27/5

Τεχνικές ολοκλήρωσης

Το πρόβλημα: Δίνεται η f και θέλουμε να βρούμε μια παράγουσα

$$\int f(x)\,dx \text{, δηλ μια } F : F' = f.$$

A) Πίνακας με βασικά ολοκληρώματα (+c)

- $\int x^{\alpha}\,dx = \frac{x^{\alpha+1}}{\alpha+1}, \quad \alpha \neq -1$
- $\alpha = -1$

$$\int \frac{1}{x}\,dx = \ln|x|$$

- $\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x$
- $\int \cos x\,dx = \sin x$
- $\int \frac{1}{x^2+1}\,dx = \arctan x$
- $\int \sin x\,dx = -\cos x$
- $\int \frac{1}{\cos^2 x}\,dx = \tan x$
- $\int \frac{1}{\eta\mu^2 x}\,dx = -\cot x$
- $\int e^x\,dx = e^x$

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B) Αντικατάσταση α' είδους

$$\int f(\varphi(x))\varphi'(x)dx \underset{u=\varphi(x)}{=\!=\!=} \int f(u)du$$

Αν βρω συνάρτηση G : $G'(u)=f(u)$
τότε για την $G(\varphi(x))=G\circ\varphi(x)$ έχω

$$(G\circ\varphi)'(x)=G'(\varphi(x))\varphi'(x)=f(\varphi(x))\cdot\varphi'(x)$$

δηλαδή η $G\circ\varphi$ είναι παράγουσα της $f(\varphi(x))\cdot\varphi'(x)$

Παραδείγματα

α) $\displaystyle\int \frac{\arctan x}{1+x^2}dx = \int \arctan x\cdot(\arctan x)'dx$

$\left(\begin{array}{l}\varphi(x)=\arctan x\\ f(u)=u\end{array}\right)$

$$\underset{du=\frac{1}{x^2+1}dx}{\overset{u=\arctan x}{=\!=\!=}} \int u\,du = \frac{u^2}{2}+c = \frac{(\arctan x)^2}{2}+c$$

β) $\displaystyle\int \tan x\,dx = \int \frac{\sin x}{\cos x}dx \underset{du=-\sin x dx}{\overset{u=\cos x}{=\!=\!=}} \int -\frac{du}{u} = -\ln|u|+c$

$=-\ln|\cos x|+c$

γ) $\displaystyle\int \frac{\cos(\sqrt{x})}{\sqrt{x}}dx \underset{du=\frac{1}{2\sqrt{x}}dx}{\overset{u=\sqrt{x}}{=\!=\!=}} \int 2\cos u\,du = 2\sin u + c =$

$\frac{dx}{\sqrt{x}}=2du$

$=2\sin(\sqrt{x})+c$

- $\cos(a+b) = \cos a \cos b - \sin a \sin b$
- $\cos 2x = \cos^2 x - \sin^2 x$

Γ) Τριγωνομετρικές ταυτότητες

- $\sin^2 x + \cos^2 x = 1$
- $1 + \tan^2 x \left(= \frac{\cos^2 x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} \right) = \frac{1}{\cos^2 x}$
- $1 + \cot^2 x = \frac{1}{\sin^2 x}$
- $\cos^2 x = \frac{1 + \cos 2x}{2}$ • $\sin^2 x = \frac{1 - \cos 2x}{2}$
- $\sin 2x = 2\sin \cancel{x} \cdot \cos x$
- $\sin(ax) \cdot \sin(bx) = \frac{\cos[(b-a)x] - \cos[(a+b)x]}{2}$
- $\cos(ax) \cdot \cos(bx) = \frac{\cos(b+a)x + \cos(a-b)x}{2}$
- $\sin(ax) \cdot \cos(bx) = \frac{\sin(a+b)x + \sin(a-b)x}{2}$

Παραδείγματα

a) $\int \cos^2 x \, dx = \int \frac{1 + \cos 2x}{2} dx = \int \frac{1}{2} dx + \int \frac{\cos 2x}{2} dx$

$= \frac{x}{2} + \frac{\sin 2x}{4} + c$

β) $\int \sin^5 x\,dx = \int (\sin^2 x)^2 \sin x\,dx = \int (1-\cos^2 x)^2 \sin x\,dx$

$\overset{u=\cos x}{\underset{du=-\sin x dx}{=}} -\int (1-u^2)^2\,du = -\int (1-2u^2+u^4)\,du$

$= -u + \frac{2u^3}{3} - \frac{u^5}{5} + c = -\cos x + \frac{2\cos^3 x}{3} - \frac{\cos^5 x}{5} + c$

γ) $\int \tan^2 x\,dx = \int \left(1 - \frac{1}{\cos^2 x}\right) dx = x - \tan x + c$

δ) $\int \cot^2 x\,dx = \int \left(-1 + \frac{1}{\sin^2 x}\right) dx = -x - \cot x + c$

Δ) <u>Ολοκληρωση κατα μερη</u>

$\int f'g = fg - \int fg'$

<u>Παραδειγματα</u>

α) $\int x \log x\,dx = \int \left(\frac{x^2}{2}\right)' \log x\,dx = \frac{x^2}{2}\log x - \int \frac{x^2}{2}(\log x)'\,dx$

$= \frac{x^2}{2}\log x - \int \frac{x^2}{2}\cdot\frac{1}{x^2}\,dx = \frac{x^2}{2}\log x - \frac{x^4}{4} + c$

β) $\int x\cos x\,dx = \int x\cdot(\sin x)'\,dx = x\sin x - \int \sin x =$

$= x\sin x + \cos x$

$$\text{β)} \quad \overbrace{\int e^x \cdot \sin x\,dx}^{I} = \int (e^x)' \cdot \sin x\,dx = e^x \cdot \sin x - \int e^x \cos x\,dx$$

$$= e^x \cdot \sin x - \int (e^x)' \cdot \cos x\,dx = \cancel{e^x \sin x}$$

$$= e^x \cdot \sin x - e^x \cdot \cos x\,dx + \int e^x (\cos x)'\,dx$$

$$= e^x \sin x - e^x \cos x - \underbrace{\int e^x \sin x\,dx}_{I}$$

Άρα

$$2I = 2\int e^x \sin x\,dx = e^x(\sin x - \cos x)$$

$$\Rightarrow \int e^x \sin x\,dx = \frac{e^x(\sin x - \cos x)}{2}$$

Άσκηση 4 (ΒΑΣΙΚΗ)

Για κάθε $n \in \mathbb{N}$

$$\int \frac{1}{(x^2+1)^{n+1}}\,dx = \frac{1}{2n}\frac{x}{(x^2+1)^n} + \frac{2n-1}{2n}\int \frac{1}{(x^2+1)^n}\,dx \quad (*)$$

Έχουμε ότι $(n=1)$

$$\int \frac{1}{x^2+1}\,dx = \arctan x + c$$

$$\int \frac{1}{(x^2+1)^2}\,dx =$$

$$\frac{1}{2}\frac{x}{(x^2+1)} + \frac{1}{2}\underbrace{\int \frac{1}{x^2+1}\,dx}_{\arctan x}$$

$$\int \frac{1}{(x^2+1)^3}\,dx = \frac{1}{4}\frac{x}{(x^2+1)^2} + \frac{3}{4}\int \frac{1}{(x^2+1)^2}\,dx$$

$$= \frac{1}{4}\frac{x}{(x^2+1)^2} + \frac{3}{8}\frac{x}{x^2+1} + \frac{3}{8}\arctan x + c$$

Απόδειξη της (*)

$$I_n = \int \frac{1}{(x^2+1)^n}\,dx = \int (x)' \frac{1}{(x^2+1)^n}\,dx = \frac{x}{(x^2+1)^n} + n\int x\,\frac{2x}{(x^2+1)^{n+1}}\,dx$$

$$= \frac{x}{(x^2+1)^n} + 2n\int \frac{x^2+1-1}{(x^2+1)^{n+1}}\,dx = \frac{x}{(x^2+1)^n} + 2n\int \frac{1}{(x^2+1)^n}\,dx - 2n\int \frac{1}{(x^2+1)^{n+1}}\,dx$$

$$= \frac{x}{(x^2+1)^n} + 2nI_n - 2nI_{n+1}$$

Άρα $2nI_{n+1} = \dfrac{x}{(x^2+1)^n} + (2n-1)I_n$

$$\Rightarrow I_{n+1} = \frac{1}{2n}\,\frac{x}{(x^2+1)^n} + \frac{2n-1}{2n}\,I_n.$$

E) Αντικατάσταση β' είδους

$$\int f(x)\,dx \underset{\substack{\text{"}x=\varphi(t)\text{"} \\ \text{"}dx=\varphi'(t)dt\text{"}}}{=\!=} \int f(\varphi(t))\,\varphi'(t)\,dt$$

και μετά θέτω $t=\varphi^{-1}(x)$

Αν βρω $G(t): G'(t) = f(\varphi(t))\cdot\varphi'(t)$
τότε θέλω την $G(\varphi^{-1}(x)) = (G\circ\varphi^{-1})(x)$

Έχω $(G\circ\varphi^{-1})(x) = G'(\varphi^{-1}(x))\,(\varphi^{-1})'(x)$

$= f(\varphi(\varphi^{-1}(x)))\cdot\varphi'(\varphi^{-1}(x))\cdot(\varphi^{-1})'(x) = f(x)$

Παραδείγματα

$\sin t = x/3$

$t = \arcsin\frac{x}{3}$

α) $\displaystyle\int \frac{dx}{x^2\sqrt{9-x^2}} \quad \frac{x=3\sin t}{dx=3\cos t\,dt}$

$\displaystyle = \int \frac{\cancel{3}\cos t}{9\sin^2 t \cdot \cancel{3}\cos t}\,dt$

$\displaystyle = -\frac{1}{9}\cdot\cot t + c = \frac{-1}{9}\cot\left(\arcsin\frac{x}{3}\right)+c$

$\displaystyle = \frac{-1}{9}\cdot\frac{\sqrt{9-x^2}}{x}+c$

Αν έχω ολοκλήρωμα που περιέχει $\sqrt{a^2-x^2}$ θέτω π.χ

$x = a\cdot\sin t$

- $\sqrt{a^2-x^2}=\sqrt{a^2\cos^2 t}$ $= a\cdot\cos t$
- $dx = a\cdot\cos t\,dt$

$t = \arccos\frac{2}{x}$

β) $\displaystyle\int \frac{\sqrt{x^2-4}}{x}\,dx \quad \frac{x=\frac{2}{\cos t}}{dx=\frac{2\sin t}{\cos^2 t}\,dt}$

$\displaystyle = \int \frac{\sqrt{\frac{4}{\cos^2 t}-4}}{\frac{2}{\cos t}}\cdot\frac{2\sin t}{\cos^2 t}\,dt$

$\displaystyle = \int 2\cdot\tan t\,\frac{\cancel{2}\sin t}{\cancel{2}\cos t}\,dt = 2\int \tan^2 t\,dt =$

$\displaystyle = 2(\tan t - t)+c = 2\left(\tan\left(\arccos\frac{2}{x}\right)-\arccos\frac{2}{x}\right)+c$

Σε ολοκληρώματα που περιέχει $\sqrt{x^2-a^2}$ θέτω $x=\frac{a}{\cos t}$

τότε

$\sqrt{\frac{a^2}{\cos^2 t}-a^2} = a\tan t$

12) Υπολογίστε το: $\displaystyle\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx$

Κάνουμε την αλλαγή μεταβλητής

$$y=\pi-x$$
$$\sin x=\sin(\pi-y)=\sin y$$
$$\cos x=\cos(\pi-y)=-\cos y$$
$$dy=-dx$$

Άρα,

$$I=\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}\,dx=-\int_{\pi}^{0}\frac{(\pi-y)\sin y}{1+\cos^2 y}\,dy=\int_0^{\pi}\frac{\pi\sin y-y\sin y}{1+\cos^2 y}\,dy$$

$$=\pi\cdot\int_0^{\pi}\frac{\sin y}{1+\cos^2 y}\,dy-I$$

Άρα $\displaystyle 2I=\pi\int_0^{\pi}\frac{\sin y}{1+\cos^2 y}\,dy\left(=\Big[-\pi\arctan(\cos y)\Big]_0^{\pi}\right)$

$$=\pi\cdot\frac{\pi}{2}\Rightarrow I=\frac{\pi^2}{4}$$

$$\int_0^{\pi}\frac{\sin x}{1+\cos^2 x}\,dx\quad \underset{du=-\sin x\,dx}{\overset{u=\cos x}{=}}$$

$$=-\int_1^{-1}\frac{du}{1+u^2}=\int_{-1}^{1}\frac{du}{1+u^2}$$

$$=\Big[\arctan u\Big]_{-1}^{1}=\frac{\pi}{4}-\left(-\frac{\pi}{4}\right)$$

$$=\frac{\pi}{2}$$

Όμοιως η 13 και η 14

18) Βρείτε τα όρια

(α) $\lim\limits_{x\to+\infty} x^3\cdot e^{-x^6}\int_0^{x^3} e^{t^2}\,dt$, (β) $\lim\limits_{x\to0^+}\frac{1}{x^4}\int_0^{x^2} e^t\cdot \sin t\,dt$

α) $\underline{y=x^3}$ $\lim\limits_{y\to+\infty} y\cdot e^{-y^2}\int_0^y e^{t^2}dt = \lim\limits_{y\to+\infty} y\int_0^y e^{t^2}dt$

Παίρνω παράγωγους

$$\frac{\int_0^y e^{t^2}dt + y\cdot e^{y^2}}{2y\cdot e^{y^2}} = \frac{1}{2} + \frac{\int_0^y e^{t^2}dt}{2y\cdot e^{y^2}} \longrightarrow \frac{1}{2}$$

0

DLH

$$\frac{e^{y^2}}{2e^{y^2}+4y^2\cdot e^{y^2}} \longrightarrow 0$$

Για το (β), αν $y=x^2$ θέτουμε το

$$\lim_{y\to0^+}\frac{\int_0^y e^t \sin t\,dt}{y^2}$$

$$\left|\int_0^y e^t \sin t\,dt\right| \le \int_0^y e^t|\sin t|\,dt \le \int_0^y e^y t\,dt = e^y\int_0^y t\,dt = \frac{y^2\cdot e^y}{2} \xrightarrow[y\to0^+]{} 0$$

DLH $\dfrac{\left(\int_0^y e^t \sin t\,dt\right)'}{(y^2)'} =$

$$= \frac{e^y \sin y}{2y} \to \frac{1}{2}$$