Mon 8 May

Αλλαγή μεταβλητών σε διπλά και τριπλά ολοκληρώματα

(1) Αλλαγή Μεταβλητών σε Διπλά ολοκληρώματα

$f:D\subsetneq\mathbb{R}^2\to\mathbb{R}$ ολοκληρώσιμη

$T:D^*(\subsetneq\mathbb{R}^2)\to D$ C^1, 1-1 και επί

$T(u,v) = (x(u,v), y(u,v))$

Τότε

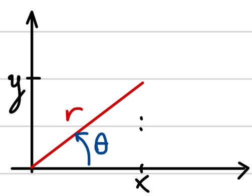
$$\iint_D f(x,y)dxfy = \iint_{D^*} f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dudv$$

Όπου $\partial(x,y)/\partial(u,v) = \begin{vmatrix} dx/du & dx/dv \\ dy/du & dy/dv \end{vmatrix}$

(2) Αλλαγή μεταβλητών από Ευκλείδειες σε Πολικές

$x=r\cos\theta$

$y=r\sin\theta$



$\partial(x,y)/\partial(u,v) = r$

$$\iint_D f(x,y)dxdy = \iint_{D^*} f(r\cos\theta, r\sin\theta)rdrd\theta$$

(3) Το Γκαουσιανό Ολοκλήρωμα

$$I = \int_{-\infty}^{\infty} e^{-x^2} dx = ?$$

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2} dx * \int_{-\infty}^{\infty} e^{-y^2} dy = \lim_{a\to\infty} \int_{-a}^{a} e^{-x^2} dx * \int_{-a}^{a} e^{-y^2} dy = \lim_{a\to\infty} \int_{-a}^{a}\int_{-a}^{a} e^{-x^2-y^2} dxdy = \lim_{a\to\infty} \iint_{[-a,a]x[-a,a]} e^{-x^2-y^2} dxdy$$

$$= \iint_{\mathbb{R}^2} e^{-x^2-y^2} dxdy = \lim_{a\to\infty} \iint_{D_a = \{(x,y):\, x^2+y^2\le a^2\}} e^{-x^2-y^2} dxdy = \lim_{a\to\infty} \int_0^{2\pi}\int_0^{a} e^{-r^2} r\, drd\theta = \lim_{a\to\infty} \int_0^{2\pi} d\theta \int_0^{a} e^{-r^2} * rd\theta$$

($du=2rd\theta$)

$$=\int_0^{a^2} e^{-u}\,du/2 = \lim_{a\to\infty} \pi(1-e^{-a^2}) = \pi$$

(4) Αλλαγή Μεταβλητών στα τριπλά Ολοκληρώματα

$f: W\subseteq\mathbb{R}^3 \to \mathbb{R}$ ολοκληρώσιμη

$T: W^* (\subseteq\mathbb{R}^3) \to W$ C^1, 1-1', επί

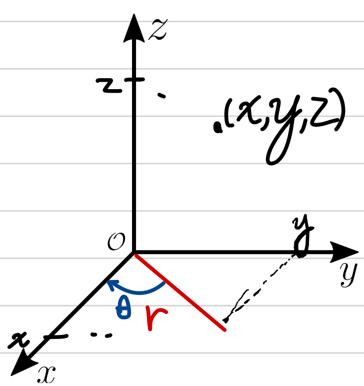
εκτός από σύνολο όγκου 0.

$T(u,v,w) = (x(u,v,w), y(u,v,w), z(u,v,w))$

Τότε

$$\iiint_W f(x,y,z)dxdydz = \iiint_{W^*} f(T(u,v,w)) \left| \overbrace{\begin{vmatrix} \partial x/\partial u & \partial x/\partial u & \partial x/\partial w \\ \partial y/\partial u & \partial y/\partial v & \partial y/\partial w \\ \partial z/\partial u & \partial z/\partial v & \partial z/\partial w \end{vmatrix}}^{\partial(x,y,z)/\partial(u,v,w)} \right| dudvdw$$

(5) Αλλαγή Μεταβλητών από Ευκλείδειες σε Κυλινδρικές



$x=r\cos\theta$, $y=r\cos\theta$, $z=z$

$$\partial(x,y,z)/\partial(r,\theta,z) = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

Άρα $\iiint_W f(x,y,z)dxdydz = \iiint_{W^*} f(r\cos\theta, r\sin\theta, z)*rdrd\theta dz$

(6) Παράδειγμα

$$I = \iiint_W (x^2+y^2)z^2\,dx\,dy\,dz$$

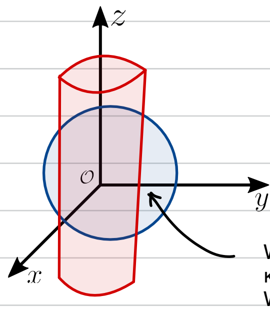
όπου

W η τομή του κυλίνδρου

$\{(x,y,z): x^2+y^2 \le 1\}$

με την σφαίρα

$\{(x,y,z): x^2+y^2+z^2 \le 4\}$



$W=\{(x,y,z),\ x^2+y^2\le 1$ και $x^2+y^2+z^2\le 4\}$

$W^*=\{(r,\theta,z),\ 0\le r\le 1,\ 0\le\theta\le 2\pi,\ -\sqrt{4-r^2}\le z\le\sqrt{4-r^2}\}$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)}$$

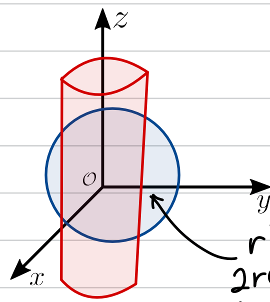
$$\text{Άρα } I = \int_0^1\int_0^{2\pi}\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r^2z^2\,dz\,d\theta\,dr \cdot r$$

$$= \underbrace{\int_0^{2\pi} d\theta}_{2\pi} \int_0^1 r^2 \cdot r \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} z^2\,dz\,dr$$

$$= 2\pi\int_0^1 r^3\,[z^3/3]_{z=-\sqrt{4-r^2}}^{\sqrt{4-r^2}}\,dr$$

$$= 2\pi\int_0^1 r^3(2/3)(4-r^2)^{3/2}\,dr$$

$$= 4\pi/3\int_0^1 r^3(4-r^2)^{3/2}\,dr$$



$r^2=u$

$2r\,dr=du$

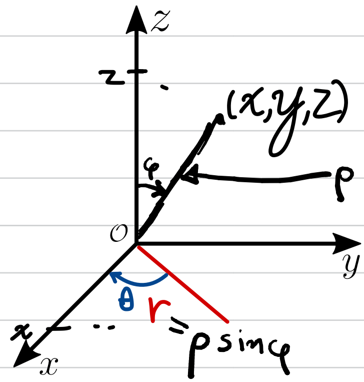
$$= \frac{4\pi}{3}\int_0^1 r^2(4-r^2)^{3/2}\,r\,dr =$$

$$= \frac{4\pi}{3}\int_0^1 u\,(4-u)^{3/2}\,\frac{du}{2} = \frac{2\pi}{3}\int_0^1 u\left(\frac{-(4-u)^{5/2}}{5/2}\right)' du$$

$$= \frac{2\pi}{3}\left[\frac{-u(4-u)^{5/2}}{5/2}\right]_{u=0}^{1} + \frac{2\pi}{3}\int_0^1 (u)'\,\frac{(4-u)^{5/2}}{5/2}\,du$$

$$= \dots = \frac{4\pi}{15}\left(-3^{5/2} - \frac{2}{7}3^{7/2} + \frac{2}{7}4^{7/2}\right).$$

(7) Αλλαγή Μεταβλητών από Ευκλείδειες σε Σφαιρικές



$x = \rho\sin\phi\cos\theta$

$y = \rho\sin\phi\sin\theta$

$z=\rho\cos\phi$

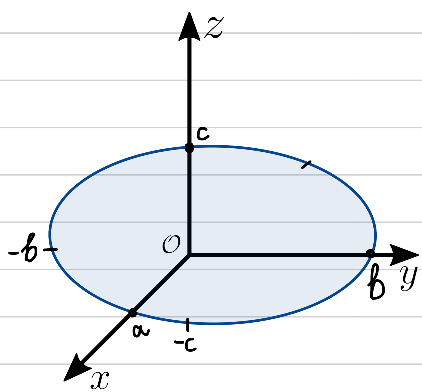
$$\begin{vmatrix} \partial x/\partial\rho & \partial x/\partial\theta & \partial x/\partial\phi \\ \partial y/\partial\rho & \partial y/\partial\theta & \partial y/\partial\phi \\ \partial z/\partial\rho & \partial z/\partial\theta & \partial z/\partial\phi \end{vmatrix} =$$

$$\begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix} =$$

$$\cos\phi \begin{vmatrix} -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \end{vmatrix} -\rho\sin\phi \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix}$$

$=-\rho^2\cos^2\phi * \sin\phi -\rho^2\sin^3\phi =-\rho^2\sin\phi$

(8) Παράδειγμα: Όγκος ελλειψοειδούς



Να βρεθεί ο όγκος του

$W=\{(x,y,z): x^2/a^2 + y^2/b^2 + z^2/c^2 \leq 1\}$

$V=\iiint_W dxdydz$

$x'=x/a,\ y'=y/b,\ z'=z/c$

$W' = \{(x',y',z'): (x')^2 + (y')^2 + (z')^2 \leq 1\}$

$$\frac{\partial(x',y',z')}{\partial(x,y,z)} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$$

Άρα $V = abc \iiint_{W'} dx'dy'dz'$.

Χρησιμοποιώ πολικές

$x'=\rho\sin\phi\cos\theta$

$y'=\rho\sin\phi\sin\theta$

$z'=\rho\cos\phi$

$W^*=\{(\rho,\theta,\Phi): 0\leq\rho\leq1\}$.

Άρα

$V=abc\int\int\int \rho^2\sin\phi d\phi d\theta d\phi$

$=abc\int_0^1 \rho^2 d\rho * \int_0^{2\pi} d\theta * \int_0^{\pi} \sin\phi d\phi$

$=abc\,[\rho^3/3]_{\rho=0}^{1}\ 2\pi\,[-\cos\phi]_{\phi=0}^{\pi}$

$=abc(1/3)2\pi*2= (4\pi abc)/3$.