Διανυσματικά πεδία.

1. Δυναμικές γραμμές (γραμμές ροής).

$$\vec{F} : \mathbb{R}^n \to \mathbb{R}^n .$$

Γραμμή ροής (Δυναμική γραμμή).

$$\vec{\sigma}(t) : \qquad \vec{\sigma}'(t) = \vec{F}(\vec{\sigma}(t))$$

2. Στροβιλισμός Διανυσματικού Πεδίου.

$$\vec{\nabla} \times \vec{F} = \hat{i}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + \hat{j}\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + \hat{k}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

* $\vec{\nabla} \times \vec{F} = 0 \Rightarrow \vec{F} = \vec{\nabla} f$.

↳ βαθμωτό δυναμικό.

$(\vec{\nabla} \times (\vec{\nabla} f) = 0)$.

3. Απόκλιση Διανυσματικού Πεδίου.

$$\vec{\nabla}\cdot\vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

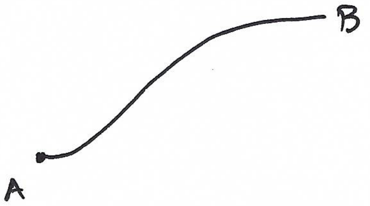
* $$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = 0$$

$$\vec{\nabla}\cdot\vec{F} = 0 \Rightarrow \vec{F} = \vec{\nabla}\times\vec{A}.$$

↳ διανυσματικό δυναμικό.

4. Επικαμπύλια ολοκληρώματα.

C :



καμπύλη με παραμετρική περιγραφή : $\vec{\sigma}(t)$.

Στοιχείο Μήκους: $dl = \left\|\frac{d\vec{\sigma}}{dt}\right\| dt$

$$\vec{dl} = \hat{T}\, dl = \frac{d\vec{\sigma}}{dt}\, dt$$

$$= \frac{\frac{d\vec{\sigma}}{dt}}{\left\|\frac{d\vec{\sigma}}{dt}\right\|} \left\|\frac{d\vec{\sigma}}{dt}\right\| dt$$

f : $$\int_A^B f(\vec{r})\, dl = \int_{t_1}^{t_2} f(\vec{\sigma}(t)) \left\|\frac{d\vec{\sigma}}{dt}\right\| dt$$ $\vec{\sigma}(t_1) = A$, $\vec{\sigma}(t_2) = B$.

$\vec{F}$: $$\int_A^B \vec{F}\cdot\vec{dl} = \int_{t_1}^{t_2} \vec{F}(\vec{\sigma}(t))\cdot\frac{d\vec{\sigma}}{dt}\, dt$$

Για $n=2$ $\quad \hat{T}, \hat{N}$ βάση του χώρου.

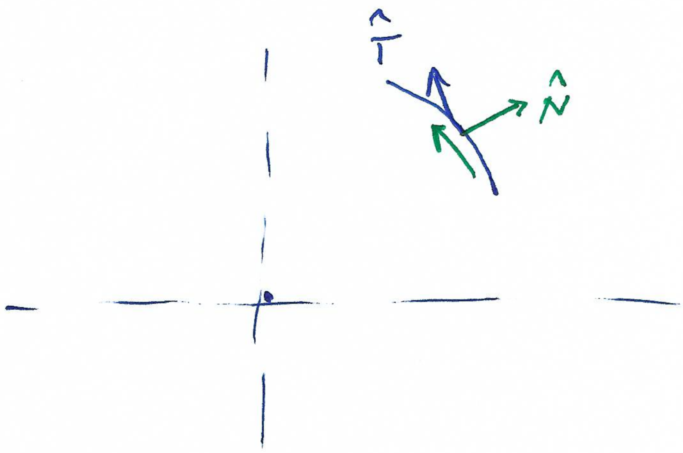
$$\vec{T} = \left(\frac{dx}{dt}, \frac{dy}{dt}\right), \quad \hat{T} = \frac{\vec{T}}{\left\|\frac{d\vec{\sigma}}{dt}\right\|}$$

$$\vec{N} = \left(+\frac{dy}{dt}, -\frac{dx}{dt}\right), \quad \hat{N} = \frac{\vec{N}}{\left\|\frac{d\vec{\sigma}}{dt}\right\|}$$

$$\int_A^B \vec{F}\cdot\hat{N}\, dl = \int_{t_1}^{t_2} \vec{F}(\vec{\sigma}(t))\cdot\hat{N}(t)\left\|\frac{d\vec{\sigma}}{dt}\right\| dt.$$

$$\int_A^B \vec{F}\cdot d\vec{l} = \int_{t_1}^{t_2}\left(F_x\frac{dx}{dt} + F_y\frac{dy}{dt}\right)dt = \int_A^B F_x dx + F_y dy$$

$$\int_A^B \vec{F}\cdot\hat{N}\, dl = \int_{t_1}^{t_2}\left(F_x\frac{dy}{dt} - F_y\frac{dx}{dt}\right)dt = \int_A^B F_x dy - F_y dx$$



π.χ. $\vec{\sigma}(t) = (\cos t, \sin t)$

$\hat{T}(t) = (-\sin t, \cos t)$

$\hat{N}(t) = (\cos t, \sin t)$

5. Επιφανειακά Ολοκληρώματα. $(n=3)$.

S : $\vec{\Phi}(u,v)$

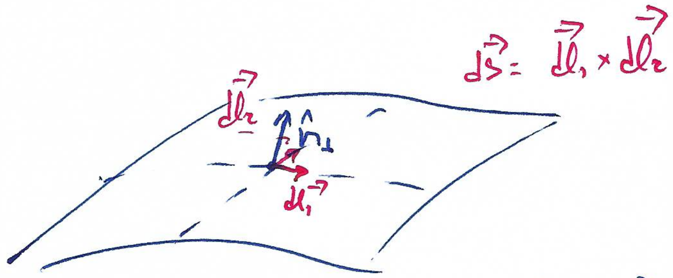
Στοιχείο εμβαδού: $\left\|\dfrac{\partial\vec{\Phi}}{\partial u}\times\dfrac{\partial\vec{\Phi}}{\partial v}\right\| du\,dv = dS$

Αλλά και: $d\vec{S} = \hat{n}_\perp\, dS = \left(\dfrac{\partial\vec{\Phi}}{\partial u}\times\dfrac{\partial\vec{\Phi}}{\partial v}\right) du\,dv$

• f : $$\int_S f\,dS = \iint_{\substack{D \\ \llcorner \{(u,v)\}}} f(\vec{\Phi}(u,v))\left\|\frac{\partial\vec{\Phi}}{\partial u}\times\frac{\partial\vec{\Phi}}{\partial v}\right\| du\,dv$$

• $\vec{F}$: Ροή Διανυσματικού Πεδίου από επιφάνεια S:

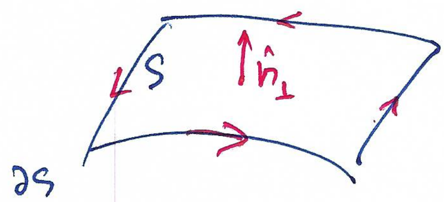
$$\int_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}(\vec{\Phi}(u,v))\cdot\left(\frac{\partial\vec{\Phi}}{\partial u}\times\frac{\partial\vec{\Phi}}{\partial v}\right) du\,dv$$



$$= -\iint_D \vec{F}(\vec{\Phi}(u,v))\cdot\left(\frac{\partial\vec{\Phi}}{\partial v}\times\frac{\partial\vec{\Phi}}{\partial u}\right) du\,dv$$

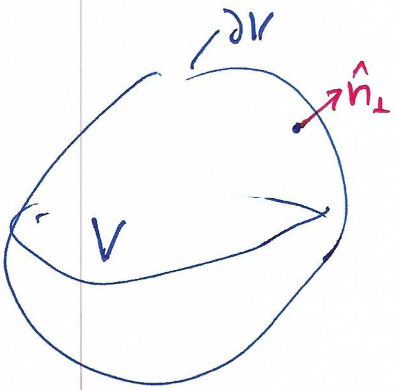
6. Θεώρημα Stokes.

$$\int_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{S} = \oint_{\partial S} \vec{F} \cdot d\vec{l}$$



7. Θεώρημα Gauss.

$$\int_V \vec{\nabla} \cdot \vec{F} \, dv = \oint_{\partial V} \vec{F} \cdot d\vec{S}$$



$$dv = dx\,dy\,dz = \left| \left(\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right) \cdot \frac{\partial \vec{r}}{\partial w} \right| du\,dv\,dw$$

Εάν $x = x(u,v,w)$

$y = y(u,v,w)$

$z = z(u,v,w)$

Δυναμικές γραμμές.

1. $\left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0\right)$.

$$\left.\begin{aligned} \frac{dx}{dt} &= -\frac{y}{x^2+y^2} \\ \frac{dy}{dt} &= \frac{x}{x^2+y^2} \end{aligned}\right\} \Rightarrow x\frac{dx}{dt} + y\frac{dy}{dt} = 0 \Rightarrow \frac{d}{dt}(x^2+y^2) = 0$$

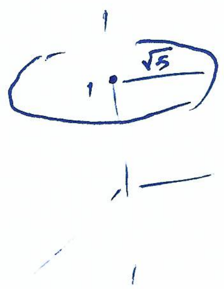
$$\Rightarrow x^2+y^2 = c_1$$

$$\frac{dz}{dt} = 0 \qquad \Rightarrow z = c_2$$

Να ευρεθεί η δυναμική γραμμή η οποία διέρχεται από το σημείο $(1, 2, 1)$.

$z = 1$

$x^2+y^2 = 5$



2. (x, x^2)

$$\frac{dx}{dt} = x \Rightarrow x = c_1 e^t \quad , \quad \begin{matrix} c_1 > 0 \Rightarrow x > 0 \\ c_1 < 0 \Rightarrow x < 0 \end{matrix}$$

$$\frac{dy}{dt} = x^2 \Rightarrow \frac{dy}{dt} = c_1^2 e^{2t} \Rightarrow y(t) = \frac{1}{2} c_1^2 e^{2t} + c_2$$

$$x = c_1 s \qquad s \geq 0$$

$$y = \frac{1}{2} c_1^2 s^2 + c_2$$

$$y = \frac{1}{2} x^2 + c_2$$

Να ευρεθεί η διαφορική γραμμή η οποία διέρχεται από το σημείο $(1, -2)$.

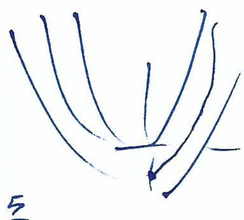
$$y = \frac{1}{2} x^2 + c_2 \Rightarrow -2 = \frac{1}{2} + c_2 \Rightarrow c_2 = -\frac{5}{2}$$

$$y = \frac{1}{2}(x^2 - 5)$$

$x = 1$ π.χ. για $s = 1 \Rightarrow c_1 = 1$

$$x = c_1 s$$

$$y = \frac{1}{2} s^2 - \frac{5}{2}$$

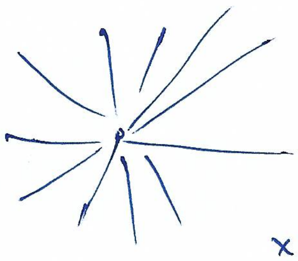


3. (x, y)

$$\left.\begin{array}{l} \frac{dx}{dt} = x \Rightarrow x = c_1 e^t \\ \frac{dy}{dt} = y \Rightarrow y = c_2 e^t \end{array}\right\} \Rightarrow \left\{\begin{array}{l} x = c_1 s \\ y = c_2 s \end{array}\right.$$

$0 < s < \infty$

$y = \frac{c_2}{c_1} x$



$(1, -2)$

$c_1 = 1$

$c_2 = -2$

$x = s$ $\quad s > 0$

$y = -2s$ $\quad y = -2x$ (ημιευθεία)

4. $(x, -y)$

$$\left.\begin{array}{l} x = c_1 e^t \\ y = c_2 e^{-t} \end{array}\right\} \Rightarrow xy = \frac{c_1 c_2}{c}$$

$x = c_1 s$ $\quad s > 0$

$y = c_2 s^{-1}$

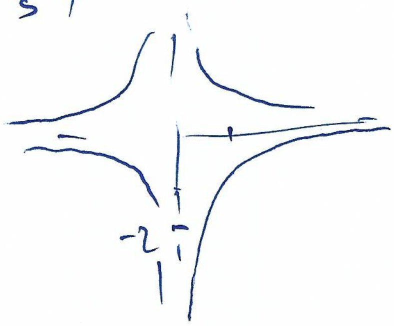
$(1, -2)$

$s = 1$

$$\left.\begin{array}{l} c_1 = 1 \\ c_2 = -2 \end{array}\right\}$$

$x = s$

$y = -\frac{2}{s}$ / $xy = -2$



5. $\vec{\nabla} f = \vec{F} \Rightarrow \int_A^B \vec{F}\cdot d\vec{l} = f(\vec{r}_B) - f(\vec{r}_A)$

ανεξάρτητο από διαδρομής.

$$\int_A^B \vec{F}\cdot d\vec{l} = \int_{t_1}^{t_2} \vec{F}(\vec{\sigma}(t))\cdot \frac{d\vec{\sigma}}{dt}\, dt$$

$$= \int_{t_1}^{t_2} \left[F_x(\vec{\sigma}(t))\frac{dx}{dt} + F_y(\vec{\sigma}(t))\frac{dy}{dt} + F_z(\vec{\sigma}(t))\frac{dz}{dt} \right] dt$$

$$= \int_{t_1}^{t_2} \left[\frac{\partial f}{\partial x}(\vec{\sigma}(t))\frac{dx}{dt} + \frac{\partial f}{\partial y}(\vec{\sigma}(t))\frac{dy}{dt} + \frac{\partial f}{\partial z}(\vec{\sigma}(t))\frac{dz}{dt} \right] dt$$

$$= \int_{t_1}^{t_2} \frac{d}{dt} f(\vec{\sigma}(t))\, dt = f(\vec{\sigma}(t_2)) - f(\vec{\sigma}(t_1)).$$

6. $\vec{\nabla}\times\vec{F} = 0$,

$$f(x,y,z) = \int_{x_0}^{x} F_x(t, y_0, z_0)\,dt + \int_{y_0}^{y} F_y(x, t, z_0)\,dt + \int_{z_0}^{z} F_z(x,y,t)\,dt$$

$$\frac{\partial f}{\partial x} = F_x(x,y_0,z_0) + \int_{y_0}^{y} \underbrace{\frac{\partial F_y}{\partial x}(x,t,z_0)}_{\frac{\partial F_x}{\partial t}} + \int_{z_0}^{z} \underbrace{\frac{\partial F_z}{\partial x}(x,y,t)}_{\frac{\partial F_x}{\partial t}}$$

$$= \cancel{F_x(x,y_0,z_0)} + \cancel{F_x(x,y,z_0)} - \cancel{F_x(x,y_0,z_0)} + F_x(x,y,z) - \cancel{F_x(x,y,z_0)}$$

$$= F_x(x,y,z).$$

Ομοίως για τις άλλες συνιστώσες.

$f(x_0,y_0,z_0) = 0$

7. $$\vec{F} = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}, 0 \right)$$

$$f = \int_{x_0}^{x} \frac{t}{t^2+y_0^2}\,dt + \int_{y_0}^{y} \frac{t}{x^2+t^2}\,dt$$

$$= \frac{1}{2}\int_{x_0^2}^{x^2} \frac{ds}{s+y_0^2} + \frac{1}{2}\int_{y_0^2}^{y^2} \frac{ds}{x^2+s} =$$

$$= \frac{1}{2}\left[\ln \frac{x^2+y_0^2}{x_0^2+y_0^2} + \ln \frac{x^2+y^2}{x^2+y_0^2} \right] = \frac{1}{2} \ln \frac{x^2+y^2}{x_0^2+y_0^2}$$

$$\frac{\partial f}{\partial x} = \frac{x}{x^2+y^2} \Rightarrow f(x,y,z) = \frac{1}{2} \ln (x^2+y^2) + h_1(y,z)$$

$$\frac{\partial f}{\partial y} = \frac{y}{x^2+y^2} \Rightarrow \frac{y}{x^2+y^2} = \frac{y}{x^2+y^2} + \frac{\partial h_1}{\partial y}(y,z) \Rightarrow \frac{\partial h_1}{\partial y} = 0 \Rightarrow$$

$$h_1 = h_1(z)$$

$$\frac{\partial f}{\partial z} = 0 \Rightarrow \frac{dh_1}{dz} = 0 \Rightarrow h_1 = C$$

$$f(x,y,z) = \frac{1}{2} \ln (x^2+y^2) + C$$

$$f(x_0,y_0,z_0) = 0 \Rightarrow C = -\frac{1}{2} \ln (x_0^2+y_0^2).$$

8. $\vec{F}: \quad \vec{\nabla}\cdot\vec{F}=0$

$\vec{F}=\vec{\nabla}\times\vec{A}$ όπου:

$$A_x=\int_0^z F_y(x,y,t)\,dt-\int_0^y F_z(x,t,0)\,dt$$

$$A_y=-\int_0^z F_x(x,y,t)\,dt$$

$$A_z=0.$$

$$\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}=0-\left(-F_x(x,y,z)\right)=F_x(x,y,z)$$

$$\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}=F_y(x,y,z)-0=F_y(x,y,z)$$

$$\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}=-\int_0^z\frac{\partial F_x}{\partial x}(x,y,t)\,dt-\int_0^z\frac{\partial F_y}{\partial y}(x,y,t)\,dt+F_z(x,y,0)$$

$$=\int_0^z\frac{\partial F_z}{\partial t}(x,y,t)\,dt+F_z(x,y,0)$$

$$=F_z(x,y,z)-F_z(x,y,0)+F_z(x,y,0)=F_z(x,y,z)$$