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Ανάλυση ΙΙ

Διαφορικός Λογισμός / Παραγώγιση

(A) Εάν $\vec{U} = (U_1, U_2, \ldots, U_n) : \mathbb{R}^n \to \mathbb{R}^n$ γραμμική απεικόνιση σε αυτήν αντιστοιχεί ο πίνακας $\begin{pmatrix} U_1(\vec{e_1}) & U_1(\vec{e_2}) & \ldots & U_1(\vec{e_n}) \\ \vdots & \vdots & & \vdots \\ U_m(\vec{e_1}) & U_m(\vec{e_2}) & \ldots & U_m(\vec{e_n}) \end{pmatrix} = \Pi$

π.χ. $U : \mathbb{R} \to \mathbb{R}$, $U(h) = ah$ $\Pi_u \to (a)$, $U(1) = a$ (1×1)

$U : \mathbb{R}^2 \to \mathbb{R}$, $U(h_1, h_2) = ah_1 + bh_2$, $\Pi_u \to (a, b)$, $U(1,0) = a$, $U(0,1) = b$ (2×1)

$\vec{U} : \mathbb{R} \to \mathbb{R}^2$, $U(h) = (ah, bh)$, $h \in \mathbb{R}$

$\Pi_u \to \begin{pmatrix} a \\ b \end{pmatrix}$ (1×2) $\begin{pmatrix} U_1(h) = ah, & U_1(1) = a \\ U_2(h) = bh, & U_2(1) = b \end{pmatrix}$

$\boxed{\vec{U}(\vec{h}) = \Pi_u \vec{h}}$ $(\Pi_u (\vec{h})^t)$

(B) (1) Εάν $f : A (\subseteq \mathbb{R}^n) \to \mathbb{R}$ διαφορίσιμη στο $x_0 \in A$ τότε ξέρουμε $df(\vec{x_0})(\vec{h}) = \nabla f(\vec{x_0}) \vec{h} = \left(\frac{\partial f(\vec{x_0})}{\partial x_1}, \ldots, \frac{\partial f(\vec{x_0})}{\partial x_n} \right) \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{pmatrix}$

$\Pi_{df(\vec{x_0})} = \nabla f(\vec{x_0})$, δηλ: $df(\vec{x_0})(\vec{e_i}) = \frac{\partial f}{\partial x_i}(\vec{x_0})$, $i = 1, \ldots, n$

(2) $\vec{f} = (f_1, f_2, \ldots, f_m) : A (\subseteq \mathbb{R}^n) \to \mathbb{R}^m$ διαφορίσιμη στο $\vec{x_0} \in A$

$d\vec{f}(\vec{x_0})(\vec{h}) = (df_1(\vec{x_0})(\vec{h}), \ldots, df_m(\vec{x_0})(\vec{h}))$, $\vec{h} \in \mathbb{R}^n$

Άρα ο πίνακας της γραμμικής $df(\vec{x_0})$

$$J_{\vec{f}}(\vec{x_0}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \ldots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \vdots & & \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \ldots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$$

Πίνακας Jacobi της $\vec{f}$ $(m \geq 2$

π.χ. $f(x) = x^3$, $x_0 = 1$ $J_f(1) = (3$

$df(1)(h) = 3h$, $h \in \mathbb{R}$

• $\vec{f}(x) = (x^2, x^3)$, $x_0 = 1$ $J_{\vec{f}}(1) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

- $f(x,y) = x^3 + y^2$, $\vec{x_0} = (1,1)$ $\quad J_f(1,1) = (3,2)$ $\quad (= \nabla f(1,1))$
- $\vec{f}(x,y) = (x^3, x^3+y^2)$ $\quad \vec{x_0} = (1,1)$ $\quad J_f(1,1) = \begin{pmatrix} 3 & 0 \\ 3 & 3 \end{pmatrix}$

(Γ) $\vec{f} : B(\subseteq \mathbb{R}^m) \to \mathbb{R}^\ell$, $\vec{\phi} : A(\subseteq \mathbb{R}^n) \to \mathbb{R}^m$

$\vec{\phi}(A) \subseteq B$

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "\vec{\phi}"] \arrow[dr, "\vec{f}\circ\vec{\phi}"'] & \mathbb{R}^m \arrow[d, "\vec{f}"] \\ & \mathbb{R}^\ell \end{tikzcd}$$

Εάν $\exists\, d\vec{\phi}(\vec{t_0})$, $d\vec{f}(\vec{\phi}(\vec{t_0}))$ Τότε

$$\exists\, d(\vec{f}\circ\vec{\phi})(\vec{t_0}) = d\vec{f}(\vec{\phi}(\vec{t_0})) \circ d\vec{\phi}(\vec{t_0})$$

Ξέρουμε από τη γραμμική άλγεβρα ότι ο πίνακας που αντιστοιχεί στην σύνθεση 2 γραμμικών απεικονίσεων είναι το γινόμενο των αντίστοιχων πινάκων.

$$\boxed{J_{\vec{f}\circ\vec{\phi}}(\vec{t_0}) = J_{\vec{f}}(\vec{\phi}(\vec{t_0})) \cdot J_{\vec{\phi}}(\vec{t_0})} \qquad (*)$$

π.χ

$$(t)\quad \begin{tikzcd} \mathbb{R} \arrow[r, "\vec{r}"] \arrow[dr, "f\circ\vec{r}"'] & \mathbb{R}^n \arrow[d, "f"] \\ & \mathbb{R} \end{tikzcd}$$

$$\frac{d(f\circ\vec{r})(\vec{t_0})}{dt} = \nabla \vec{f}(\vec{r}(\vec{t_0})) \cdot \vec{r}\,'^{T}(t_0) =$$

$$= \nabla f(\vec{r}(t_0)) \cdot \vec{r}\,'(t_0)$$

εσωτερικό γινόμενο

$$\begin{tikzcd} (u,v) \arrow[r, "\vec{\phi}"] \arrow[dr, "f\circ\vec{\phi}"'] & (x,y) \arrow[d, "f"] \\ & \mathbb{R} \end{tikzcd}$$

$\vec{\phi}(u,v) = (x(u,v), y(u,v))$

$$\left(\frac{\partial(f\circ\vec{\phi})}{\partial u}, \frac{\partial(f\circ\vec{\phi})}{\partial v} \right)\Bigg|_{(u_0,v_0)} = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)\Bigg|_{\vec{\phi}(u_0,v_0)} \cdot \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}\Bigg|_{(u_0,v_0)}$$

Π.χ

$$\begin{tikzcd} (x,y) \arrow[r, "\phi"] \arrow[dr, "f\circ\phi"'] & \mathbb{R} \arrow[d, "f"] & (t) \\ & \mathbb{R} \end{tikzcd}$$

$$\left(\frac{\partial(f\circ\phi)}{\partial x}, \frac{\partial(f\circ\phi)}{\partial y} \right)\Bigg|_{(x_0,y_0)} = f'(\phi(x_0,y_0)) \cdot \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y} \right)\Bigg|_{(x_0,y_0)}$$

Ασκήσεις

(1) Η θερμοκρασία στο σημείο (x,y,z) του χώρου είναι $x\cdot y + z$. Ποιος ο ρυθμός μεταβολής της θερμοκρασίας στην έλικα $\vec{r} = (\text{συν}t, \text{ημ}t, t)$. Ιδιαιτέρως $t_0 = 0$.

Λύση

$$\begin{tikzcd} (t)\ \mathbb{R} \arrow[r, "\vec{r}"] \arrow[dr, "T\circ\vec{r}\ (T)"'] & \mathbb{R}^3 \arrow[d, "T"] \\ & \mathbb{R} \end{tikzcd}$$

$$\frac{d(T\circ\vec{r})(t_0)}{dt} = \nabla T(\vec{r}(t_0)) \cdot \vec{r}\,'(t_0) =$$

$$= (y, x, 1)\Big|_{(\text{συν}t_0, \text{ημ}t_0, t_0)} \cdot (-\text{ημ}(t), \text{συν}t, 1) =$$

$$= (-\text{ημ}^2 t_0 + \text{συν}^2 t_0 + 1) \quad \text{Άρα} \quad \frac{d(T\circ\vec{r})}{dt}(0) = 2.$$

(2) $\phi(x,y) = \dfrac{x}{y}$, $f : \mathbb{R} \to \mathbb{R}$ διαφορίσιμες

$\dfrac{\partial f\circ\phi}{\partial x}$, $\dfrac{\partial f\circ\phi}{\partial y}$ στο (x_0, y_0)

$$\begin{tikzcd}
(x,y) \arrow[r, "\phi"] \arrow[dr, "f\circ\phi"'] & \mathbb{R} \; (t) \arrow[d, "f"] \\
& \mathbb{R}
\end{tikzcd}$$

Άρα $\left(\dfrac{\partial(f\circ\phi)}{\partial x}, \dfrac{\partial(f\circ\phi)}{\partial y}\right)_{(x_0,y_0)} = f'(\phi(x_0,y_0)) \cdot \left(\dfrac{\partial\phi}{\partial x}, \dfrac{\partial\phi}{\partial y}\right)_{(x_0,y_0)}$

$$\frac{\partial(f\circ\phi)}{\partial x}(x_0, y_0) = f'\left(\frac{x_0}{y_0}\right)\frac{1}{y_0}$$

$$\frac{\partial(f\circ\phi)}{\partial y}(x_0, y_0) = f'\left(\frac{x_0}{y_0}\right)\left(-\frac{x_0}{y_0^2}\right)$$

(3) $\dfrac{\partial w}{\partial r}$, $\dfrac{\partial w}{\partial s}$, $w = x + 2y + z^2$, $x = \dfrac{r}{s}$, $y = r^2 + \ln s$

$z = 2r$

$$\begin{tikzcd}
(r,s)\ \mathbb{R}^2 \arrow[r, "\vec{\phi}"] \arrow[dr, "w\circ\vec{\phi}\ (w)"'] & (x,y,z)\ \mathbb{R}^3 \arrow[d, "w"] \\
& \mathbb{R}
\end{tikzcd}$$

$$\left(\frac{\partial(w\circ\vec{\phi})}{\partial r}, \frac{\partial(w\circ\vec{\phi})}{\partial s}\right)_{(r_0,s_0)} = \left(\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial w}{\partial z}\right)_{(\vec{\phi}(r_0,s_0))} \begin{pmatrix} \frac{1}{s_0} & -\frac{r_0}{s_0^2} \\ 2r_0 & \frac{1}{s_0} \\ 2 & 0 \end{pmatrix}$$

Άρα $\dfrac{\partial(w\circ\vec{\phi})}{\partial r}(r_0, s_0) = \dots = \dfrac{1}{s} + 12r$

$\dfrac{\partial(w\circ\vec{\phi})}{\partial r}(r_0, s_0) = \dots = \dfrac{2}{s_0} - \dfrac{r_0}{s_0^2}$

(4)* $F(r,\phi) = f(r\,\text{συν}\phi,\ r\,\text{ημ}\phi)$

Ν.Σ.Ο (i) $\|\nabla f\|^2 = \left(\frac{\partial F}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial F}{\partial \phi}\right)^2$

(ii) $\nabla^2 f = \frac{\partial^2 F}{\partial r^2} + \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \phi^2}$ (εκτός)

$\left(\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2}\right)$

Λυση

$$\begin{tikzcd}
(r,\phi)\ \mathbb{R}^2 \arrow[r, "\vec{\Phi}"] \arrow[dr, "(F\equiv f)"'] & \mathbb{R}^2\ (x,y) \arrow[d, "f"] \\
& \mathbb{R}
\end{tikzcd}$$

$\vec{\Phi}(r,\phi) = (r\,\text{συν}\phi,\ r\,\text{ημ}\phi)$

$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$, $\|\nabla f\| = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2$ (1)

Είναι τώρα $\left(\frac{\partial F}{\partial r}, \frac{\partial F}{\partial \phi}\right) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)\begin{pmatrix} \text{συν}\phi & r(-\text{ημ}\phi) \\ \text{ημ}\phi & r\,\text{συν}\phi \end{pmatrix}$

$$\frac{\partial F}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = \text{συν}\phi\frac{\partial f}{\partial x} + \text{ημ}\phi\frac{\partial f}{\partial y}$$

$$\frac{\partial F}{\partial \phi} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \phi} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \phi} = -r\,\text{ημ}\phi\frac{\partial f}{\partial x} + r\,\text{συν}\phi\frac{\partial f}{\partial y}$$

$$\left(\frac{\partial F}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial F}{\partial \phi}\right)^2 = \left[\text{συν}^2\phi\left(\frac{\partial f}{\partial x}\right)^2 + \cancel{2\,\text{ημ}\phi\,\text{συν}\phi\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}} + \text{ημ}^2\phi\left(\frac{\partial f}{\partial y}\right)^2\right] +$$

$$+\left[\text{ημ}^2\phi\left(\frac{\partial f}{\partial x}\right)^2 - \cancel{2\,\text{ημ}\phi\,\text{συν}\phi\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}} + \text{συν}^2\phi\left(\frac{\partial f}{\partial y}\right)^2\right] =$$

$= \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2$ (2)

Άρα (1) = (2)

(5) Θ.Μ.Τ.Δ.Λ (ΘΕΩΡΗΜΑ ΜΕΣΗΣ ΤΙΜΗΣ ΔΙΑΦΟΡΙΚΟΥ ΛΟΓΙΣΜΟΥ)

(i) Έστω ότι η $f : A(\subseteq \mathbb{R}^n) \to \mathbb{R}$ διαφορίσιμη.

Έστω $\vec{a}, \vec{b} \in A : [\vec{a}, \vec{b}] \subseteq A$

Τότε $\exists\ \vec{\xi} \in (\vec{a}, \vec{b}) : f(\vec{b}) - f(\vec{a}) = df(\vec{\xi})(\vec{b} - \vec{a}) =$

$= \nabla f(\xi)(\vec{b} - \vec{a})$

(ii) $g =$ διαφορίσιμη στο κυρτό σύνολο K

$(\vec{x}_0, \vec{y}_0 \in K \Rightarrow [\vec{x}_0, \vec{y}_0] \subseteq K)$

$\nabla g(\vec{x}) = \vec{0} \quad \forall\ \vec{x} \in K \Rightarrow g =$ σταθερή

$\nabla \phi = \nabla h \Rightarrow \phi - h = c$

Θεωρούμε το ευθ. τμήμα $[\vec{a}, \vec{b}]$ έχει δ. εξίσωση

$\vec{r}(t) = \vec{a} + t(\vec{b} - \vec{a}), \ t \in [0,1]$

$\vec{r}'(t) = \vec{b} - \vec{a}$

$g = f \circ \vec{r} : [0,1] \to \mathbb{R}$ | $g(1) - g(0) = g'(t_0)(1-0)$ (ΘΜΤ) για κάποιο $t_0 \in (0,1)$

Άρα $f(\vec{b}) - f(\vec{a}) = g'(t_0) = \nabla f(\vec{r}(t_0)) \cdot \vec{r}'(t_0) =$

$= \nabla f(\vec{r}(t_0)) \cdot (\vec{b} - \vec{a})$

(6) **Θ. EULER** Έστω U αν. σύνολο $\subseteq \mathbb{R}^n$ με $\{t\vec{x}, t \in (0,+\infty), \vec{x} \in U\} \subseteq U$. Αν f διαφορίσιμη συν. ομογενής(*) βαθμού a με ((*) $f(t\vec{x}) = t^{\alpha} f(\vec{x})$, $t>0$) $f : U \subseteq \mathbb{R}^n \to \mathbb{R} \iff \vec{x} \cdot \nabla f(\vec{x}) = a f(\vec{x})$

Π.χ. Η βαρυτική δυναμική συνάρτηση $V(x,y,z) = -\dfrac{GMm}{\sqrt{x^2+y^2+z^2}}$ $(x,y,z) \in \mathbb{R}^3 \setminus (0,0,0)$ είναι ομογενής βαθμού $a = -1$

ΕΠΟΜΕΝΩΣ $x\dfrac{\partial V}{\partial x} + y\dfrac{\partial V}{\partial y} + z\dfrac{\partial V}{\partial z} = -V$ (που πράγματι ισχύει).

| Υποθ. ($\Rightarrow$) $\vec{x}_0 =$ σταθ. $\vec{\varepsilon}(t) = t\vec{x}_0,\ t>0$. $g(t) = f \circ \vec{\varepsilon}(t) \Rightarrow \nabla f(\vec{\varepsilon}(t)) \cdot \vec{\varepsilon}'(t) \overset{(*)}{=} \alpha t^{\alpha-1} f(\vec{x}_0)$. Βάζουμε $t=1$

($\Leftarrow$) Θεωρούμε $F(t) = \dfrac{f(t\vec{x}_0)}{t^{\alpha}},\ t>0 \overset{(\text{υποθ})}{\Rightarrow} F'(t) = 0,\ t>0 \Rightarrow f(t\vec{x}_0) = ct^{\alpha},\ f(\vec{x}_0) = c \Rightarrow f(t\vec{x}_0) = t^{\alpha} f(\vec{x}_0)$