# Μια ασκηση
# σχετικα με το Φασματικο Θεωρημα

Let $A = A^* \in \mathcal{B}(H)$. Recall the functional calculus[1]

$$\Phi : C(\sigma(A)) \to \mathcal{B}(H) : f \mapsto f(A)$$

which is an isometric *-homomorphism given by $f(A) = \lim_n p_n(A)$ where $(p_n)$ is any sequence of polynomials converging to $f$ uniformly on $\sigma(A)$.

• If $H_1 \subseteq H$ is a (closed) $A$-invariant subspace of $H$, and $A_1 := A|_{H_1} \in \mathcal{B}(H_1)$, show that $\sigma(A_1) \subseteq \sigma(A)$.

Consider the corresponding functional calculus

$$\Phi_1 : C(\sigma(A_1)) \to \mathcal{B}(H_1) : g \mapsto g(A_1)\,.$$

• For $f \in C(\sigma(A))$, write $f_1 := f|_{\sigma(A_1)}$, so that $f \mapsto f_1 : C(\sigma(A)) \to C(\sigma(A_1))$ is the restriction map. Show that $f_1(A_1) = f(A)|_{H_1}$.

Conclude that if the fuction $f \in C(\sigma(A))$ vanishes on $\sigma(A_1)$, then the operator $f(A)$ vanishes on $H_1$.

Given a unit vector $x \in H$, let $H_x := \overline{\text{span}\{A^n x : n \in \mathbb{Z}_+\}}$ be the cyclic subspace of $x$ for $A$. Recall that the Riesz representation theorem shows the existence of a unique (positive regular) Borel measure $\mu_x$ on $\sigma(A)$ such that

$$\int_{\sigma(A)} f d\mu_x = \langle f(A)x, x\rangle_H \quad \text{for all } f \in C(\sigma(A))\,.$$

• Show that the support of $\mu_x$ (i.e. the complement of the largest open set on which $\mu_x$ vanishes) is contained in $\sigma(A_1)$ (here $A_1 := A|_{H_x}$).

This shows that $L^p(\sigma(A), \mu_x) = L^p(\sigma(A_1), \mu_x)$ for any $p$.

[1] funspec modifed 25 Μαΐου 2025