\frac{|I_m|}{I_M} \leqq c_q P^{\Delta},$$

where $\Delta=k/2-\frac{1}{4}k-\frac{1}{4}+\epsilon+k(1-(1/k))^l-\frac{1}{2}(k-\frac{1}{2})(1-(1-(1/k))^l)$. We now choose $l$ so that $\Delta<0$; this will ensure that $|I_m|/I_M\to 0$. Simplify the expression for $\Delta$; it becomes

$$\Delta = (1-(1/k))^l((3k/2)-\tfrac{1}{4}) - \tfrac{1}{4}k + \epsilon.$$

Thus $\Delta$ will be negative if $(1-(1/k))^l 3k/2<\frac{1}{4}k$, i.e., if

$$(1-(1/k))^l < \exp(-6\log k^2).$$

Now $(1-(1/k))^l=\exp\{l\log(1-(1/k))\}<e^{-l/k}$, so it suffices to take $-l/k>-\log 6k^2$, i.e., $l<2k\log k+k\log 6$. In this case $r(N)>0$ for all sufficiently large values of $N$, and the number of $k$th powers required is $4k+3l<4k+6k\log k+k\log 216$. Therefore

$$G(k) \leqq 6k\log k + k(4+\log 216).$$

Since Vinogradov's proof appeared, a great deal of work has been done in trying to improve the estimates for $G(k)$. This work is exceedingly complicated involving intricate analysis and very delicate estimations of trigonometric sums. The best known results are the following:

$$G(2)=4;\quad G(3)\leqq 7;\quad G(4)=16;\quad G(5)\leqq 23;\quad G(6)\leqq 36;$$
$$G(7)\leqq 52;\quad G(8)\leqq 73;\quad G(9)\leqq 99;\quad G(10)\leqq 122.$$

Chen [**52**] showed that $G(k)\leqq k(3\log k+5.2)$ and Vinogradov [**63**] showed that $G(k)<k(2\log k+4\log\log k+2\log\log\log k+13)$ for all "large" $k$, (here "large" means in excess of 170,000). Full references are in the bibliography.

**4. The determination of** $g(k)$. The estimates for $G(k)$ obtained by Vinogradov lead to a determination of the precise value of $g(k)$ for any given $k \geqq 6$. In principle the method of proof is quite simple. Once one has an upper bound for $G(k)$ which is less than our lower bound $[(3/2)^k]+2^k-2$ for $g(k)$, then from some integer $c(k)$ (which can be calculated in terms of $k$ by carrying out Vinogradov's proof and estimating all the constants and error terms explicitly in terms of $k$) each integer is a sum of fewer than $g(k)$ $k$th powers. It is then a finite calculation to check that each integer less than $c(k)$ can be expressed as a sum of at most $g(k)$ $k$th powers.

The detailed work was carried out in a series of papers by Dickson, Pillai, Rubugunday, and Niven. Their final conclusions are summed up in the following theorem:

THEOREM 4.1. *If $k \geqq 6$ and if the following inequality holds*

$$\text{(i)} \qquad 3^k - 2^k + 2 < (2^k - 1)[(3/2)^k],$$

*then $g(k) = [(3/2)^k]+2^k-2$. However, if*

$$\text{(ii)} \qquad 3^k - 2^k + 2 \geqq (2^k - 1)[(3/2)^k],$$

*then we define $N(k)$ by $N(k) = [(3/2)^k]\cdot[(4/3)^k]+[(3/2)^k]+[(4/3)^k]$ and the conclusion is now*

$$g(k) = [(3/2)^k] + [(4/3)^k] + 2^k - 3 \qquad \text{if } 2^k < N(k),$$

*or*

$$g(k) = [(3/2)^k] + [(4/3)^k] + 2^k - 2 \qquad \text{if } 2^k = N(k).$$

It has been verified by Stemmler [**48**] that condition (i) holds for all $k$ in the range $6 \leqq k \leqq 200{,}000$, and (i) is conjectured to hold for every $k \geqq 6$. Evidence in favour of this conjecture was provided by Mahler [**40**] who proved, using deep theorems from the theory of diophantine approximations, that there were at most only a finite number of values of $k$ for which condition (i) does *not* hold. Unfortunately Mahler's method of proof does not enable us to find this set of integers, if indeed the set is nonempty.

Chen [22] using extremely detailed arguments, showed that $g(5)=37$, Dickson [24] showed that $g(3)=9$, and Auluck [**51**] showed that each integer $\geqq c$ can be written as a sum of at most 19 4th powers, where $\log_{10}\log_{10} c \leqq 88.39$. Thus, in principle, $g(4)$ can be calculated.

**5. Proof of Hilbert's theorem.** We give a complete proof of Hilbert's theorem.

THEOREM 5.1. *Each positive integer can be expressed as a sum of at most $g(k)$ $k$th powers of positive integers, for $k=1, 2, 3, \cdots$. Moreover $g(k)$ depends only on $k$, not on the integer being represented.*

Our method of proof is based on Hilbert's method, but the arguments have been much simplified, mainly by Hausdorff, Stridsberg, and Ellison. It will turn out to be easier to prove the following theorem, which is equivalent to Theorem 5.1.

THEOREM 5.2. *There are integers* $A>0$, $M>0$ *and positive rationals* $\lambda_1, \cdots, \lambda_M$, *depending only on* $k$, *such that each integer* $N \geqq A$ *can be written in the form*

$$N = \sum_{i=1}^{M} \lambda_i n_i^k, \quad \textit{where} \quad n_i \in \mathbf{Z}^+ \quad \textit{for} \quad 1 \leqq i \leqq M.$$

We now show that Theorems 5.1 and 5.2 are equivalent. Obviously if Theorem 5.1 is true, it implies that Theorem 5.2 is true with $A=1$, $M=g(k)$, and $\lambda_1 = \cdots = \lambda_M = 1$. Now suppose Theorem 5.2 is true. Let $\sigma$ be the least common multiple of the denominators of the $\lambda_i$. Then $\sigma\lambda_i = \sigma_i \in \mathbf{Z}^+$. If $X$ is a positive integer and $X \geqq \sigma A$, then $X = N\sigma + \theta$, where $0 \leqq \theta < \sigma$ and $N \geqq A$. By Theorem 5.2 we have

$$N = \sum_{i=1}^{M} \lambda_i n_i^k,$$

so

$$X = N\sigma + \theta = \sum_{i=1}^{M} \sigma_i n_i^k + \theta.$$

Now $\theta = \sum 1^k < \sigma$, so each integer $X \geqq \sigma A$ can be expressed as a sum of at most $\{\sigma - 1 + \sum_{i=1}^{M} \sigma_i\}$ $k$th powers of positive integers. Hence each positive integer can be expressed as a sum of at most

$$g(k) \leqq \left\{ A\sigma + \sigma - 1 + \sum_{i=1}^{M} \sigma_i \right\}$$

$k$th powers of positive integers, which is Theorem 5.1.

It is Theorem 5.2 which we shall actually prove. The key result is Hilbert's lemma.

LEMMA 5.3. *For each positive integer* $k$ *there are positive rational numbers* $\lambda_0, \cdots, \lambda_N$, *where* $N = (2k+1) \cdots (2k+4)/24$, *and integers* $\alpha_{11}, \cdots, \alpha_{N1}$, $\alpha_{12}, \cdots, \alpha_{N5}$ *such that*

$$(x_1^2 + \cdots + x_5^2)^k = \sum_{i=0}^{N} \lambda_i (\alpha_{i1} x_1 + \cdots \alpha_{i5} x_5)^{2k}.$$

*Proof.* To prove this lemma we use some elementary properties of convex bodies in $\mathbf{R}^N$. The properties which we need are very intuitive, so we put their proofs (Propositions 5.4 and 5.5) into an appendix, so as not to interrupt the main argument.

It is immediate that the set of homogeneous forms of degree $2k$ in 5 variables with real coefficients forms a vector space $V$ of dimension $N = (2k+1) \cdots (2k+4)/24$ over $\mathbf{R}$. Here $N$ is just the number of coefficients in the general form of degree $2k$ in $x_1, \cdots, x_5$.

Consider now the set of vectors $S$ of $V$ given by the forms $L=L(\alpha)=(\alpha_1x_1 \cdots +\alpha_5x_5)^{2k}$ for all sets of $\alpha_1, \cdots, \alpha_5 \in \boldsymbol{Q}$, the rational numbers. Let $h(S)$ be the convex hull of $S$ (i.e., $h(S)$ is the smallest convex set in $V$ that contains $S$). We now need our first geometric result.

PROPOSITION 5.4. *If $S$ is any subset of a real vector space $V$, then any vector $\boldsymbol{a}\in h(S)$ can be written in the form $\boldsymbol{a}=\sum_{i=0}^{N}\lambda_i s_i$, where $N=\dim V$, $s_i\in S$, $\lambda_i\in \boldsymbol{R}$, $\sum\lambda_i=1$, and $\lambda_i\geqq 0$ for $i=0, \cdots, N$.*

*Moreover if $\boldsymbol{a}$ is a rational vector, i.e., all its coordinates are rational, and all the vectors in $S$ are rational, then all the numbers $\lambda_0, \cdots, \lambda_N$ can be chosen to be rationals.*

Thus, to prove the lemma it will suffice to show that a rational multiple of the vector $(x_1^2+ \cdots +x_5^2)^k$ is in $h(S)$. For if it is, then Proposition 5.4 tells us that we can write $(x_1^2+ \cdots +x_5^2)^k$ as $\sum_{i=0}^{N}\lambda_i(\alpha_{i1}x_1+ \cdots +\alpha_{i5}x_5)^{2k}$ where $\lambda_0, \cdots, \lambda_N \in \boldsymbol{Q}^+$ and $\alpha_{11}, \cdots, \alpha_{N1}, \cdots, \alpha_{N5}\in \boldsymbol{Q}$.

To show that $(x_1^2+ \cdots +x_5^2)^k \in h(S)$ we need our second geometric result.

PROPOSITION 5.5. *The centre of gravity of a continuous mass distribution in a bounded subset $S$ of a real vector space $V$ always lies in the interior of the convex hull of the set.*

Let $T$ be the set of vectors in $V$ given by the forms $L=(\alpha_1x_1+ \cdots +\alpha_5x_5)^{2k}$ with $\alpha_i\in\boldsymbol{R}$ for $1\leqq i\leqq 5$ and $\alpha_1^2+ \cdots +\alpha_5^2\leqq 1$. Then $T\subset h(T)\subset h(S)$. So the centre of gravity of the mass distribution of unit density throughout $T$ and zero elsewhere certainly lies in $h(S)$. The centre of gravity of such a mass distribution is given by

$$g = \int_R (\alpha_1x_1 + \cdots + \alpha_5x_5)^{2k}d\alpha_1 \cdots d\alpha_5 \Big/ \int_R d\alpha_1 \cdots d\alpha_5,$$

where $R$ is the region of $\boldsymbol{R}^{(5)}$ defined by $\alpha_1^2+ \cdots +\alpha_5^2\leqq 1$. To evaluate the integral we change the variables as follows. Put

$$\begin{aligned} t_1 &= \beta_{11}\alpha_1 + \cdots + \beta_{15}\alpha_5 \\ &\vdots \\ t_5 &= \beta_{51}\alpha_1 + \cdots + \beta_{55}\alpha_5, \end{aligned}$$

where $\beta_{1i}=x_i(x_1^2+ \cdots +x_5^2)^{-1/2}$ for $1\leqq i\leqq 5$. The remaining entries in the matrix $(\beta_{ij})$ which defines the transformation are chosen in such a way that the matrix $(\beta_{ij})$ is orthogonal. The expression for $g$ now becomes:

$$g = c_1(x_1^2 + \cdots + x_5^2)^k \int_R t_1^{2k}\, dt_1 \cdots dt_5,$$

where $c_1^{-1}=\int_R dt_1 \cdots dt_5>0$. Thus we have,

$$g = c(x_1^2 + \cdots + x_5^2)^k,$$

where $c=c_1\int_R t_1^{2k}dt_1 \cdots dt_5>0$.

Since $O\in h(S)$ and $h(S)$ is convex, therefore $\lambda g\in h(S)$, where $\lambda$ is any number in the range $[0, 1]$. In particular we take $\lambda=r/c$, where $r$ is a positive rational in the range $0<r<c$. Hence $r(x_1^2+\cdots+x_5^2)^k$, a rational vector, is in $h(S)$. Thus, Lemma 5.3 follows from Proposition 5.4.

We shall need the following three corollaries of Lemma 5.3:

COROLLARY 1. *For any positive integers $k$ and $y$ there are integers $\alpha_0, \cdots, \alpha_N$' $\beta_1, \cdots, \beta_N$ and positive rationals $\lambda_0, \cdots, \lambda_N$, with $N$ and the $\lambda_i$ depending only on $k$, such that $(x_1^2+y)^k=\sum_{i=0}^N \lambda_i(\alpha_i x_1+\beta_i)^{2k}$.*

*Proof.* This follows immediately from the lemma and Lagrange's four squares theorem.

COROLLARY 2. *If Theorem 5.2 is true for $k=m$, then it is true for $k=2m$.*

*Proof.* This follows immediately from Corollary 1 on putting $x_1=0$. For if each integer $P\geqq A$ can be written as $P=\sum_{i=0}^M \mu_i y_i^k$, where the $\mu_i\in Q^+$ for $0\leqq i\leqq M$ depend only on $k$ and the $y_i\in Z^+$ for $0\leqq i\leqq M$ depend on $P$, then $P$ can be written as

$$P=\sum_{i=0}^{M}\mu_i\sum_{j=0}^{N}\lambda_j n_j^{2k}=\sum_{i=0}^{MN}\nu_i m_i^{2k}$$

where the $\nu_i$ depend only on $k$.

Before going on to Corollary 3 we shall introduce a convenient shorthand notation. If we have a sum of the form $n=\sum_{i=1}^M \lambda_i n_i^k$, where $M$ and the $\lambda_i\in Q^+$ *depend only on $k$ not on $n$*, then we denote it by $n=\sum(k)$. With this notation if $a=\sum(k)$ and $b=\sum(k)$ then $a+b=\sum(k)$ and if $a=\sum(2k)$ then $a=\sum(k)$. If the reader prefers he can always write out the expressions in full. (Warning: Have a large sheet of paper ready.)

COROLLARY 3. *If $r, m, x, T\in Z^+$ and $r<m$, $x^2<T$, then there is an equality of the form*

$$\sum_{\nu=1}^{r-1} B_{\nu r}\, x^{2\nu}T^{2\nu}+x^{2r}T^{m-r}=\sum(m).$$

*The $B_{\nu r}$ are positive integers and are explicit functions of $m$ and $r$ only.*

*Proof.* Put $k=m+r$ in Corollary 1 and then differentiate with respect to $x_1$ $2r$ times to get:

$$\begin{aligned} x_1^{2r}(x_1^2+y)^{m-r}+\sum_{\nu=1}^{r-1}B_{\nu r}\,x_1^{2\nu}(x_1^2+y)^{2\nu} &= \sum_{i=0}^{N}\lambda_i(2m+2r)!\,\alpha_i^{2r}(\alpha_i x_1+\beta_i)^{2m}/(2r)! \\ &= \sum(2m)=\sum(m). \end{aligned}$$

Put $x_1=x$ and $y=T-x^2$ to obtain the result.

*Proof of Theorem* 5.2. The basic idea of the proof is to show that we can find integers $A$, $N_0$ depending only on $k$, such that if $T \geqq N_0$ is an integer then each integer in the range $AT^k \leqq n \leqq A(T+1)^k$ can be written in the form $n = \sum (k)$. Since every integer greater than $AN_0^k$ is contained in such an interval by making a suitable choice of $T$, this will show that each integer greater than $AN_0^k$ can be written in the form $\sum (k)$. This is just Theorem 5.2, which as we have already seen implies Theorem 5.1.

LEMMA 5.6. *If $T > N_1(k)$, then each integer in the range $AT^k \leqq n < A(T+1)^k$ can be written in the form*

$$n = AT^k + b_1T^{k-1} + \cdots + b_{k-1}T + b_k,$$

*where $b_1, \cdots, b_k$ are integers satisfying $0 \leqq b_i < T$, $1 \leqq i \leqq k$.*

*Proof.* It is only necessary to show that if $T$ is large enough, then $AT^k + (T-1)(T^{k-1} + \cdots + T) > A(T+1)^k$, for we then have the result by writing $n$ in the scale of $T$. The inequality is

$$AT^k + T^k - T > A(T+1)^k,$$

and this is certainly satisfied if $T$ is large enough, say $T \geqq N_1$.

We now go on to show that if $T$ is large enough, then each integer of the form $AT^k + b_1T^{k-1} + \cdots + b_{k-1}T$, where the $b_i$ are integers in the range $0 \leqq b_i \leqq T-1$, can be written in the form $\sum (k)$. The proof is by induction on $k$. For $k=1$ the result is trivial and for $k=2$ the result is contained in Lagrange's four squares theorem.

Lagrange's theorem asserts that each positive integer can be expressed as a sum of at most four squares of integers. A proof can be found in almost any elementary book on number theory. There is a very neat geometrical proof in J. W. S. Cassels's book *Geometry of Numbers*, page 99.

Suppose that Theorem 5.2 is true for all integers $k \leqq m-1$. Then by Corollary 2 of Lemma 5.5, Theorem 5.2 is true for all even integers less than $2m$. As Theorem 5.2 is equivalent to Theorem 5.1, our hypothesis implies that Waring's theorem holds for all even integers less than $2m$.

Let $T$, $N_{m-\nu}$ ($\nu = 1, 2, \cdots, m-1$) be positive integers with $N_{m-\nu} \leqq T$, which will be chosen explicitly later. By our induction hypothesis there is an integer $r$, depending only on $m$, and integers $x_{ij} \geqq 0$ such that

$$\sum_{i=1}^{r} x_{ij}^{2j} = N_{m-j} \qquad \text{for } j = 1, 2, \cdots, m-1. \tag{1}$$

We can take $r = \max(g(2k), 1 \leqq k \leqq m-1)$. Substitute these values of $x_{ij}$ for $x$ in the identity of Corollary 3. Add the resulting equalities. We obtain

$$\sum_{\nu=0}^{j-1} B_{\nu j} T^{m-\nu} \sum_{i=1}^{r} x_{ij}^{2\nu} + N_{m-j} T^{m-j} = \sum (m), \tag{2}$$

for $j=1, \cdots, m-1$. Let $c_{\nu,j}=B_{\nu j}\sum_{i=1}^{r} x_{ij}^{2\nu}$, and sum (2) over all $j$, to obtain

$$\sum_{j=1}^{m-1}\left\{\sum_{\nu=0}^{j-1} c_{\nu j}T^{m-\nu}+N_{m-j}T^{m-j}\right\}=\sum(m). \tag{3}$$

Write (3) as a polynomial in $T$, say

$$a_mT^m+\cdots+a_1T=\sum(m), \tag{4}$$

where $a_1=N_1$, $a_i=N_i+\sum_{j=1}^{i-1} c_{m-1,m-j}$ for $i=2, \cdots, m-1$ and $a_m=r\cdot(B_{0,1}+\cdots+B_{0,m-1})=A_1-1$ say. Note that $A_1$ depends only on $m$, since $r$ depends only on $m$.

As $\sum_{i=1}^{r} x_{ij}^{2m-2}\leqq T$ we have $x_{ij}\leqq T^{1/(2m-2)}$ for all $i$ and $j$. Hence we certainly have

$$\sum_{i=1}^{r} x_{ij}^{2\nu}\leqq rT^{(m-2)/(m-1)} \quad \text{for } 1\leqq \nu\leqq m-2 \quad\text{and}\quad 1\leqq j\leqq m-1.$$

Consequently

$$\sum_{j=1}^{m-1} c_{\nu,j}\leqq Br(m-1)T^{(m-2)/(m-1)},$$

where $B=\max B_{\nu,j}$. We now assume that

$$T>Br(m-1)T^{(m-2)/(m-1)}, \quad\text{i.e.,}\quad T>\{Br(m-1)\}^{m-1}.$$

This means that $T>a_i-N_i\geqq 0$ for $1\leqq i<m$.

If we are now given any integers $b_i$ (for $1\leqq i<m$) in the range $0\leqq b_i<T$, then we successively choose the $N_1, \cdots, N_{m-1}$ as follows:

$$a_{m-1}\equiv b_{m-1} \pmod T, \qquad 0\leqq N_{m-1}\leqq T.$$

If $a_{m-1}\geqq T$ with this choice of $N_{m-1}$, then we find $N_{m-2}$ such that $1+a_{m-2}\equiv b_{m-2} (\text{mod } T)$ and $0\leqq N_{m-2}\leqq T$. However, if $a_{m-1}<T$, then we find $N_{m-2}$ such that $a_{m-1}\equiv b_{m-1} (\text{mod } T)$ and $0\leqq N_{m-2}\leqq T$. Continuing in this way we choose $N_{m-3}, \cdots, N_2$, and in the final step we choose $N_1$ so that $a_1\geqq T$. We have now shown that for all choices of $b_1, \cdots, b_{m-1}$ in the range $0\leqq b_i<T$ for $1\leqq i\leqq m-1$,

$$A_1T^m+b_1T^{m-1}+\cdots+b_{m-1}T=\sum(m) \tag{5}$$

if $T>\{Br(m-1)\}^{m-1}$.

The proof of Theorem 5.2 now follows easily. We replace $T$ by $(T+1)$ and have the particular result:

$$A_1(T+1)^m+c_m(T+1)=\sum(m), \tag{6}$$

where $c_m$ is any integer in the range $0\leqq c_m<T$. If $c_1, \cdots, c_m$ are given integers in the range $0\leqq c_i<T$ for $1\leqq i\leqq m$, then by choosing the integers $b_1, \cdots, b_{m-1}$ in (5) suitably and adding to (6) we obtain

(7) $$(2A_1+1)T^m+c_1T^{m-1}+\cdots+c_{m-1}T+c_m=\sum(m).$$

Thus, if $T>N_0=\max\{\{Br(m-1)\}^m,\ N_1(2A_1+1)\}$, we have shown that each integer $n$ in the range $AT^m\leqq n<A(T+1)^m$ can be written in the form $n=\sum(m)$. Since every integer $n>AN_0^m$ lies in such an interval, by an appropriate choice of $T$ we have proved Theorem 5.2 and hence Theorem 5.1.

*Appendix on Convexity*: The best general reference is H. G. Eggleston, *Convexity*, Cambridge Tract number 47. We refer to this book for the proofs of the following plausible assertions.

When thinking of a convex body $C$ in $R^n$ it is usually convenient to consider it as a subset of the "smallest" linear variety containing it; for example we always think of a circle as lying in a plane rather than as a subset of $R^{(3)}$ or $R^{(4)}$, etc. It is then a theorem that a convex body has an interior with respect to this space (see H. G. E. page 13). A *support hyperplane* to a convex body $C$ is a hyperplane which intersects the closure of $C$ but does not intersect the interior of $C$. It is an easy lemma that through each point of the frontier of $C$ there passes at least one support hyperplane (see H. G. E. page 20). If $S$ is any point set in $R^{(N)}$, the *convex hull*, $h(S)$, of $S$ is the intersection of all convex sets $S$ which contain $S$. Carathéodory's theorem asserts that if $y\in h(S)$, then $y$ can be expressed in the form $y=\sum_{i=0}^n\lambda_is_i$, where $s_i\in S$ and $\lambda_i\in R^+$ for $0\leqq i\leqq n$. In addition $\sum_{i=0}^n\lambda_i=1$. (See H. G. E. page 35.)

Proposition 5.4 is a trivial consequence of Carathéodory's theorem, for if $y$ and all the elements of $S$ are rational vectors in $R^n$ then we have a set of linear equations for $\lambda_0,\cdots,\lambda_n$ with rational coefficients, hence $\lambda_0,\cdots,\lambda_n$ are positive rationals. Our second geometrical result, Proposition 5.5, is also easy to prove. For if the centre of gravity $g$ of the mass distribution lies in the exterior of $h(S)$ or on the frontier of $h(S)$, then in the first case we can pass a hyperplane between $g$ and $h(S)$ and in the second case we can pass a support hyperplane through $g$. Taking moments about the hyperplane leads to a contradiction in either case.

**6. Generalizations of Waring's problem.** The problems which one could say are generalizations of the original Waring problem are legion. There just is not enough space here to describe them. Consequently, I shall confine myself to just a few chosen more or less at random.

The first may be called the prime Waring problem; it is just like the classical problem only we restrict the integers $x_i$ in $\sum_{i=1}^N x_i^k$ to be prime numbers. The analytic method can be used to solve this problem and a good account of its solution, together with its many ramifications is to be found in the book by Hua [111].

A fairly natural generalization is to ask: "If $f(x)$ is an integral valued polynomial which takes the value 1, then can each positive integer be expressed as a sum of a bounded number of values of $f(x)$?" The classical Waring's problem corresponds to the case $f(x)=x^k$. In a sense this problem goes back to Fermat, who in 1640 characteristically asserted: "A positive integer is triangular or the sum of 2 or 3 triangular numbers; square or the sum of 2, 3, or 4 squares; pentagonal or the sum of 2, 3, 4, 5 pentagonal numbers; etc."

The expression for the $n$th $r$-gonal number is

$$P_r(n) = \tfrac{1}{2}(r-2)(n^2-n)+n \qquad \text{for } r>2.$$

This assertion also occurs in Waring's book, just preceding 'Waring's problem'! Fermat's problem was solved by Cauchy, the proof is quite elementary. Kamke [104] gave a solution of the more general problem, his argument being based on Hilbert's method. Later, the analytic machinery was brought to bear on the problem and a number analogous to $G(k)$ appeared. Hua [97] gives upper bounds for this number, similar in character to Vinogradov's estimates for $G(k)$.

Perhaps the most natural generalization of Waring's problem is to ask the question about algebraic number fields or even about arbitrary fields. Siegel [77] and [78] tackled the problem for number fields. He showed that if $A_k$ is the set of algebraic integers of the number field $K$ which *can* be written as a sum of $k$th powers of algebraic integers, then there is a bound $g(k, K)$ depending on $k$ and $K$ such that each integer $\theta \in A_k$ can be written in the form $\theta = \sum_{i=1}^N \alpha_i^k$, where $N \leqq g(k, K)$ and $\alpha_1, \cdots, \alpha_n$ are algebraic integers.

In general $A_k$ does not consist of all the algebraic integers of the field $K$. For example take $K = Q(\sqrt{2})$ and look at $A_2$. The integers of $K$ are all of the form $a+b\sqrt{2}$, $a, b \in Z$; the square of an integer is of the form $a^2+2b^2+2ab\sqrt{2}$ so if $\alpha = u+v\sqrt{2}$ is an integer and $v \not\equiv 0 \pmod 2$ then $\alpha$ cannot be written as a sum of squares of integers in $K$. Siegel also obtained an asymptotic formula of the number of solutions of the equation $\theta = \sum_{i=1}^N \alpha_i^k$ analogous to Theorem 1.2. He also conjectured that the bound $g(k, K)$ was independent of the number field $K$, a result subsequently proved by Birch, [69] and [81].

Waring's problem for general fields did not receive much treatment in the literature until recently when Ellison [86] showed by very elementary methods the following two theorems:

THEOREM 6.1. *Let $k$ be a fixed positive integer. Suppose that $K$ is a real field with the following properties*:

(1) *Each totally positive element in $K$ can be expressed as a sum of $s$ squares in $K$, where $s$ depends only on $K$.*

(2) *For each totally positive element $\alpha$ in $K$, there exists a $\beta$, depending on $\alpha$, such that $s\alpha/(s+2k) \prec \beta^k \prec \alpha$ for all orderings $\prec$ of $K$.*

*Then each totally positive $\alpha$ in $K$ can be written in the form $\alpha = \sum_{i=1}^N a_i^k$, where $a_i \in K$ for $1 \leqq i \leqq N$ and $N \leqq g(k, K) < \infty$.*

THEOREM 6.2. *If $K$ is a non-real field of characteristic 0, and $-1$ is a sum of squares in $K$, then Waring's theorem is true for all exponents.*

As a generalization of Waring's problem in an entirely different direction, one can take a sequence $n_1 \leqq n_2 \leqq \cdots$ of positive integers and ask whether every positive integer $N$ can be written in the form $N=\sum_{i=1}^{r} x_i^{n_i}$, where $r$ is less than some bound depending only on the sequence $\{n_i\}$.

It turns out that there is a very nice characterization of such sequences proved by Scourfield [127]:

*A necessary and sufficient condition that such a bound exists is* $\sum_{i=1}^{\infty} 1/n_i = \infty$.

There is a simpler problem in a similar vein. Let $r(n)$ denote the least integer $r$ such that the equation $N=u_1+\cdots+u_s$, $s\leqq r$, is soluble for every positive integer $N$, where each $u_i$ is an integer of the form $x_i^m$ with $m\geqq n$. Pillai [125] showed that $r(n)\leqq 2^n+k-1$ for all $n\geqq 32$, where $k=[\log l/\log 2]$ and $l=[(3/2)^k]$.

Another problem, known as the "easier" Waring problem, considers the representation of an integer $n$ in the form $n=\pm x_1^k\pm\cdots\pm x_s^k$. It is easy enough to prove that the analogue of $g(k)$ exists, but obtaining more precise information is largely an unsolved problem.

**References.** The literature associated with Waring's problem and its various ramifications is enormous. This list is by no means complete, though it is extensive enough to provide a solid base for those who wish to explore the literature in great detail.

As an extra aid to the intrepid explorer I also give references to reviewing journals, where a short critical summary of the relevant paper will be found. For papers published between 1940 and 1969 I give the Mathematical Reviews number, e.g., MR. 20 33. Papers which were published prior to 1940 have either the *Zentralblatt für Mathematik* review number or the *Jahrbuch über die Fortschritte der Mathematik* review number, e.g., ZB. 8 4 and F. 40 237 respectively.

**General books.**

1. H. Davenport, Analytic methods for diophantine equations and approximations, Ann Arbor Publishers, 1962.

2. L. E. Dickson, History of the theory of numbers, Carnegie Institution of Washington, 1923.

3. G. H. Hardy and J. E. Littlewood, The collected works of G. H. Hardy, vol. 1, Oxford University Press, New York, 1966.

4. H. Halberstam and K. F. Roth, Sequences, Clarendon Press, Oxford, 1966.

5. I. M. Vinogradov, The method of trigonometric sums in the theory of numbers, Interscience, London, 1954.

**Hilbert's method.**

6. F. Hausdorff, Zur Hilbertschen Lösung des Waringschen Problems, Math. Ann., 67 (1909) 301–302. F. 40 237.

7. D. Hilbert, Beweis für die Darstellbarkeit der ganzen Zahlen durch eine feste Anzahl $n$-ter Potenzen (Waringsches Problem), Math. Ann., 67 (1909) 281–300. F. 40 236.

8. G. Frobenius, Über den Stridsbergs Beweis des Waringschen Satzes, Berl. Ber., 1 (1912) 666–670. F. 43 238.

9. A. Oppenheim, Hilbert's proof of Waring's theorem, Messenger of Math., 58 (1929) 153–158. F. 55–109.

10. R. Remak, Bemerkung zu Herrn Stridsberg's Beweis des Waringschen Theorems, Math. Ann., 72 (1912) 153–156. F. 43 238.

11. G. J. Rieger, Zur Hilbertschen Lösung des Waringschen Problems: Abschätzung von $g(n)$, Arch. der Math., 4 (1953) 275–281. MR. 15 289.

12. E. Schmidt, Zum Hilbertschen Beweise des Waringschen Theorems, Math. Ann., 74 (1913) 271–274. F. 44 211.

13. E. Stridsberg, Sur la démonstration de M. Hilbert du théorème de Waring, Math. Ann. 72 (1912) 145–152. F. 43 238.

**Linnik's method.**

14. A. O. Gelfond and Ju. V. Linnik, Elementary methods in the theory of numbers, Pergamon Press, New York, 1966.

15. A. Khinchine, Three pearls of number theory, Graylock Press, Rochester, 1952.

16. A. V. Kuzel', Elementary solution of Waring's Problem with polynomials by the method of Ju. V. Linnik, Uspehi Mat. Nauk., (N.S.) 11 (1956) 165–168. MR. 18 466.

17. Ju. V. Linnik, An elementary solution of a problem of Waring by Schnirelmann's method, Rec. Math., Mat. Sb., 12 (54) (1943) 225–230. MR. 5 200.

18. J. Necaev, Waring's problem for polynomials, Amer. Math. Soc. Transl., (2) 3 (1956) 39–89. MR. 17 1045.

19. D. J. Newman, A simplified proof of Waring's conjecture, Michigan Math. J., 7 (1960) 291–295. MR. 22 10967.

20. G. J. Rieger, Zu Linniks Lösung des Waringschen Problems: Abschätzung von $g(n)$, Math. Z., 60 (1954) 213–234. MR. 16 114.

21. G. J. Rieger, Über eine Verallgemeinerung des Waringsches Problems, Math. Z., 58 (1953) 281–283. MR. 15 13.

**Estimates for $g(k)$.**

22. Jing-jun Chen, Waring's problem for $g(5)$, Sci. Sinica, 13 (1964) 1547–1568. MR. 34 135.

23. S. Chowla, On $g(k)$ in Waring's problem, Proc. Lahore Philos. Soc., 6 (1944) 16–17. MR. 7 145.

24. L. E. Dickson, Simpler proofs of Waring's Theorem on cubes with various generalizations, Trans. Amer. Math. Soc., 30 (1928) 1–18. F. 54 177.

25. ———, Proof of Waring's theorem for fifth powers, Bull. Amer. Math. Soc., 37 (1931) 549–553. F. 57 196.

26. ———, Recent progress on Waring's theorem and its generalizations, Bull. Amer. Math. Soc., 39 (1933) 701–727. ZB. 8 5.

27. ———, Minimal decompositions into $n$th powers, Amer. J. Math., 55 (1933) 593–602. ZB. 8 4.

28. ———, Waring's problem for ninth powers, Bull. Amer. Math. Soc., 40 (1934) 487–493. ZB. 9 299.

29. ———, Universal Waring's theorem for eleventh powers, J. London Math. Soc., 9 (1935) 201–206. ZB. 12 290.

30. ———, Cyclotomy, higher congruences and Waring's problem, Amer. J. Math., 57 (1935) 463–474. ZB. 12 290.

31. ———, Researches on Waring's problem, Carnegie Institution of Washington, 1923.

32. ———, Ideal Waring's problem for twelfth powers, Duke Math. J., 2 (1936) 192–204. ZB. 14 251.

33. ———, Proof of the ideal Waring's theorem for exponents 7–180. Amer. J. Math., 58 (1936) 521–529. ZB. 14 251.

34. ———, Solution of Waring's problem, Amer. J. Math., 58 (1936) 530–535. ZB. 14 251.

35. ———, All integers except 23 and 239 are sums of 8 cubes, Bull. Amer. Math. Soc., 45 (1939) 588–591. ZB. 21 391.

36. M. Gelbke, A propos de $g(k)$ dans le problème de Waring, Leningrad, Bull. Acad. Sci., 7 (1933) 631–640, F. 59 950.

37. R. D. James, The value of the number $g(k)$ in Waring's problem, Trans. Amer. Math. Soc., (1934) 395–444. F. 60 132.

38. ———, Waring's problem for odd powers, Proc. London Math. Soc., 37 (1934) 257–291. ZB. 9 54.

39. ———, The constants in Waring's problem for odd powers, Bull. Amer. Math. Soc., 41 (1935) 689–694. ZB. 12 340.

40. K. Mahler, On the fractional parts of powers of real numbers, Mathematika, 4 (1957) 122–124. MR. 20 33.

41. I. Niven, An unsolved case of the Waring problem, Amer. J. Math., 66 (1944) 137–143. MR. 5 142.

42. S. S. Pillai, On Waring's problem, J. Indian Math. Soc., 2 (1933) 16–44. F. 62 1132.

43. ———, On Waring's problem I, Annamalai Univ. J., 5 (1935) 145–166. F. 62 1132.

44. ———, On Waring's problem III and IV, Annamalai Univ. J., 6 (1936) 50–64. ZB. 15 343.

45. ———, On Waring's problem V, J. Indian Math. Soc., 2 (1936) 213–214 (see page following 130). ZB. 17 5.

46. ———, On Waring's problem: $g(6)=73$, Proc. Indian Acad. Sci., Sec. A., 12 (1940) 30–40. MR. 2 146.

47. R. K. Rubugunday, On $g(k)$ in Waring's problem, J. Indian Math. Soc., 6 (1942) 192–198. MR. 5 142.

48. R. M. Stemmler, The ideal Waring theorem for exponents 401–200,000, Math. Comp., 18 (1964) 144–146. MR 28 3019.

49. E. Trost, Eine Bemerkung zum Waringschen Problem, Elem. Math., 13 (1958) 73–75. MR. 20 3839.

50. H. S. Zuckerman, New results for the number $g(n)$ in Waring's problem, Amer. J. Math., 58 (1936) 545–552. ZB. 14 250.

Estimates for $G(k)$.

51. F. C. Auluck, On Waring's problem for biquadratics, Proc. Indian Acad. Sci., Sec. A., 11 (1940) 437–450. MR. 2 35.

52. Jing-jun Chen, On Waring's problem for $n$th powers, Acta. Math. Sinica, 8 (1958) or Chinese Math. Acta, 8 (1967) 849–853. MR. 36 6367.

53. H. Davenport, On Waring's problem for cubes, Acta Math., 71 (1939) 123–143. MR. 1 4.

54. ———, On Waring's problem for fourth powers, Ann. of Math., 40 (1939) 731–747. MR. 1 42.

55. ———, On Waring's problem for fifth and sixth powers, Amer. J. Math., 64 (1942) 199–207. MR. 3 162.

56. M. Gelbcke, Zum Waringsche Problem, Math. Ann., 105 (1931) 637–652. F. 57 214.

57. Ju. V. Linnik, On the representation of large numbers as sums of seven cubes, Rec. Math., Mat. Sb. 12 (54) (1943) 218–224. MR. 4 142.

58. V. Narasimhamurti, On Waring's problem for 8th, 9th and 10th powers, J. Indian Math. Soc., 5 (1941) 122. MR. 3 162.

59. G. Pall, Representation by quadratic forms, Canad. Math. J., 1 (1949) 344–364. MR. 11 643.

60. K. Sambasiva Rao, On Waring's problem for smaller powers, J. Indian Math. Soc., 5 (1941) 117–121. MR. 3 162.

61. Kwang-Chang Tong, On Waring's problem, Advancement in Math., 3 (1957) 602–607. MR. 20 3838.

62. I. M. Vinogradov, On Waring's problem, Ann. of Math., 36 (1935) 395–405. F. 61 150.

63. ———, On an upper bound for $G(n)$, Izv. Akad. Nauk SSSR. Ser. Mat., 23 (1959) 637–642. MR. 22 699.

64. G. L. Watson, A proof of the seven cube theorem, J. London Math. Soc., 26 (1951) 153–156. MR. 13 915.

65. ———, A simple proof that all large numbers are sums of at most eight cubes, Math. Gaz., 37 (1953) 209–211. MR. 15 13.

66. A. E. Western, Numbers representable by four or five cubes, J. London Math. Soc., 1 (1926) 244–250. F. 52 171.

Algebraic number fields.

67. P. T. Bateman and R. M. Stemmler, Waring's problem for algebraic number fields and primes of the form $(p^r-1)/(p^n-1)$, Illinois J. Math., 6 (1962) 142–156. MR. 25 2059.

68. M. Bhaskaran, Sums of $m$th powers in algebraic and Abelian number fields, Arch. Math., (Basel) 17 (1966) 497–504. MR. 34 4242.

69. B. J. Birch, Waring's problem in algebraic number fields, Proc. Cambridge Philos. Soc., 57 (1961) 449–459. MR. 26 1306.

70. O. Körner, Über das Waringsche Problem in algebraischen Zahlkörpern, Math. Ann., 144 (1961) 224–238. MR. 25 56.

71. ———, Ganze algebraischen Zahlen als Summen von Polynomwerten, Math. Ann., 149 (1962) 97–104. MR. 26 2423.

72. I. Niven, Integers of quadratic fields as sums of squares, Trans. Amer. Math. Soc., 48 (1940) 405–417. MR. 2 147.

73. ———, Sums of fourth powers of Gaussian integers, Bull. Amer. Math. Soc., 47 (1941) 923–926. MR. 3 162.

74. ———, Sums of $n$th powers of quadratic integers, Duke Math. J., 8 (1941) 441–457. MR. 3 67.

75. G. J. Rieger, Zum Waringschen Problem für algebraische Zahlen und Polynomen, J. Reine Angew. Math., 195 (1956) 108–120. MR. 17 463.

76. ———, Elementare Lösung des Waringschen Problems für algebraische Zahlkörper mit der verallgemeinerten Linnikschen Methode, Math. Ann., 148 (1962) 83–88. MR. 26 106.

77. C. L. Siegel, Additive Theorie der Zahlkörper II, Math. Ann., 88 (1923) 184–210. F. 49 128.

78. ———, Generalizations of Waring's problem to algebraic number fields, Amer. J. Math., 66 (1944) 122–136. MR. 5 200.

79. ———, Sums of $m$th powers of algebraic integers, Ann. of Math., (2) 46 (1946) 313–339. MR. 7 49.

80. T. Tatuzawa, On Waring's problem in an algebraic number field, J. Math. Soc. Japan, 10 (1958) 322–341. MR. 20 5763.

**$p$-adic fields.**

81. B. J. Birch, Waring's problem for $p$-adic number fields, Acta Arith., 9 (1964) 169–176. MR. 29 3462.

82. J-R. Joly, Sur le problème de Waring pour un exposant premier dans certains anneaux locaux, C. R. Acad. Sci. Paris, 262 (1966) A1438–A1441. MR. 35 6664.

83. C. P. Ramanujan, Sums of $m$th powers in $p$-adic rings, Mathematika, 10 (1963) 137–146. MR. 29 5811.

**Finite fields.**

84. I. Chowla, On Waring's problem (mod $p$), Proc. Nat. Inst. Sci. India. Part A, 13 (1943) 195–220. MR. 7 242.

85. S. Swartz, On Waring's problem for finite fields, Quart. J. Math., Oxford Ser., 19 (1948) 123–128. MR. 9 572.

**General fields.**

86. W. J. Ellison, Waring's problem for fields, to appear shortly.

87. L. Tornheim, Sums of $n$th powers in fields of prime characteristic, Duke Math. J., 4 (1938) 359–362. ZB. 19 3.

**The polynomial Waring problem.**

88. H. Chatland, The asymptotic Waring problem for homogeneous polynomial summands, Ann. of Math., 39 (1938) 49–57. ZB. 18 107.

89. L. E. Dickson, Waring's problem for cubic functions, Trans. Amer. Math., Soc., 36 (1934) 1–12. ZB. 8 297.

90. ———, A new method for Waring's theorem with polynomial summands, Trans. Amer. Math. Soc., 36 (1934) 731–748. ZB. 10 295.

91. ———, Universal Waring theorems with cubic summands, Acta Arith., 1 (1935) 184–196. ZB. 13 104.

92. ———, A new method for Waring's theorem with polynomial summands, Trans. Amer. Math. Soc., 39 (1936) 205–208. ZB. 14 10.

93. L. E. Dickson, Waring theorems of a new type, Amer. J. Math., 58 (1936) 241–248. ZB. 13 346.

94. ———, New Waring's theorems for polygonal numbers, Quart. J. Math. Oxford Ser., 8 (1937) 62–65. ZB. 16 391.

95. L. W. Griffiths, Representation as sums of multiples of generalized polygonal numbers, Amer. J. Math., 58 (1936) 769–782. ZB. 15 200.

96. ———, Universal functions of polygonal numbers, this MONTHLY, 49 (1942) 107–110. MR. 3 268.

97. L. K. Hua, On Waring's problem with polynomial summands, Amer. J. Math., 58 (1936) 553–562. ZB. 14 294.

98. ———, On Waring's problem with polynomial summands, J. Chinese Math. Soc., 1 (1936) 23–61. ZB. 15 388.

99. ———, On Waring's problem, Tôhoku Math. J., 42 (1936) 210–225. ZB. 16 155.

100. ———, On a generalized Waring problem, Proc. London Math. Soc., 43 (1937) 161–182. F. 63 127.

101. ———, On a generalization of Waring's problem II, J. Chinese Math. Soc., 2 (1940) 175–191. MR. 2 348.

102. M. G. Humphreys, On the Waring problem with polynomial summands, Duke Math. J., 1 (1935) 361–375. F. 61 153.

103. R. D. James, The representation of integers as sums of pyramidal numbers. Math. Ann., 109 (1933) 196–199. F. 59 177.

104. E. Kamke, Verallgemeinerungen des Waring-Hilbertschen Satzes, Math. Ann., 83 (1921) 85–112. F. 48 142.

105. ———, Über die Zerfällung rationaler Zahlen in rationale Polynomwerte, Math. Z., 12 (1922) 323–328. F. 48 143.

106. ———, Zum Waringsche Problem für rationale Zahlen und Polynome, Math. Ann., 87 (1922) 238–245. F 48 179.

107. S. S. Pillai, On Waring's problem VI, Annamalai Univ. J., 7 (1937) 171–197. ZB. 16 245.

108. ———, On Waring's problem VIII, J. Indian Math. Soc., (N.S.) 3 (1939) 205–220. ZB. 22 115.

109. G. L. Watson, Sums of eight values of a cubic polynomial, J. London Math. Soc., 27 (1952) 217–224. MR. 14 280.

110. G. C. Weber, Waring's problem for cubic functions, Trans. Amer. Math. Soc., 36 (1934) 493–510. F. 60 139.

**The prime Waring problem.**

111. L. K. Hua, Additive Prime Number Theory, Amer. Math. Soc., 1965.

112. S. S. Pillai, Symposium on Waring's problem, Math. Student, 7 (1939) 165–168.

113. ———, On Waring's problem IX, J. Indian Math. Soc., 3 (1939) 221–225. F. 65 146.

114. ———, On Waring's problem with powers of primes, Proc. Indian Acad. Sci. Sect. A, 9 (1939) 29–34. F. 65 146.

115. ———, On Waring's problem with powers of primes, Proc. Indian Acad. Sci. Sect. A, 12 (1940) 202–204. MR. 2 35.

116. ———, On Waring's problem with powers of primes, J. Indian Math. Soc., 8 (1944) 18–20 MR. 6 57.

117. K. F. Roth, On Waring's problem for cubes, Proc. London Math. Soc., 53 (1951) 268–279. MR. 13 112.

**The easier Waring problem.**

118. H. Davenport, Note on sums of fourth powers, J. London Math. Soc., 16 (1941) 3–4. MR. 3 162.

119. W. H. J. Fuchs and E. M. Wright, The 'easier' Waring problem, Quart. J. Math. Oxford Ser., 10 (1939) 190–209. ZB. 22 115.

120. W. Hunter, The representation of numbers by sums of fourth powers, J. London Math. Soc., 16 (1941) 177–179. MR. 3 162.

121. T. Rai, Easier Waring problem, J. Sci. Res. Banaras Hindu Univ., 1 (1951) 5–12. MR. 13 626.

122. R. M. Stemmler, The easier Waring problem in algebraic number fields, Acta Arith., 6 (1960) 447–468. MR. 23 3131.

**Waring's problem with varying exponents.**

123. G. A. Freiman, Solution of Waring's problem in a new form, Uspehi Mat. Nauk., 4 (1949) 193. MR. 11 162.

124. M. Haberzetle, The Waring problem with summands $x^m$, $m \geqq n$, Duke Math. J., 5 (1939) 49–57. ZB. 22 115.

125. S. S. Pillai, Waring's problem with indices $\geqq n$, Proc. Indian Acad. Sci. Sec. A, 12 (1940) 41–45. MR. 2 35.

126. K. F. Roth, A problem in additive number theory, Proc. London Math. Soc., 53 (1951) 381–395. MR. 13 14.

127. E. J. Scourfield, A generalization of Waring's problem, J. London Math., Soc., 35 (1960) 98–116. MR. 22 2591.

128. B. J. Segal, Waringsches Problem für Potenzen mit gebrochenen und irrationalen Exponenten, Trav. Inst. Steklov, 5 (1934) 73–86. ZB. 9 299.

129. G. K. Stanley, The representation of a number as a sum of one square and a number of $k$th powers, Proc. London Math. Soc., 31 (1931) 512–553. F. 56 174.

130. K. Thanigasalam, Asymptotic formulae in a generalized Waring problem, Proc. Cambridge Philos. Soc., 63 (1967) 87–98. MR. 34 2551.

131. ———, On additive number theory, Acta Arith., 13 (1967) 237–258. MR. 36 5096.

132. ———, A generalization of Waring's problem for prime powers, Proc. London Math. Soc., 16 (1966) 193–212. MR. 34 5790.

**Miscellaneous topics.**

133. H. Davenport and H. Heilbronn, On Waring's problem: two cubes and one square, Proc. London Math. Soc., 43 (1937) 73–104. ZB. 16 246.

134. L. E. Dickson, The converse of Waring's problem, Bull. Amer. Math. Soc., 40 (1934) 711–714. ZB. 10 103.

135. ———, A new method for universal Waring theorems with details for seventh powers, this MONTHLY, 41 (1934) 547–555. ZB. 10 294.

136. ———, Polygonal numbers and related Waring problems, Quart. J. Math. Oxford Ser., 5 (1934) 283–290. ZB. 10 391.

137. ———, Universal Waring's theorems, Mh. Math. Phys., 43 (1936) 391–400. ZB. 14 102.

138. ———, Waring's problem and its generalizations, Ann. of Math., 37 (1936) 293–316. ZB. 13 391.

139. ———, A generalization of Waring's Problem, Bull. Amer. Math. Soc., 42 (1936) 525–529. ZB. 14 345.

140. ———, The Waring problem and its generalizations, Bull. Amer. Math. Soc., (1936) 833–842. ZB. 15 343.

141. H. Heilbronn, On the representation of a rational as a sum of four squares by means of regular functions, J. London Math. Soc., 39 (1964) 72–76. MR. 28 3003.

142. L. K. Hua, On Waring's problem with cubic polynomial summands, J. Indian Math. Soc., 4 (1940) 127–135. MR 2 348.

143. J-R Joly, Sur les puissances $d^{\text{ièmes}}$ des éléments d'un anneau commutatif, C. R. Acad. Sci. Paris, 261 (1965) 3259–3262. MR. 32 2432.

144. E. Kamke, Über die simulte Zerfällung ganzer Zahlen in $l$-te und $n$-te Potenzen, Zbl Math., 152 (1922) 30–32. F. 48 143.

145. A. A. Karacuba, Waring's problem for a congruence modulo a power of a prime, Vestnik Moskov. Univ. Ser. I, Mat. Meh., 4 (1962) 28–38. MR. 26 97.

146. K. Sambasiva Rao, On a particular representation of integers as sums of $k$th powers, J. Indian Math. Soc., 3 (1939) 262–265. MR 1 135.