By a **space** we will always mean a Hausdorff topological space (X, τ).

A **compactification** of X is a pair (Y, f) where Y is compact and $f : X \to Y$ is a homeomorphism with dense image. Two compactifications (Y, f) and (Z, g) of X are **equivalent** if there is a homeomorphism $h : Y \to Z$ that leaves X 'pointwise fixed', i.e. for all $x \in X$, $h(f(x)) = g(x)$.

If (X, τ) is a locally compact (i.e. every point has a compact neighbourhood), non-compact space, the **one-point compactification** (X_∞, τ_∞) of (X, τ) is $X_\infty = X \cup \{\infty\}$ where ∞ is a point not in X and

$$\tau_\infty = \{U \subseteq X : U \in \tau\} \cup \{X_\infty \setminus F : F \subseteq X, \text{ compact}\} \cup \{X_\infty\}.$$

Alternatively, a base of neighbourhoods at each point x of X is the set $\{U \in \tau : x \in U\}$, while a base of neighbourhoods of ∞ consists of the complements of compact subsets of X. It is easy to verify that this is a compactification of (X, τ).

Let (X, τ) be a locally compact, non-compact space. Consider the abelian C*-algebra

$$\mathcal{A} = C_0(X) = \{f : X \to \mathbb{C} \text{ continuous s.t. } \forall \epsilon > 0 \exists K \text{ compact s.t. } \|f\|_{K^c} < \epsilon\}$$

(here $\|f\|_{K^c} = \sup\{|f(x)| x \in K^c\}$).

If $\mathcal{B} = \tilde{\mathcal{A}}$ is the unitization of $\mathcal{A}$, we will prove that the character space (Y, w^*) of $\mathcal{B}$ is (a compactification equivalent to) the one-point compactification of (X, τ).

Recall that $\mathcal{B} = \mathcal{A} \oplus \mathbb{C}$ is an abelian C*-algebra with the operations

$$\begin{aligned}(f, \lambda) + (g, \mu) &= (f + g, \lambda + \mu) \\ (f, \lambda) \cdot (g, \mu) &= (fg + \mu f + \lambda g, \lambda\mu) \\ (f, \lambda)^* &= (f^*, \bar{\lambda})\end{aligned}$$

and the norm

$$\|(f, \lambda)\| = \sup\{\|fg + \lambda g\| : g \in \mathcal{A}, \|g\| \leq 1\}.$$

Now each $f \in \mathcal{A}$ extends to a continuous[1] function $\tilde{f}$ on X_∞ by setting $\tilde{f}(\infty) = 0$. Define a map

$$\pi : \mathcal{B} \to C(X_\infty) : (f, \lambda) \to f + \lambda$$

and verify it is an isometric *-isomorphism (to prove that it is onto, take $g \in C(X_\infty)$ and define $(f, \lambda) \in \mathcal{B}$ by $f(x) = g(x) - g(\infty)$ for $x \in X$ and $\lambda = g(\infty)$; continuity of g at ∞ shows that $f \in C_0(x)$).

[1] the condition $\|f\|_{K^c} < \epsilon$ ensures continuity of $\tilde{f}$ at ∞

Given $x \in X$ we define ϕ_x to be the character (verify) on $\mathcal{B}$ given by $\phi_x(f, \lambda) = f(x) + \lambda$ and we set $\phi_\infty(f, \lambda) = \lambda$.

We claim that

$$Y = \{\phi_x : x \in X_\infty\}.$$

Proof Suppose, by way of contradiction, that there exists

$$\phi \in Y \setminus \{\phi_x : x \in X_\infty\}$$

and let $\mathcal{J} = \ker\phi$. For all $x \in X_\infty$, since $\phi_x \neq \phi$, there exists $a_x = (f_x, \lambda_x) \in \mathcal{J}$ such that $\phi_x(a_x) \neq 0$.[2] Thus the continuous function $g_x \equiv \tilde{f}_x + \lambda_x \in C(X_\infty)$ does not vanish at x. Hence there is an open neighbourhood U_x of x so that $g_x|_{U_x}$ never vanishes. The open cover $\{U_x : x \in X_\infty\}$ has a finite subcover, $\{U_1, \ldots, U_n\}$. Let $\{g_1, \ldots, g_n\}$ be the corresponding functions. Since $\pi(\mathcal{J})$ is an ideal, $|g_k|^2 = \bar{g}_k g_k \in \pi(\mathcal{J})$ so $g \equiv \sum_{k=1}^n |g_k|^2 \in \pi(\mathcal{J})$. But g never vanishes; for if $x \in X_\infty$, there is some k with $x \in U_k$, and then $g(x) \geq |g_k(x)|^2 > 0$. But then $\frac{1}{g}$ is a continuous function on X_∞ and so $\mathbf{1} = \frac{1}{g} \cdot g \in \pi(\mathcal{J})$. This gives $\phi(\mathbf{1}) = 0$, a contradiction. $\square$

Therefore we have a bijection

$$h : X_\infty \to Y \quad \text{gievn by} \quad h(x) = \begin{cases} \phi_x & \text{if } x \in X \\ \phi_\infty & \text{if } x = \infty \end{cases}$$

We prove that h is continuous. Suppose $x_i \to x$ in X_∞. If $x \neq \infty$ then $x_i \in X$ eventually and so $h(x_i) = \phi_{x_i}$ eventually and $h(x) = \phi_x$. Now for all $(f, \lambda) \in \mathcal{B}$ we have $f(x_i) \to f(x)$ by continuity of f and so

$$\phi_{x_i}((f, \lambda)) = f(x_i) + \lambda \to f(x) + \lambda = \phi_x((f, \lambda))$$

showing that $\phi_{x_i} \overset{w*}{\to} \phi_x$.

If $x_i \to \infty$ then for every $f \in C_0(X)$ we have $\tilde{f}(x_i) \to 0$. Indeed given $\epsilon > 0$ there is a compact set $K \subseteq X$ with $\|f\|_{K^c} < \epsilon$; but $U \equiv X_\infty \setminus K$ is a neighbourhood of ∞, so $x_i \in U$ eventually and thus $|\tilde{f}(x_i)| < \epsilon$ eventually. Therefore

$$\phi_{x_i}((f, \lambda)) = f(x_i) + \lambda \to 0 + \lambda = \phi_\infty((f, \lambda))$$

so $\phi_{x_i} \overset{w*}{\to} \phi_\infty$ which completes the proof that h is continuous on X_∞.

Thus h is a continuous bijection between compact spaces, so it must be a homeomorphism.

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[2] otherwise $\ker\phi \subseteq \ker\phi_x$ which implies that $\phi_x = \phi$ (write any $a \in \mathcal{B}$ as $a = (a - \phi(a)\mathbf{1}) + \phi(a)\mathbf{1}$, observe that $a - \phi(a)\mathbf{1} \in \ker\phi \subseteq \ker\phi_x$ so $\phi_x(a) = \phi_x(a - \phi(a)\mathbf{1}) + \phi(a)\phi_x(\mathbf{1}) = \phi(a)$).

Let $g : X \to Y$ be the restriction of h to X. Since X is dense in X_∞ and h is a homeomorphism it follows that $g(X)$ is dense in Y, so (Y, g) is a compactification of X. Since $h(id(x)) = h(x) = g(x)$ for all $x \in X$, the compactifications (Y, g) and (X_∞, id) are equivalent.

This concludes the argument.

Two irrelevant remarks [3]

Remark 1 *If (X, τ) is a locally compact space and $F \subseteq U \subsetneqq X$ where F is compact and U is open, there is a compact neighbourhood K of F contained in U (i.e. $F \subseteq K^o \subseteq K \subseteq U$) and there is a continuous function $f : X \to [0, 1]$ such that $f(x) = 1$ for $x \in F$ and $f(y) = 0$ for $y \notin U$.*

Proof F and $F' = X_\infty \setminus U$ are disjoint closed sets in the compact, hence normal space X_∞. Hence there are V, V' disjoint open subsets of X_∞ such that $F \subseteq V$ and $F' \subseteq V'$. Set $K = X_\infty \setminus V'$.

To obtain f, apply Urysohn to F and F' to find $g : X_\infty \to [0, 1]$ such that $f(x) = 1$ for $x \in F$ and $f(y) = 0$ for $y \in F'$ and let $f = g|_X$. $\square$

Remark 2 *The topology of a compact space (K, τ) is determined by $C(K)$. More precisely, if $\mathcal{F} \subseteq C(K)$ is a separating family, then the weakest topology τ_F on K making all members of $\mathcal{F}$ continuous coincides with τ.*

Proof Since the elements of $\mathcal{F}$ are τ-continuous, clearly $\tau_F \leq \tau$. Thus the identity $id : (X, \tau) \to (X, \tau_F)$ is continuous. Since every τ-closed set F is τ-compact, its image under id will be τ_F-compact. If τ_F is Hausdorff, then F will be τ_F-closed. Hence the two topologies will have the same closed sets, so they will coincide.

The fact that τ_F is Hausdorff follows because $\mathcal{F}$ separates K. Indeed, if $x \neq y$ there is $f \in \mathcal{F}$ such that that $f(x) \neq f(y)$. There are disjoint open disks V_x, V_y in $\mathbb{C}$ such that $f(x) \in V_x$ and $f(y) \in V_y$. Then $f^{-1}(V_x)$ and $f^{-1}(V_y)$ are disjoint τ_F-open (because f is τ_F-continuous) neighbourhoods of x and y. $\square$

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[3] $\alpha\phi o\upsilon\ \tau\alpha\ \epsilon\gamma\rho\alpha\psi\alpha$...