Group C*-algebras

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Recall[1] that if G is a nonempty set, the linear space

$$c_{oo}(G) = \{f : G \to \mathbb{C} : \operatorname{supp} f \text{ finite }\}$$

has a Hamel basis consisting of the functions $\{\delta_t : t \in G\}$ where

$$\delta_t(s) = \begin{cases} 1, & s = t \\ 0, & s \neq t \end{cases}$$

Thus every $f \in c_{oo}(G)$ is a finite sum

$$f = \sum_{t \in G} f(t)\delta_t.$$

The Hilbert space $\ell^2(G)$ is the completion of $c_{oo}(G)$ with respect to the scalar product

$$\langle f, g \rangle = \sum_{t \in G} f(t)\overline{g(t)}$$

and then $\{\delta_t : t \in G\}$ becomes an orthonormal basis of $\ell^2(G)$.

In case G is a group, the group operations $(s, t) \to st$ and $t \to t^{-1}$ extend linearly to make $c_{oo}(G)$ into a *-algebra: we define $\delta_s * \delta_t = \delta_{st}$ and $(\delta_t)^* = \delta_{t^{-1}}$, so that

$$f * g = \left(\sum_s f(s)\delta_s\right) * \left(\sum_t g(t)\delta_t\right) = \sum_{s,t} f(s)g(t)\delta_{st}$$

and

$$f^* = \left(\sum_s f(s)\delta_s\right)^* = \sum_s \overline{f(s)}\delta_{s^{-1}}$$

[1] gpstar, 16/1/07

in other words (setting $r = st$)

$$f * g = \sum_r \left(\sum_s f(s) g(s^{-1}r) \right) \delta_r = \sum_r \left(\sum_t f(rt^{-1}) g(t) \right) \delta_r$$

and (changing s to $r = s^{-1}$)

$$f^* = \sum_r \overline{f(r^{-1})} \delta_r.$$

Thus

$$(f * g)(r) = \sum_s f(s) g(s^{-1}r) = \sum_t f(rt^{-1}) g(t) \qquad (r \in G)$$

and

$$(f^*)(r) = \overline{f(r^{-1})} \qquad (r \in G).$$

We may also complete $c_{oo}(G)$ with respect to the ℓ^1 norm

$$\|f\|_1 = \sum_t |f(t)|$$

to obtain the Banach space $\ell^1(G)$. Note that because of the relations

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1$$
$$\text{and} \quad \|f^*\|_1 = \|f\|_1$$

(the proof of the first one is easy[2] and the second one is obvious) the multiplication and the involution extend continuously to $\ell^1(G)$, which becomes a Banach algebra with isometric involution, although rarely a C*-algebra.

For example if e, s and s^2 are different elements of G and $f = \delta_{s^{-1}} + \delta_e - \delta_s$ then $\|f\|_1 = 1 + 1 + 1$ and

$$f^* * f = (\delta_s + \delta_e - \delta_{s^{-1}})(\delta_{s^{-1}} + \delta_e - \delta_s) = -\delta_{s^{-2}} + 3\delta_e - \delta_{s^2}$$

hence $\|f^* * f\|_1 = 1 + 3 + 1$.

In order to equip $c_{oo}(G)$ with a suitable C*-norm, we study its *-representations on Hilbert space.

[2] $\sum_r \left|\sum_s f(s) g(s^{-1}r)\right| \leq \sum_r \sum_s |f(s)|.|g(s^{-1}r)| = \sum_s \sum_r |f(s)|.|g(s^{-1}r)| = \sum_s |f(s)|. \sum_r |g(s^{-1}r)| = \sum_s |f(s)| \sum_t |g(t)|$

The left regular representation Let $H = \ell^2(G)$. Each $t \in G$ defines a unitary operator λ_t on H by the formula

$$\lambda_t(\sum_s \xi(s)\delta_s) = \sum_s \xi(s)\delta_{ts} \qquad (\xi = \sum_s \xi(s)\delta_s \in \ell^2(G)).$$

For example, if $G = \mathbb{Z}$ then $\lambda_n = U^n$ where U is the bilateral shift, $U(\delta_n) = \delta_{n+1}$, on $\ell^2(\mathbb{Z})$.

Making the change of variable $r = ts$, we find

$$(\lambda_t\xi)(r) = \xi(t^{-1}r) \quad (r \in G).$$

Note that λ_t is a well-defined linear isometry, because

$$\|\lambda_t(\xi)\|_2^2 = \sum_r |\xi(t^{-1}r)|^2 = \sum_s |\xi(s)|^2 = \|\xi\|_2^2 .$$

Also $\lambda_e = I$ (the identity operator) and

$$\lambda_t\lambda_s = \lambda_{ts}$$

because

$$\lambda_t(\lambda_s\delta_r) = \lambda_t(\delta_{sr}) = \delta_{tsr} = \delta_{(ts)r} = \lambda_{ts}(\delta_r)$$

for each $r \in G$. Since the operators involved are bounded and linear and the $\{\delta_t\}$ span $\ell^2(G)$ the claim follows.

In particular it follows that each λ_t is invertible with inverse $(\lambda_t)^{-1} = \lambda_{t^{-1}}$ and so it is an onto isometry, i.e. a unitary, with $(\lambda_t)^* = \lambda_{t^{-1}}$. Thus we have a group homomorphism

$$G \to \mathcal{U}(\mathcal{B}(H))$$

into the group of unitary operators on $H = \ell^2(G)$. This is called *a unitary representation of G on H.*

The unitary representation λ immediately extends to a *-representation, also denoted by λ, of the *-algebra $c_{oo}(G)$ on $\ell^2(G)$. More precisely, given $f = \sum_t f(t)\delta_s \in c_{oo}(G)$ we define

$$\lambda(f) = \sum_t f(t)\lambda_t$$

$$\text{i.e.} \quad (\lambda(f)\xi)(r) = \sum_t f(t)\xi(t^{-1}r) \quad (\xi \in \ell^2(G)).$$

This is a bounded operator because

$$\|\lambda(f)\| = \left\| \sum_t f(t)\lambda_t \right\| \leq \sum_t |f(t)| \, \|\lambda_t\| = \sum_t |f(t)| = \|f\|_1$$

since each λ_t is unitary. In fact this inequality shows that λ extends to a (contractive) map $\ell^1(G) \to B(\ell^2(G))$.

The fact that λ is a *-representation immediately follows from the properties of its restriction to G:

$$\lambda\left(\left(\sum_t f(t)\delta_t\right) * \left(\sum_s g(s)\delta_s\right)\right) = \lambda\left(\sum_{t,s} f(t)g(s)\delta_{ts}\right) = \sum_{t,s} f(t)g(s)\lambda_{ts}$$

$$= \sum_{t,s} f(t)g(s)\lambda_t\lambda_s = \left(\sum_t f(t)\lambda_t\right) \cdot \left(\sum_s g(s)\lambda_s\right) = \lambda(f)\lambda(g)$$

$$\text{and} \quad \lambda\left(\left(\sum_t f(t)\delta_t\right)^*\right) = \lambda\left(\sum_t \overline{f(t)}\delta_{t^{-1}}\right) = \sum_t \overline{f(t)}\lambda_{t^{-1}}$$

$$= \sum_t \overline{f(t)}\lambda_t^* = \left(\sum_t f(t)\lambda_t\right)^* .$$

The above calculations can be carried out for any unitary representation of G. The details are left as an exercise.

Proposition 1 *There is a bijective correspondence between unitary representations of G and *-representations of $c_{oo}(G)$:*
If $\pi : G \to \mathcal{U}(\mathcal{B}(H))$ is any unitary representation of the group G, the formula

$$\tilde{\pi}\left(\sum_t f(t)\delta_t\right) = \sum_t f(t)\pi(t)$$

*defines a unital *-representation of $c_{oo}(G)$ (and of $\ell^1(G)$) on the same Hilbert space H which is $\|\cdot\|_1$-contractive.*

*Conversely, every unital $\|\cdot\|_1$-contractive *-representation ρ of $c_{oo}(G)$ (or of $\ell^1(G)$) defines a unitary representation π by 'restriction': $\pi(t) = \rho(\delta_t)$ satisfying $\tilde{\pi} = \rho$.*

We usually use the symbol π for $\tilde{\pi}$.

Definition 1 *Let Σ be the set of all $\|\cdot\|_1$-contractive *-representations (π, H_π) of $c_{oo}(G)$ (equivalently, of $\ell^1(G)$).*

The C-norm on $c_{oo}(G)$ (or $\ell^1(G)$) is defined by the formula*

$$\|f\|_* = \sup\{\|\pi(f)\| : \pi \in \Sigma\}.$$

The group C-algebra $C^*(G)$ is defined to be the completion of $c_{oo}(G)$ (or equivalently of $\ell^1(G)$) with respect to this norm.*

Remarks 2 First of all, the set Σ is non-empty: it contains the left regular representation.

Clearly $\|\cdot\|_*$ is a seminorm on $c_{oo}(G)$, being the supremum of seminorms, all of which are (by definition) bounded by $\|\cdot\|_1$, hence so is $\|\cdot\|_*$. Also, $\|\cdot\|_*$ satisfies the C*-identity, because all the seminorms $f \to \|\pi(f)\|$ satisfy it.[3]

But why is $\|\cdot\|_*$ a norm? In other words, why is it true that $\|f\|_* > 0$ whenever $f \in c_{oo}(G)$ is nonzero?

The reason is that the left regular representation is *faithful* on $c_{oo}(G)$ and $\ell^1(G)$; thus if $f \in \ell^1(G)$ is nonzero then $\lambda(f) \neq 0$ and so $\|f\|_* \geq \|\lambda(f)\| > 0$.

Indeed if $f = \sum_t f(t)\delta_t \in \ell^1(G)$ is nonzero then there exists $s \in G$ with $f(s) \neq 0$ and then[4]

$$\langle \lambda(f)\delta_e, \delta_s \rangle_{\ell^2(G)} = \left\langle \sum_t f(t)\lambda_t(\delta_e), \delta_s \right\rangle = \sum_t f(t) \langle \delta_t, \delta_s \rangle = f(s)$$

because the δ_s are orthonormal in $\ell^2(G)$. Thus $\lambda(f) \neq 0$.

The usefulness of $C^*(G)$ comes from the following property, whose proof is an immediate consequence of the previous proposition and the fact that $c_{oo}(G)$ is a dense *-subalgebra of $C^*(G)$.

Proposition 3 *There is a bijective correspondence between unitary representations of G and unital *-representations of the group C*-algebra $C^*(G)$.*

In particular, the left regular representation λ extends to a *-representation of $C^*(G)$ on $\ell^2(G)$. However, the fact that λ is faithful on $c_{oo}(G)$ does NOT mean that its extension remains faithful on $C^*(G)$!

[3] $\|\pi(f^* * f)\| = \|\pi(f)^*\pi(f)\| = \|\pi(f)\|^2$.

[4] since $\sum_t |f(t)| < \infty$, the sum $\sum_t f(t)\lambda_t$ converges (absolutely) in the norm of $B(\ell^2(G))$.

The image $\lambda(C^*(G))$ in $\mathcal{B}(\ell^2(G))$ is a C*-algebra; it equals the closure of $\lambda(c_{oo}(G))$ in the norm of $\mathcal{B}(\ell^2(G))$ and is called *the reduced C*-algebra* $C_r^*(G)$ of G.

In many cases, for example when G is abelian, λ is faithful on $C^*(G)$, so that $C_r^*(G) \simeq C^*(G)$ (isometrically and *-isomorphically). In general, however, $C_r^*(G)$ is a quotient of $C^*(G)$ and does not 'contain' all unitary representations of G.

Example 4 Let $G = \mathbb{F}_2$ be the free group in two generators a and b; that is, any element of G is a (finite) 'word' of the form $a^n b^m a^k b^j$ where $n, m, k, j \in \mathbb{Z}$ and there are no relations between a and b. It is known that the reduced C*-algebra $C_r^*(\mathbb{F}_2)$ is simple, i.e. it has no nontrivial closed two-sided ideals. Thus all of its representations are isomorphisms; Since $C_r^*(\mathbb{F}_2)$ is obviously infinite-dimensional, it cannot have finite dimensional representations. On the other hand, the group $\mathbb{F}_2$ does have unitary representations on finite-dimensional spaces: Just take any two unitary $n \times n$ matrices U and V and define $\pi(a) = U$ and $\pi(b) = V$. Since there are no relations between a and b, this extends to a unitary representation of $\mathbb{F}_2$ on $\mathbb{C}^n$; for example $\pi(a^n b^m a^k b^j) = U^n V^m U^k V^j$. Hence $C^*(\mathbb{F}_2)$ does have nontrivial finite-dimensional representations: therefore it cannot be isomorphic to $C^*(\mathbb{F}_2)$.

Thus, $C_r^*(\mathbb{F}_2)$ is a proper quotient of $C^*(\mathbb{F}_2)$.

Example 5 Let $G = \mathbb{Z}$. If we represent each $n \in \mathbb{Z}$ by the function $\zeta_n(z) = z^n$, $z \in \mathbb{T}$, the convolution product $\zeta_n * \zeta_m = \zeta_{n+m}$ becomes pointwise product, involution becomes complex conjugation, and the elements of $c_{oo}(G)$ become trigonometric polynomials. Hence if $\mathcal{P} \subset C(\mathbb{T})$ is the set of trigonometric polynomials we have a *-isomorphism

$$c_{oo}(\mathbb{Z}) \to \mathcal{P} : \sum_n f(n)\delta_n \to p_f, \text{ where } p_f(z) \equiv \sum_n f(n) z^n.$$

Note that, as observed earlier, the left regular representation λ is generated by $\lambda(1) = U$, the bilateral shift on $\ell^2(\mathbb{Z})$, which is unitarily equivalent to multiplication by ζ on $L^2(\mathbb{T})$. Therefore, for each $f \in c_{oo}(\mathbb{Z})$, $\lambda(f) = p_f(U)$ is unitarily equivalent to the multiplication operator M_{p_f} acting on $L^2(\mathbb{T})$ and so

$$\|\lambda(f)\| = \left\|M_{p_f}\right\| = \|p_f\|_\infty = \sup\{|p_f(z)| : z \in \mathbb{T}\}.$$

It follows that the closure $C_r^*(\mathbb{Z})$ of $\lambda(c_{oo}(\mathbb{Z})$ is isometrically isomorphic to the sup-norm closure of the trigonometric polynomials, namely $C(\mathbb{T})$.

*We will show that $C^*(\mathbb{Z})$ is isometrically *-isomorphic with $C(\mathbb{T})$.*

Since $c_{oo}(\mathbb{Z})$ is $\|\cdot\|_*$-dense in $C^*(\mathbb{Z})$ and $\mathcal{P}$ is $\|\cdot\|_\infty$-dense in $C(\mathbb{T})$ (Stone-Weierstrass) it suffices to show that the norm $\|f\|_*$ on $c_{oo}(\mathbb{Z})$ coincides with the sup norm $\|p_f\|_\infty$ of $C(\mathbb{T})$. For this, since we just proved that $\|p_f\|_\infty = \|\lambda(f)\| \le \|f\|_*$, it is enough to prove the reverse inequality, namely that if π is any unitary representation of $\mathbb{Z}$ on some Hilbert space, then

$$\|\pi(f)\| \le \|p_f\|_\infty$$

for any $f = \sum_n f(n)\delta_n \in c_{oo}(\mathbb{Z})$.

Indeed let $V = \pi(1)$; this is a unitary operator and

$$\pi(f) = \sum_n f(n)\pi(n) = \sum_n f(n)V^n = p_f(V).$$

Now $p_f(V)$ is a normal operator and hence its norm equals its spectral radius. By the spectral mapping theorem,

$$\sigma(p_f(V)) = \{p_f(z) : z \in \sigma(V)\} \subseteq \{p_f(z) : z \in \mathbb{T}\}$$

because V is unitary and so $\sigma(V) \subseteq \mathbb{T}$. Thus

$$\|\pi(f)\| = \|p_f(V)\| \le \sup\{|p_f(z)| : z \in \mathbb{T}\} = \|p_f\|_\infty .$$

Abelian groups The situation of this last example generalizes to arbitrary abelian groups. Briefly, if G is an abelian group, then of course $c_{oo}(G)$ is abelian, and hence so is $C^*(G)$. Thus $C^*(G) \simeq C(K)$, where K is the compact space of multiplicative linear functionals on $C^*(G)$ with the weak* topology. We identify the space K:

Define the *set of characters of G*

$$\widehat{G} = \Gamma = \{\gamma : G \to \mathbb{T} : \text{ homomorphism}\}.$$

With the topology of pointwise convergence, it is not hard to see that this is a compact space (a closed subspace of the Cartesian product $\mathbb{T}^G$) and it is a group with pointwise operations. In fact it can be shown to be a topological group (the group operations are continuous). It is called *the dual group* of G.

Any $\gamma \in \Gamma$ is a *-representation of G on the Hilbert space $\mathbb{C}$ (since $\gamma(t)\gamma(s) = \gamma(ts)$ and $\gamma(t^{-1}) = (\gamma(t))^{-1} = \overline{\gamma(t)}$) and thus extends (Proposition

3) to a *-representation $\tilde{\gamma}$ of $C^*(G)$ on $\mathbb{C}$, i.e. a multiplicative linear functional on $C^*(G)$. Conversely, any multiplicative linear functional on $C^*(G)$ restricts to a character on G. Thus there is a bijection between the set Γ of characters of G and the set K of multiplicative linear functionals on $C^*(G)$. We claim that this bijection is a homeomorphism; since both spaces are compact and Hausdorff, it suffices to prove that it is continuous.

Let $\gamma_i \to \gamma$ in Γ; this means $\gamma_i(t) \to \gamma(t)$ for each $t \in G$. To prove that $\tilde{\gamma}_i \to \tilde{\gamma}$ in K, we need to prove that $\tilde{\gamma}_i(a) \to \tilde{\gamma}(a)$ for all $a \in C^*(G)$. Fix such an a. Since $c_{oo}(G)$ is dense in $C^*(G)$, given $\epsilon > 0$ there exists $f \in c_{oo}(G)$ with $\|a - f\|_* < \epsilon$. Now each $\tilde{\gamma}_i$ and $\tilde{\gamma}$ has norm 1, and so

$$|\tilde{\gamma}_i(a - f) - \tilde{\gamma}(a - f)| \le 2\,\|a - f\|_* < 2\epsilon.$$

On the other hand, if f is a finite sum $\sum_t f(t)\delta_t$, we have

$$|\tilde{\gamma}_i(f) - \tilde{\gamma}(f)| = \left|\sum_t f(t)(\gamma_i(t) - \gamma(t))\right| \le \sum_t |f(t)|.|\gamma_i(t) - \gamma(t)|.$$

Now since $\gamma_i(t) \to \gamma(t)$ for each $t \in G$, there is i_o such that $|\gamma_i(t) - \gamma(t)| < \epsilon$ for each $i \ge i_o$ and each t in the finite support of f. Combining with the previous inequality we conclude that

$$|\tilde{\gamma}_i(a) - \tilde{\gamma}(a)| < (2 + \|f\|_1)\epsilon$$

whenever $i \ge i_o$; thus $\tilde{\gamma}_i \to \tilde{\gamma}$ in the weak*-topology.

This concludes the proof that Γ and K are homeomorphic; we henceforth identify K with Γ and now we can conclude by Gelfand theory that $C^*(G) \simeq C(\Gamma)$. In fact the *-isomorphism is given by $a \to \hat{a}$, where

$$\hat{a}(\gamma) = \tilde{\gamma}(a), \quad a \in C^*(G)$$

and in particular

$$\hat{f}(\gamma) = \sum_s f(s)\gamma(s), \quad f \in c_{oo}(G).$$

Haar measure on Γ We now wish to equip Γ with a suitable probability measure μ and form $L^2(\Gamma, \mu)$. We first define a state:

$$\omega : c_{oo}(G) \to \mathbb{C} : f \to f(e)$$

Clearly this is linear[5] and $\omega(\mathbf{1}) = \omega(\delta_e) = \delta_e(e) = 1$. We check positivity:

$$\begin{aligned}\omega(f^* * f) =&(f^* * f)(e) = \sum_s f^*(s)f(s^{-1}e) = \sum_s \overline{f(s^{-1})}f(s^{-1}) = \\ &\sum_s |f(s^{-1})|^2 = \sum_s |f(s)|^2 \geq 0 \end{aligned} \tag{1}$$

for all $f = \sum_s f(s)\delta_s \in c_{oo}(G)$.

Note also that ω is continuous in the norm of $C^*(G) \simeq C(\Gamma)$: Indeed

$$|\omega(f)| = |f(e)| = |\langle \lambda(f)\delta_e, \delta_e \rangle| \leq \|\lambda(f)\| \, \|\delta_e\|_2^2 = \|\lambda(f)\| \leq \|f\|_* = \left\| \hat{f} \right\|_\infty$$

when $f \in c_{oo}(G)$. Therefore ω extends to a continuous linear form on the completion $C(\Gamma)$ and the extension is a state. By the Riesz representation theorem, there exists a unique Borel probability measure μ on the compact space Γ such that

$$\omega(a) = \int_\Gamma \hat{a}(\gamma) d\mu(\gamma) \qquad \text{for all } \ a \in C^*(G). \tag{2}$$

Lemma 6 *The measure* μ
(i) is left invariant*, i.e.* $\mu(\gamma E) = \mu(E)$ *for every Borel subset of* Γ *and any* $g \in \Gamma$ *(where* $\gamma E = \{\gamma\gamma' : \gamma' \in E\}$*), and*
(ii) has full support*, i.e.* $\mu(U) > 0$ *for every nonempty open set* $U \subseteq \Gamma$.

Proof (i) Fix $\gamma \in \Gamma$. We claim that

$$\int_\Gamma \hat{f}(\gamma^{-1}\gamma') d\mu(\gamma') = \int_\Gamma \hat{f}(\gamma') d\mu(\gamma') \quad \text{for all } \ f \in c_{oo}(G).$$

Indeed, seting $g(s) = \overline{\gamma(s)}f(s)$ we easily find that $\hat{g}(\gamma') = \hat{f}(\gamma^{-1}\gamma')$ and so $\int_\Gamma \hat{f}(\gamma^{-1}\gamma')d\mu(\gamma') = \int_\Gamma \hat{g}(\gamma')d\mu(\gamma') = g(e) = \overline{\gamma(e)}f(e) = f(e)$.

Since $c_{oo}(G)$ is dense in $C(\Gamma)$ it follows that

$$\int_\Gamma a(\gamma^{-1}\gamma') d\mu(\gamma') = \int_\Gamma a(\gamma') d\mu(\gamma') \quad \text{for all } \ a \in C(\Gamma).$$

[5] This is not multiplicative: the product on $c_{oo}(G)$ is not pointwise multiplication, it is convolution

By uniqueness of μ this implies

$$\int_\Gamma \chi_E(\gamma^{-1}\gamma')d\mu(\gamma') = \int_\Gamma \chi_E(\gamma')d\mu(\gamma') \quad \text{for every Borel set } \; E \subseteq \Gamma.$$

But since $\chi_E(\gamma^{-1}\gamma') = \chi_{\gamma E}(\gamma')$, claim (i) follows.

(ii) Let $U \subseteq \Gamma$ be a nonempty open set. Observe that $\{\gamma U : \gamma \in \Gamma\}$ is an open cover of Γ (the map $\gamma' \to \gamma\gamma'$ is a homeomorphism) and so there is a finite subcover $\{\gamma_i U : i = 1, \dots, n\}$. Now $\mu(\gamma_i U) = \mu(U)$ by left invariance, hence

$$\mu(\Gamma) = \mu\left(\bigcup_{i=1}^n \gamma_i U\right) \leq \sum_{i=1}^n \mu(\gamma_i U) = n\mu(U).$$

Since $\mu(\Gamma) > 0$ it follows that $\mu(U) > 0$. $\square$

The Fourier transform It follows from (2) that for $f \in c_{oo}(G)$ (remembering that the Gelfand transform is a *-morphism, so that $\widehat{g * f} = \hat{g}\hat{f}$) we have

$$\omega(f^* * f) = \int_\Gamma \widehat{f^* * f} d\mu = \int \widehat{f^*}\hat{f} d\mu = \int \overline{\hat{f}}\hat{f} d\mu = \int |\hat{f}(\gamma)|^2 d\mu(\gamma). \qquad (3)$$

Combine this with (1) to conclude that

$$\int_\Gamma |\hat{f}(\gamma)|^2 d\mu(\gamma) = \sum_s |f(s)|^2 \qquad \text{for all } \; f \in c_{oo}(G).$$

This equality shows (if we write $L^2(\Gamma)$ for $L^2(\Gamma, \mu)$) that the linear map

$$(c_{oo}(G), \|\cdot\|_{\ell^2(G)}) \to (C(\Gamma), \|\cdot\|_{L^2(\Gamma)}) : f \to \hat{f}$$

is isometric and has dense range, and thus extends to a *unitary bijection*

$$F : \ell^2(G) \to L^2(\Gamma)$$

which is called the *Fourier transform.*

Finally, if $f \in c_{oo}(G)$ and $\xi \in c_{oo}(G) \subset \ell^2(G)$ we have

$$F(\lambda(f)\xi) = F(f * \xi) = \widehat{f * \xi} = \hat{f}\hat{\xi} = M_{\hat{f}}\hat{\xi} = M_{\hat{f}}F\xi$$

where M_g denotes multiplication by g on $L^2(\Gamma)$. The operators $F\lambda(f)$ and $M_{\hat{f}}F$ are both bounded operators on $\ell^2(G)$ and coincide on the dense subspace $c_{oo}(G)$; therefore they are equal:

$$F\lambda(f) = M_{\hat{f}}F \quad \text{or} \quad F\lambda(f)F^* = M_{\hat{f}}$$

(F is unitary). It follows that $\|\lambda(f)\| = \|M_{\hat{f}}\|$. But, since μ has full support,[6] $\|M_{\hat{f}}\| = \|\hat{f}\|_\infty$. Thus finally

$$\|\lambda(f)\| = \|\hat{f}\|_\infty \quad \text{for all} \quad f \in c_{oo}(G)$$

so that the left regular representation is isometric on $c_{oo}(G)$. Therefore its extension to $C^*(G) \simeq C(\Gamma)$ is also isometric, hence injective, and implements a *-isomorphism between $C^*(G)$ and $C^*_r(G)$. We summarize:

Theorem 7 *If G is an abelian group and $\Gamma = \widehat{G}$, then $C^*(G) \simeq C(\Gamma)$ and the Fourier transform $F : \ell^2(G) \to L^2(\Gamma)$ implements a unitary equivalence between the left regular representation λ of $C^*(G)$ on $\ell^2(G)$ and the multiplication representation $g \to M_g$ of $C(\Gamma)$ on $L^2(\Gamma)$. Hence λ is isometric and so $C^*(G) \simeq C^*_r(G)$.*

[6]If U is an open set on which $|\hat{f}| \geq \|\hat{f}\|_\infty - \epsilon$, then $\xi = \chi_U$ is a nonzero element of $L^2(\Gamma)$ and $\|M_{\hat{f}}\|\|\xi\|_2 \geq \|M_{\hat{f}}\xi\|_2 \geq (\|\hat{f}\|_\infty - \epsilon)\|\xi\|_2$.