ΜΑΘΗΜΑ 22

ΠΡΟΤΕΙΝΟΜΕΝΕΣ ΑΝΤΙΚΑΤΑΣΤΑΣΕΙΣ.

R ρητή συνάρτηση

1 $R(\cos t, \sin t)$.

θέτουμε

$$u = \tan\frac{x}{2} \Rightarrow \begin{cases} \sin x = \dfrac{2u}{1+u^2} \\ \cos x = \dfrac{1-u^2}{1+u^2} \end{cases}$$

$$\Downarrow$$

$$\arctan u = \frac{x}{2}$$

$$\Downarrow$$

$$2\arctan u = x \Rightarrow dx = 2\, d\arctan u =$$

$$= 2\,\frac{du}{1+u^2}.$$

Παράδ:

$$\int \frac{1+\sin x}{1-\cos x}dx = \int \frac{1+\frac{2y}{1+y^2}}{1-\frac{1-y^2}{1+y^2}} \cdot \frac{2dy}{1+y^2} =$$

$$\left(y = \tan\frac{x}{2} \Rightarrow \right.$$

$$\Rightarrow \begin{cases} \sin x = \dfrac{2y}{1+y^2} \\ \cos x = \dfrac{1-y^2}{1+y^2} \end{cases}$$

$$\left. \text{και } dx = 2\,\frac{dy}{1+y^2} \right)$$

$$= \int \frac{1+2y+y^2}{1+y^2-1+y^2} \cdot \frac{2dy}{1+y^2} =$$

$$= \int \frac{\cancel{2}(1+y)^2}{\cancel{2}y^2(1+y^2)}dy$$

$$= \int \frac{(1+y)^2}{y^2(1+y^2)}dy.$$

Αναλύουμε σε απλά κλάσματα:

$$\frac{(1+y)^2}{y^2(1+y^2)} = \frac{a}{y}+\frac{\beta}{y^2}+\frac{\gamma y+\delta}{1+y^2} = \frac{ay+\beta}{y^2}+\frac{\gamma y+\delta}{1+y^2} =$$

$$= \frac{ay^3+ay+\beta y^2+\beta+\gamma y^3+\delta y^2}{y^2(1+y^2)} =$$

$$= \frac{(a+\gamma)y^3+(\beta+\delta)y^2+ay+\beta}{y^2(1+y^2)} \Rightarrow$$

$$\Rightarrow \begin{cases} a+\gamma=0 \Rightarrow \gamma=-2 \\ \beta+\delta=1 \Rightarrow \delta=0 \\ a=2 \\ \beta=1 \end{cases}$$

$$\int \frac{(1+y)^2\,dy}{y^2(1+y^2)} = 2\int\frac{dy}{y}+\int\frac{dy}{y^2}-\int\frac{2y\,dy}{1+y^2} =$$

$$= 2\ln|y| - \frac{1}{y} - \int\frac{d(y^2+1)}{1+y^2} =$$

$$= 2\ln|y| - \frac{1}{y} - \ln(1+y^2)+c$$

$$= 2\ln\left|\tan\frac{x}{2}\right| - \cot\frac{x}{2} - \ln\left(1+\tan^2\frac{x}{2}\right)+c.$$

② $R(x, \sqrt{1-x^2})$

$x = \sin t$ ή $u = \sqrt{1-x^2} \Rightarrow u^2 = 1-x^2 \Rightarrow$

$\Rightarrow 2u\,du = -2x\,dx$

$\Rightarrow \boxed{u\,du = -x\,dx.}$

Πχ: $\displaystyle\int \frac{dx}{x\sqrt{1-x^2}} = \int \frac{x\,dx}{x^2\sqrt{1-x^2}} = -\int \frac{u\,du}{(1-u^2)\cdot u} =$

$\displaystyle = -\int \frac{du}{(1+u)(1-u)} = \int \frac{du}{(u-1)(u+1)}$

$\displaystyle = \frac{1}{2}\left[\int \frac{du}{u-1} - \int \frac{du}{u+1}\right] =$

$\displaystyle = \frac{1}{2}\ln|u-1| - \frac{1}{2}\ln|u+1| + c$

$\displaystyle = \frac{1}{2}\ln\left|\sqrt{1-x^2}-1\right| - \frac{1}{2}\ln\left|\sqrt{1-x^2}+1\right| + c.$

3 $\underline{R(x, \sqrt{x^2-1})}$

$x = \frac{1}{\cos t}$ ή $\boxed{u = x + \sqrt{x^2-1}}$ $\Rightarrow$

$u - x = \sqrt{x^2-1} \Rightarrow$

$u^2 + \cancel{x^2} - 2ux = \cancel{x^2} - 1 \Rightarrow$

$u^2 + 1 = 2xu \Rightarrow$

$\boxed{x = \frac{u^2+1}{2u}}$

και $u - x = u - \frac{u^2+1}{2u} = \frac{u^2-1}{2u} \Rightarrow$

$\boxed{\sqrt{x^2-1} = \frac{u^2-1}{2u}}$

και $dx = \frac{(2u)^2 - 2(u^2+1)}{4u^2} du = \frac{4u^2 - 2u^2 - 2}{4u^2} du =$

$= \frac{2(u^2-1)}{4u^2} = \frac{u^2-1}{2u^2} du \Rightarrow$

$\boxed{dx = \frac{u^2-1}{2u^2} du}$

Πχ

$\int \frac{dx}{x\sqrt{x^2-1}} = \int \frac{1}{x} \cdot \frac{1}{\sqrt{x^2-1}} \cdot dx =$

$= \int \frac{2u}{u^2+1} \cdot \frac{2u}{\cancel{u^2-1}} \cdot \frac{\cancel{u^2-1}}{2u^2} du =$

$= \int \frac{2du}{u^2+1} = 2\arctan u + c =$

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(4) $R(x, \sqrt{x^2+1})$

$x = \tan t$ ή $\boxed{u = x + \sqrt{x^2+1}} \Rightarrow$

$u - x = \sqrt{x^2+1} \Rightarrow$

$u^2 + x^2 - 2xu = x^2 + 1 \Rightarrow$

$u^2 - 1 = 2xu \Rightarrow$

$$\boxed{x = \frac{u^2-1}{2u}}$$

και $\sqrt{x^2+1} = u - x = u - \frac{u^2-1}{2u} = \frac{2u^2 - u^2 + 1}{2u} \Rightarrow$

$$\Rightarrow \boxed{\sqrt{x^2+1} = \frac{u^2+1}{2u}}$$

και $dx = \frac{2u \cdot 2u - 2(u^2-1)}{4u^2} du =$

$= \frac{4u^2 - 2u^2 + 2}{4u^2} du = \frac{2(u^2+1)}{4u^2} du \Rightarrow$

$$\Rightarrow \boxed{dx = \frac{u^2+1}{2u^2} du}$$

πχ

$$\int (x+1)\sqrt{x^2+1}\,dx =$$

$$= \int \left(\frac{u^2-1}{2u}+1\right)\cdot\frac{u^2+1}{2u}\cdot\frac{u^2+1}{2u^2}\,du =$$

$$= \int \frac{(u^2+2u+1)\cdot(u^2+1)^2}{8u^4}\,du =$$ (σπίτι).

$$= \frac{1}{8}\int \frac{u^6+2u^5+u^4+4u^3-u^2+2u-1}{u^4}\,du =$$

$$= \frac{1}{8}\left[\int (u^2+2u+1)\,du + \int \frac{4u^3-u^2+2u-1}{u^4}\,du\right] =$$

$$\cancel{= \frac{1}{8}\left[\frac{u^3}{3}+u^2+u+4\ln|u| - \dots\right.}$$

$$= \frac{1}{8}\left[\int (u^2+2u+1)\,du + 4\int\frac{du}{u} - \int\frac{du}{u^2} + 2\int\frac{du}{u^3} - \int\frac{du}{u^4}\right]$$

$$= \frac{1}{8}\left[\frac{u^3}{3}+u^2+u+4\ln|u| + \frac{1}{u} - \frac{1}{u^2} + \frac{1}{3u^3}\right]$$

και αντικαθιστώ το u με x.

Σχόλιο Σε κάποιες περιπτώσεις μπορεί να δουλέψουν και άλλοι μετασχ., πχ ο μετασχ $u = \sqrt{\dots}$