

Συμπλήρωμα για τον $\mathcal{C}^1$

(Με αφορμή μια ερώτηση στο μάθημα.)

Λήμμα 1. Αν $A \in \mathcal{B}(H_2, H_3)$ και $B \in \mathcal{C}_1(H_1, H_2)$, τότε ο AB ανήκει στον $\mathcal{C}_1(H_1, H_3)$.

Απόδειξη. Let $A = V|A|$ be the polar decomposition of A . Then $AB = V|A|B : H_1 \rightarrow H_2 \rightarrow H_3$.

(i) We first prove that $|A|B : H_1 \rightarrow H_2$ is trace class. Indeed, we know that B factorizes as the product $B = CD$ of Hilbert-Schmidt operators ($C = W|B|^{1/2}$ and $D = |B|^{1/2}$ where $B = W|B|$ is the polar decomposition of B). We claim that $|A|C : H_1 \rightarrow H_2$ is Hilbert-Schmidt.¹ If $\{x_n\}$ is an orthonormal basis of H_1 , we have

$$\begin{aligned} \text{Tr}(|A|C)^*(|A|C) &= \sum_n \langle |A|C x_n, |A|C x_n \rangle = \sum_n \| |A|C x_n \|^2 \\ &\leq \sum_n \| |A| \|^2 \|C x_n\|^2 = \|A\|^2 \sum_n \|C x_n\|^2 \\ &= \|A\|^2 \sum_n \langle C x_n, C x_n \rangle = \|A\|^2 \text{Tr}(C^*C) < \infty \end{aligned}$$

which proves the claim.

It follows that $|A|B = (|A|C)D$ is the product of two Hilbert-Schmidt operators, hence is trace class.

(ii) Writing $T := |A|B : H_1 \rightarrow H_2$ which is trace class, it remains to prove that $VT : H_1 \rightarrow H_3$ is trace class. It suffices to prove that $(VT)^* = T^*V^*$ is trace class. Now $V^* : H_3 \rightarrow H_2$ is a partial isometry. Observe that

$$\begin{aligned} |T^*V^*|^2 &= VTT^*V^* = V|T^*|^2V^* \\ \text{hence } |T^*V^*| &= V|T^*|V^* \end{aligned}$$

by uniqueness of square roots (both $|T^*V^*|$ and $V|T^*|V^*$ are positive and their squares are equal). Let $\{e_n : n \in \mathbb{N}_1\}$ be an orthonormal basis of $(\ker V^*)^\perp$ and $\{e_n : n \in \mathbb{N}_2\}$ an orthonormal basis of $\ker V^*$. Then $\{e_n : n \in \mathbb{N}_1 \cup \mathbb{N}_2\}$ is an orthonormal basis of H_3 and we have

$$\begin{aligned} \text{Tr}(|T^*V^*|) &= \sum_{n \in \mathbb{N}} \langle |T^*V^*| e_n, e_n \rangle \\ &= \sum_{n \in \mathbb{N}} \langle V|T^*|V^* e_n, e_n \rangle = \sum_{n \in \mathbb{N}} \langle |T^*|V^* e_n, V^* e_n \rangle \\ &= \sum_{n \in \mathbb{N}_1} \langle |T^*|V^* e_n, V^* e_n \rangle + 0. \end{aligned}$$

Now setting $f_n := V^* e_n$ we have an orthonormal family in H_2 which we can extend to an orthonormal basis of H_2 and so

$$\text{Tr}(|T^*V^*|) = \sum_{n \in \mathbb{N}_1} \langle |T^*| f_n, f_n \rangle \leq \sum_{n \in \mathbb{N}} \langle |T^*| f_n, f_n \rangle = \text{Tr}(|T^*|)$$

which is finite, since T^* is trace class (because T is). Thus $(VT)^*$ is trace class and hence so is VT . □

Πόρισμα 2. Αν $A \in \mathcal{B}(H_0, H_1)$ και $B \in \mathcal{C}_1(H_1, H_2)$, τότε ο BA ανήκει στον $\mathcal{C}_1(H_0, H_2)$.

Απόδειξη. Αρκεί να δείξουμε ότι $(BA)^* \in \mathcal{C}_1(H_2, H_0)$. Αλλά $(BA)^* = A^*B^*$ και $A^* \in \mathcal{B}(H_1, H_0)$ ενώ $B^* \in \mathcal{C}_1(H_2, H_1)$. Επομένως $A^*B^* \in \mathcal{C}_1(H_2, H_0)$ από το Λήμμα. □

¹Recall that a bounded operator $X : H_1 \rightarrow H_2$ is Hilbert Schmidt iff $\|X\|_2^2 := \sum_n \|Xx_n\|^2$ is finite for some (hence any) orthonormal basis $\{x_n\}$ of H_1 .