

Ανάδωτον II

Αδοσιδωτη Παράγωγιση

(A) Εάν $\vec{u} = (u_1, u_2, \dots, u_n) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ γραμμική απεικόνιση
 σε αυτήν αντιστοιχεί ο πίνακας $\begin{pmatrix} u_1(\vec{e}_1) & u_1(\vec{e}_2) & \dots & u_1(\vec{e}_n) \\ \vdots & \vdots & \ddots & \vdots \\ u_m(\vec{e}_1) & u_m(\vec{e}_2) & \dots & u_m(\vec{e}_n) \end{pmatrix} = J_{\vec{u}}$

$n \times n \quad u : \mathbb{R} \rightarrow \mathbb{R}, \quad u(h) = ah \quad \Pi_u \rightarrow (a), \quad u(1) = a. \quad (1 \times 1)$

$u : \mathbb{R}^2 \rightarrow \mathbb{R}, \quad u(h_1, h_2) = ah_1 + bh_2, \quad \Pi_u \rightarrow (a, b), \quad u(1, 0) = a$
 $u(0, 1) = b \quad (2 \times 1)$

$\vec{u} : \mathbb{R} \rightarrow \mathbb{R}^2, \quad u(h) = (ah, bh), \quad h \in \mathbb{R}$

$\Pi_u \rightarrow \begin{pmatrix} a \\ b \end{pmatrix} \quad (1 \times 2) \quad \begin{pmatrix} u_1(h) = ah, \quad u_2(h) = bh \\ u_2(h) = bh, \quad u_2(h) = bh \end{pmatrix}$

$\vec{u}(\vec{h}) = \Pi_u \vec{h} \quad (\Pi_u(\vec{h}))^t$

(B) (1) Εάν $f : A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ Διαφορίσιμη στο $x_0 \in A$ τότε
 έχουμε $df(\vec{x}_0)(\vec{h}) = \nabla f(\vec{x}_0) \vec{h} = \left(\frac{\partial f(\vec{x}_0)}{\partial x_1}, \dots, \frac{\partial f(\vec{x}_0)}{\partial x_n} \right) \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{pmatrix}$

$\Pi_{df(\vec{x}_0)} = \nabla f(\vec{x}_0)$, όπου: $df(\vec{x}_0)(\vec{e}_i) = \frac{\partial f}{\partial x_i}(\vec{x}_0), \quad i = 1, \dots, n$

(2) $\vec{f} = (f_1, f_2, \dots, f_m) : A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$ Διαφορίσιμη στο $\vec{x}_0 \in A$

$df(\vec{x}_0)(\vec{h}) = (df_1(\vec{x}_0)(\vec{h}), \dots, df_m(\vec{x}_0)(\vec{h}))$, $\vec{h} \in \mathbb{R}^n$

Άρα ο πίνακας της γραμμικής απεικόνισης $df(\vec{x}_0)$

$J_{\vec{f}}(\vec{x}_0) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \dots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \dots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$ Πίνακας Jacobι της \vec{f} (m x n)
 π.χ $f(x) = x^3, x_0 = 1 \quad J_f(1) = (3)$
 $df(1)(h) = 3h, h \in \mathbb{R}$

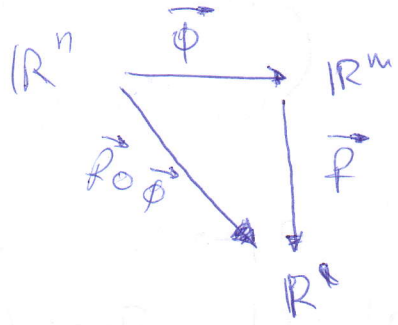
$\vec{f}(x) = (x^2, x^3), \quad x_0 = 1 \quad J_{\vec{f}}(1) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

• $f(x, y) = x^3 + y^2$, $\vec{x}_0 = (1, 1)$ $J_f(1, 1) = (3, 2)$ ($= \nabla f(1, 1)$)

• $\vec{F}(x, y) = (x^3, x^3 + y^2)$ $\vec{x}_0 = (1, 1)$ $J_{\vec{F}}(1, 1) = \begin{pmatrix} 3 & 0 \\ 3 & 2 \end{pmatrix}$

(Γ) $\vec{F} : B (\subseteq \mathbb{R}^m) \rightarrow \mathbb{R}^p$, $\vec{\Phi} : A (\subseteq \mathbb{R}^n) \rightarrow \mathbb{R}^m$

$\vec{\Phi}(A) \subseteq B$



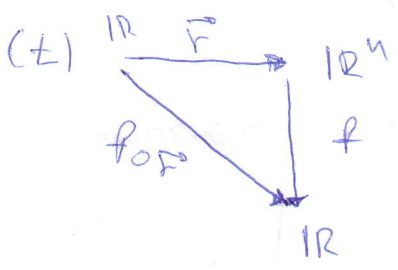
Εάν $\exists d\vec{\Phi}(\vec{t}_0)$, $d\vec{F}(\vec{\Phi}(\vec{t}_0))$ Τότε

$\exists d(\vec{F} \circ \vec{\Phi})(\vec{t}_0) = d\vec{F}(\vec{\Phi}(\vec{t}_0)) \circ d\vec{\Phi}(\vec{t}_0)$

Ξέρουμε από τη γραμμική άλγεβρα ότι ο πίνακας που αντιστοιχεί στην σύνθεση 2 γραμμικών απεικονίσεων είναι το γινόμενο των αντιστοιχών πινάκων.

$J_{\vec{F} \circ \vec{\Phi}}(\vec{t}_0) = J_{\vec{F}}(\vec{\Phi}(\vec{t}_0)) \cdot J_{\vec{\Phi}}(\vec{t}_0)$ (*)

π.χ

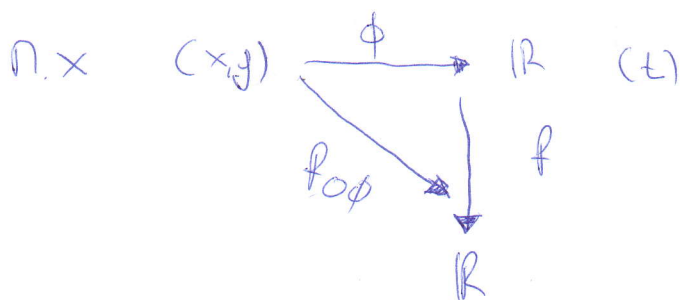


$\frac{d(f \circ F)(t_0)}{dt} = \nabla \vec{F}(\vec{r}(t_0)) \cdot \vec{r}'(t_0) =$
 $= \nabla f(\vec{r}(t_0)) \cdot \vec{r}'(t_0)$
 ↑
 εσωτερικό γινόμενο

$(u, v) \xrightarrow{\phi} (x, y) \quad \vec{\phi}(u, v) = (x(u, v), y(u, v))$
 $\downarrow f_{\phi} \quad \downarrow f$
 $\mathbb{R}^2 \quad \mathbb{R}^2$

$$\left(\frac{\partial (f \circ \phi)}{\partial u}, \frac{\partial (f \circ \phi)}{\partial v} \right)_{(u_0, v_0)} = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)_{\phi(u_0, v_0)}$$

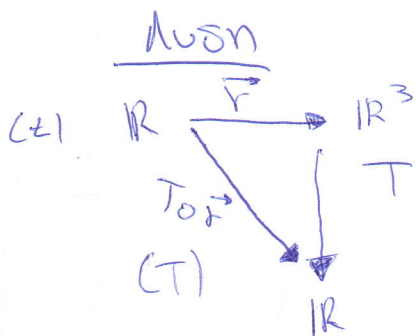
$$\cdot \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}_{(u_0, v_0)}$$



$$\left(\frac{\partial (f \circ \phi)}{\partial x}, \frac{\partial (f \circ \phi)}{\partial y} \right)_{(x_0, y_0)} = f'(\phi(x_0, y_0)) \cdot \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y} \right)_{(x_0, y_0)}$$

Ασκήσεις

(1) Η θερμοκρασία σε σημείο (x, y, z) του χώρου είναι $x \cdot y + z$. Ποιος ο ρυθμός μεταβολής της θερμοκρασίας στην επιφάνεια $\vec{r} = (\cos t, \sin t, t)$. Ίσως αρχικά $t_0 = 0$.



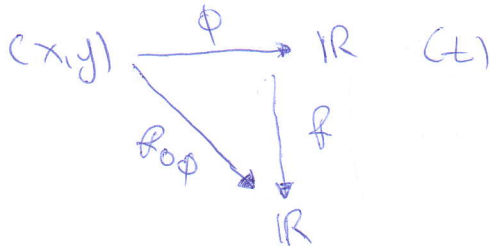
$$\frac{d(T \circ \vec{r})(t_0)}{dt} = \nabla T(\vec{r}(t_0)) \cdot \vec{r}'(t_0) =$$

$$= (y, x, 1) \Big|_{(\cos t_0, \sin t_0, t_0)} \cdot (-\sin t_0, \cos t_0, 1) =$$

$$= (-\sin^2 t_0 + \cos^2 t_0 + 1) \quad \text{Άρα} \quad \frac{d(T \circ \vec{r})}{dt} \Big|_{t_0} = 2.$$

(2) $\phi(x,y) = \frac{x}{y}$, $f: \mathbb{R} \rightarrow \mathbb{R}$ διαφορίσιμες

$\frac{\partial (f \circ \phi)}{\partial x}$, $\frac{\partial (f \circ \phi)}{\partial y}$ στο (x_0, y_0)



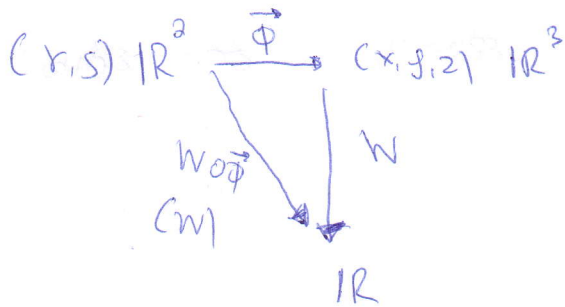
Αρα $\left(\frac{\partial (f \circ \phi)}{\partial x}, \frac{\partial (f \circ \phi)}{\partial y} \right)_{(x_0, y_0)} = f'(\phi(x_0, y_0)) \cdot \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y} \right)_{(x_0, y_0)}$

$\frac{\partial (f \circ \phi)}{\partial x} (x_0, y_0) = f' \left(\frac{x_0}{y_0} \right) \frac{1}{y_0}$

$\frac{\partial (f \circ \phi)}{\partial y} (x_0, y_0) = f' \left(\frac{x_0}{y_0} \right) \left(-\frac{x_0}{y_0^2} \right)$

(3) $\frac{\partial W}{\partial r}$, $\frac{\partial W}{\partial s}$, $W = x + 2y + z^2$, $x = \frac{r}{s}$, $y = r^2 + \cos s$

$z = 2r$



$\left(\frac{\partial (W \circ \vec{\phi})}{\partial r}, \frac{\partial (W \circ \vec{\phi})}{\partial s} \right)_{(r_0, s_0)} = \left(\frac{\partial W}{\partial x}, \frac{\partial W}{\partial y}, \frac{\partial W}{\partial z} \right)_{(\vec{\phi}(r_0, s_0))} \begin{pmatrix} \frac{1}{s_0} & -\frac{r_0}{s_0^2} \\ 2r_0 & \frac{1}{s_0} \\ 2 & 0 \end{pmatrix}$

Αρα $\frac{\partial (W \circ \vec{\phi})}{\partial r} (r_0, s_0) = \dots = \frac{1}{s} + 4r$

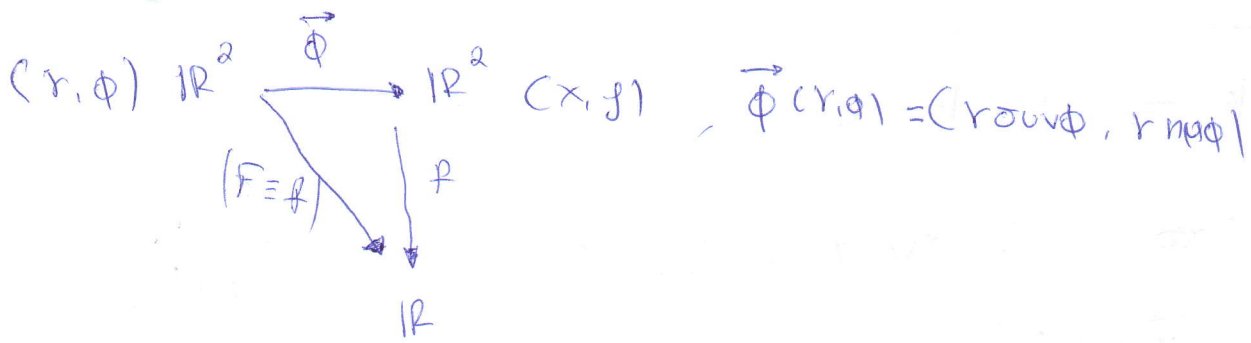
$\frac{\partial (W \circ \vec{\phi})}{\partial s} (r_0, s_0) = \dots = \frac{2}{s} - \frac{r}{s^2}$

(4)* $F(r, \phi) = f(r \cos \phi, r \sin \phi)$

N.S.O (i) $\|\nabla f\|^2 = \left(\frac{\partial F}{\partial r}\right)^2 + \frac{1}{r^2} \left(\frac{\partial F}{\partial \phi}\right)^2$

(iii) $\nabla^2 f = \frac{\partial^2 F}{\partial r^2} + \frac{1}{r} \frac{\partial F}{\partial r} + \frac{1}{r^2} \frac{\partial^2 F}{\partial \phi^2}$ (C.F. & C.S.)
 $\left(\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2}\right)$

Λυσή



$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$, $\|\nabla f\| = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2$ (1)

Επιπλέον έχουμε $\left(\frac{\partial F}{\partial r}, \frac{\partial F}{\partial \phi}\right) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \begin{pmatrix} \cos \phi & -r \sin \phi \\ \sin \phi & r \cos \phi \end{pmatrix}$

$\frac{\partial F}{\partial r} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial r} = \cos \phi \frac{\partial f}{\partial x} + \sin \phi \frac{\partial f}{\partial y}$

$\frac{\partial F}{\partial \phi} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial \phi} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \phi} = -r \sin \phi \frac{\partial f}{\partial x} + r \cos \phi \frac{\partial f}{\partial y}$

$\left(\frac{\partial F}{\partial r}\right)^2 + \frac{1}{r^2} \left(\frac{\partial F}{\partial \phi}\right)^2 = \left[\cos^2 \phi \left(\frac{\partial f}{\partial x}\right)^2 + 2 \sin \phi \cos \phi \frac{\partial f}{\partial x} \frac{\partial f}{\partial y} + \sin^2 \phi \left(\frac{\partial f}{\partial y}\right)^2 \right] +$
 $+ \left[\sin^2 \phi \left(\frac{\partial f}{\partial x}\right)^2 - 2 \sin \phi \cos \phi \frac{\partial f}{\partial x} \frac{\partial f}{\partial y} + \cos^2 \phi \left(\frac{\partial f}{\partial y}\right)^2 \right] =$

$= \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2$ (2) Άρα (1) = (2)

(5) Θ. Μ. Τ. Δ Λ (ΘΕΩΡΗΜΑ ΜΕΣΗΣ ΤΙΜΗΣ ΔΙΑΦΟΡΙΚΟΥ ΛΟΓΙΣΜΟΥ)

(i) Έστω ότι η $f : A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ διαφορίσιμη.

Έστω $\vec{a}, \vec{b} \in A : [\vec{a}, \vec{b}] \subseteq A$

Τότε $\exists \vec{\xi} \in (\vec{a}, \vec{b}) : f(\vec{b}) - f(\vec{a}) = df(\vec{\xi})(\vec{b} - \vec{a}) = \nabla f(\vec{\xi})(\vec{b} - \vec{a})$

(ii) $g =$ διαφορίσιμη ~~σε~~ στο κυρτό σύνολο K

$(\vec{x}_0, \vec{f}_0 \in K \Rightarrow [\vec{x}_0, \vec{f}_0] \subseteq K)$

$\nabla g(\vec{x}) = \vec{0} \quad \forall \vec{x} \in K \Rightarrow g = \text{σταθερή}$

$\nabla \phi = \nabla h \Rightarrow \phi - h = c$

Θεωρούμε το ευθ. τμήμα $[\vec{a}, \vec{b}]$ έχει δ εξίσωση

$\vec{r}(t) = \vec{a} + t(\vec{b} - \vec{a}), t \in [0, 1]$

$\vec{r}'(t) = \vec{b} - \vec{a}$

$g = f \circ \vec{r} : [0, 1] \rightarrow \mathbb{R} \quad \left| \quad \begin{array}{l} g(1) - g(0) = g'(t_0)(1-0) \quad (\text{ΘΜΤ}) \\ \text{για κάποιο } t_0 \in (0, 1) \end{array} \right.$

Άρα $f(\vec{b}) - f(\vec{a}) = g'(t_0) = \nabla f(\vec{r}(t_0)) \cdot \vec{r}'(t_0) = \nabla f(\vec{r}(t_0)) \cdot (\vec{b} - \vec{a})$

(6) Θ. EULER Έστω U αν. σύνολο $\subseteq \mathbb{R}^n$ με $\{t\vec{x}, t \in (0, +\infty), \vec{x} \in U\} \subseteq U$
 $\subseteq U$ Αν f διαφορίσιμη συν. ομογενής βαθμού α με $f(t\vec{x}) = t^\alpha f(\vec{x})$
 $f : U \subseteq \mathbb{R}^n \rightarrow \mathbb{R} \Leftrightarrow \vec{x} \cdot \nabla f(\vec{x}) = \alpha f(\vec{x})$

Π.χ. Η βαροσική δυναμική συνάρτηση $V(x, y, z) = -\frac{GMm}{\sqrt{x^2 + y^2 + z^2}}$
 $(x, y, z) \in \mathbb{R}^3, (0, 0, 0)$ είναι ομογενής βαθμού $\alpha = -1$

ΕΠΟΜΕΝΟΣ $x \frac{\partial V}{\partial x} + y \frac{\partial V}{\partial y} + z \frac{\partial V}{\partial z} = -V$ (που πράγματι
ΛΟΧΩΕΙ). $\left| \begin{array}{l} \text{Υπόθεση } \vec{x}_0 = \text{σταθ.} \quad \vec{x}(t) = t\vec{x}_0, t > 0 \quad g(t) = f \circ \vec{x}(t) \Rightarrow \nabla f(\vec{x}(t)) \cdot \vec{x}'(t) = \alpha t^{\alpha-1} f(\vec{x}_0) \quad \text{Βρίσκουμε } t=1 \\ (\Leftrightarrow) \text{Θεωρούμε } F(t) = \frac{f(t\vec{x}_0)}{t^\alpha}, t > 0 \xrightarrow{(100\%) } F'(t) = 0, t > 0 \Rightarrow f(t\vec{x}_0) = ct^\alpha, f(\vec{x}_0) = c \Rightarrow f(t\vec{x}_0) = t^\alpha f(\vec{x}_0) \end{array} \right.$