Discrete crossed products

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Classical dynamical systems The motion[1] [2] of a classical mechanical system is presumed to be described by a differential equation

$$\frac{dx}{dt} = F(x) \tag{DE}$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$ governs the dynamics. Under suitable smoothness conditions on F (locally Lipschitz?) given any 'initial' point $y \in \mathbb{R}^n$ there exists a unique solution $x : \mathbb{R} \to \mathbb{R}^n$ of (DE) satisfying the initial condition

$$x(0) = y. \tag{IC}$$

To emphasize dependence on the initial condition, we change notation and consider the solution to be a function

$$x : \mathbb{R}^{1+n} \to \mathbb{R}^n : \ (t, y) \to x(t, y)$$

such that $t \to x(t, y)$ satisfies (DE) and $x(0, y) = y$ for all $y \in \mathbb{R}^n$. Moreover this function is continuous, that is, it depends continuously on time as well as the initial conditions.

Now fix $s \in \mathbb{R}$ and the initial point $y \in \mathbb{R}^n$ and consider the function

$$z : t \to x(s + t, y).$$

This also satisfies (DE) and also $z(0) = x(s, y)$. But there is a *unique* solution satisfying these two conditions, namely $t \to x(t, z(0))$. Therefore we have

$$z(t) = x(t, z(0)), \quad \text{that is } \ x(s + t, y) = x(t, x(s, y)) \quad \text{for all } \ t \in \mathbb{R}. \tag{1}$$

Changing point of view, we consider time $t \in \mathbb{R}$ as a parameter: for each $t \in \mathbb{R}$, (DE) defines a map

$$\phi_t : \mathbb{R}^n \to \mathbb{R}^n : \ y \to x(t, y)$$

mapping any 'initial' point y to its location at time t.

Then equation (1) becomes

$$\begin{aligned} \phi_{t+s}(y) =& \phi_t(\phi_s(y)) \quad \text{for all } \ y \in \mathbb{R}^n \\ \text{i.e. } \ \phi_{t+s} =& \phi_t \circ \phi_s. \end{aligned}$$

[1] This section is based on [1].

[2] semcrsd, 15/4/07

Therefore we have an *action* of the group $\mathbb{R}$ by continuous maps (in fact homeomorphisms) of $\mathbb{R}^n$: the *dynamical system* is $(\mathbb{R}^n, \phi, \mathbb{R})$ where $\phi : \mathbb{R} \to \mathrm{Homeo}(\mathbb{R}^n)$ is a homomorphism from the group $\mathbb{R}$ into the group of homeomorphisms $\mathbb{R}^n \to \mathbb{R}^n$.

In some cases there are 'integrals' of (DE), namely functions $H : \mathbb{R}^n \to \mathbb{R}$ satisfying

$$\sum_{i=1}^{n} \frac{\partial H}{\partial x_i} F_i = 0 \qquad \text{where } \; F = (F_1, F_2, \ldots, F_n)$$

is the function in (DE). It follows (chain rule) that H is constant along solution curves:

$$\frac{d}{dt} H(x(t,y)) = 0.$$

This means that any hypersurface $S_c = \{x \in \mathbb{R}^n : H(x) = c\}$, where H takes the constant value c, has the property that if the initial point y is in S_c (i.e. if $H(x(0,y)) = c$) then the solution $x(t,y)$ remains on S_c for all time t. Thus the transformation group $\{\phi_t : t \in \mathbb{R}\}$ leaves S_c (globally) invariant, so one may study the (restricted) dynamical system $(S_c, \phi, \mathbb{R})$.

Sometimes one studies the 'continuous' dynamical system $(S_c, \phi, \mathbb{R})$ by replacing the 'continuous' family $\{\phi_t : t \in \mathbb{R}\}$ with a 'discrete sampling' $\{\phi_{nT} : n \in \mathbb{Z}\}$ (where T is the 'scale'). This is the same as $\{\psi^n : n \in \mathbb{Z}\}$ where $\psi = \phi_T$ and $\psi^2 = \psi \circ \psi = \phi_{2T}$, $\psi^{-2} = \psi^{-1} \circ \psi^{-1}$ etc. This gives an action of the group $\mathbb{Z}$ by the iterates of ψ and ψ^{-1}. One denotes this 'discrete' dynamical system by (S_c, ψ).

A priori, there is no mathematical reason to consider only the groups $\mathbb{R}$ and $\mathbb{Z}$:

Definition 1 *A* ***(classical) dynamical system*** *is a triple (X, ϕ, G) where X is a set, G is a group and ϕ is a group homomorphism into the group of bijections of X. One is generally interested in the following two cases:*

A ***topological dynamical system*** *is (X, ϕ, G) where X is a topological space (usually locally compact, Hausdorff) and $\phi(G)$ a group of homeomorphisms of X.*

A ***measure preserving dynamical system*** *is (X, ϕ, G) where $X = (X, \mathcal{S}, \mu)$ is a measure space and $\phi(G)$ a group of measurable bijections of X which preserve the measure μ.*

When G has a (non-discrete) topology, some continuity restriction on the action is usually imposed. For instance when G is locally compact Hausdorff, for a topological dynamical system (X, ϕ, G) one usually assumes that the map $G \times X \to X : (t, x) \to \phi_t(x)$ is continuous; or one may assume continuity of the map $G \to X : t \to \phi_t(x)$ for each fixed $x \in X$.

Alternatively, X could be a differentiable manifold and $\phi(G)$ a group of diffeomorphisms, or X could be an analytic manifold and $\phi(G)$ a group of holomorphic maps.

C*-dynamical systems The study of a classical dynamical system (X, ψ) is equivalent to the study of the pair $(\mathcal{C},\alpha)$, where $\mathcal{C}$ is the commutative C*-algebra $C_o(X)$ and

α is the *-automorphism[3] of $\mathcal{C}$ defined by $\alpha(a) = a \circ \psi^{-1}$ $(a \in \mathcal{C})$. More generally, if G is a group of homeomorphisms of X (so that for $g, h \in G$ we have $g(h(x)) = (gh)(x)$ for all $x \in X$) then the map $\alpha : G \to Aut(\mathcal{C})$ from G to the group of *-automorphisms of $\mathcal{C}$ given by $\alpha_g(a) = a \circ g^{-1}$ $(a \in \mathcal{C}, g \in G)$ is a group homomorphism.

A classical dynamical system is supposed to describe the dynamical behaviour of a classical mechanical system, in which the observable quantities are functions on phase space. In a quantum mechanical system, the observables are operators on a Hilbert space, or just elements of an abstract C*-algebra: they are non-commuting objects. The dynamical behaviour is described by the action of a group (or semigroup, when the action is not 'reversible') of automorphisms of the algebra of observables.

Definition 2 *A (discrete)* ***C*-dynamical system*** *is a triple $(\mathcal{C}, \alpha, G)$ where $\mathcal{C}$ is a C*-algebra, G is a group and $\alpha : G \to Aut(\mathcal{C})$ is a homomorphism of G into the group of *-automorphisms of $\mathcal{C}$.*

Example 1 Let $g \to U_g$ be a unitary representation of a group G on a Hilbert space H. Then letting $\mathrm{ad}_g(A) = U_g A U_g^*$ $(A \in \mathcal{B}(H))$ we obtain a C*-dynamical system $(\mathcal{B}(H), \mathrm{ad}, G)$. Moreover if $\mathcal{C} \subseteq \mathcal{B}(H)$ is any subalgebra which is invariant under $\{\mathrm{ad}_g : g \in G\}$, then denoting by α_g the restriction of ad_g to $\mathcal{C}$ we obtain a quantum dynamical system $(\mathcal{C}, \alpha, G)$.

Thus a unitary representation U of G on H gives rise to a dynamical system $(\mathcal{C}, \alpha, G)$ for every algebra $\mathcal{C} \subseteq \mathcal{B}(H)$ which is *normalized*[4] by the group $\{U_g : g \in G\}$. Conversely, given a dynamical system $(\mathcal{C}, \alpha, G)$ one may ask whether it is of the above form, in other words whether the automorphism α is *implemented* by a unitary group.

Definition 3 *Let $\mathcal{C} \subseteq \mathcal{B}(H)$ be a C*-algebra, G a group. A unitary representation U of G on H is said to* implement *an action α of G on $\mathcal{C}$ if*

$$\alpha_g(A) = U_g A U_g^* \qquad (A \in \mathcal{C},\ g \in G).$$

Failing that, one may ask whether it is possible to 'change Hilbert space' so as to achieve unitary implementation. We will show that this is always possible, provided one is willing to enlarge H to a direct sum of $|G|$ copies of itself.

Covariant representations

Definition 4 *A* ***covariant representation*** *of a C*-dynamical system $(\mathcal{C}, \alpha, G)$ on a Hilbert space H is a pair (π, U) where π is a *-representation of $\mathcal{C}$ on H, U is a unitary representation of G on (the same) H and π and U are connected by the* covariance condition*:*

$$\pi(\alpha_g(a)) = U_g \pi(a) U_g^* \qquad (a \in \mathcal{C}, g \in G). \tag{C}$$

[3]A *-automorphism of $\mathcal{C}$ is a linear bijection $\alpha : \mathcal{C} \to \mathcal{C}$ which preserves the product and the involution. Recall that a *-automorphism of a C*-algebra is automatically isometric.

[4]A unitary U *normalizes* $\mathcal{C}$ when $U\mathcal{C}U^* \subseteq \mathcal{C}$.

Example 2 Let Ω be a compact Hausdorff space, G a group of homeomorphisms of Ω, let μ be a G-invariant Borel probability measure on Ω (that is $\mu(gE) = \mu(E)$ for all $g \in G$ and $E \subseteq \Omega$ Borel). Let $\mathcal{A} = C(\Omega)$ and $\alpha_g(a) = a \circ g^{-1}$.

Represent $\mathcal{A}$ on $H = L^2(\Omega, \mu)$ as multiplication operators:

$$\pi(a)\xi = a\xi \qquad (a \in \mathcal{A},\ \xi \in H).$$

Represent G on H by composition:

$$U_g\xi = \xi \circ g^{-1}$$

(the fact that each U_g is unitary follows from the fact that μ is G-invariant).

We verify condition (C): for all $\xi \in H$,

$$\begin{aligned} U_g\pi(a)\xi &= U_g(a\xi) = (a \circ g^{-1}).(\xi \circ g^{-1}), \\ \pi(\alpha_g(a))U_g\xi &= \pi(a \circ g^{-1})(\xi \circ g^{-1}) = (a \circ g^{-1}).(\xi \circ g^{-1}) \\ \text{so} \quad U_g\pi(a) &= \pi(\alpha_g(a))U_g. \end{aligned}$$

We say that G **acts ergodically** on Ω if the only G-invariant Borel sets are the null and co-null sets. More precisely, G acts ergodically if, whenever $E \subseteq \Omega$ is a Borel set such that $\mu(E \triangle (gE)) = 0$ for all $g \in G$ ($\triangle$ denotes symmetric difference), we have $\mu(E) = 0$ or $\mu(\Omega \backslash E) = 0$.

Proposition 3 *If G acts ergodically, then the pair (π, U) is irreducible, that is the only closed subspaces of H which are invariant under all $\pi(a)$ ($a \in A$) and all U_g ($g \in G$) are $\{0\}$ and H.*

Proof. Let $P \in \mathcal{B}(H)$ be invariant under both π and U. Then, first P must commute with all $\pi(a)$. We claim that P commutes with multiplication operators $\pi(\chi_E)$ by characteristic functions of Borel sets $E \subseteq \Omega$.

Indeed, given $E \subseteq \Omega$, by Lusin[5] (or regularity of μ) for each $n \in \mathbb{N}$ we can find $f_n \in C(\Omega)$ with $\|f_n\|_\infty \leq 1$ such that $\mu(\{\omega : |\chi_E(\omega) - f_n(\omega)|\}) < \frac{1}{n}$. It is then easily verified that for all $\xi \in L^2(\Omega, \mu)$ one has $\lim_n \|f_n\xi - \chi_E\xi\|_2 = 0$ and therefore

$$\begin{aligned} P\pi(\chi_E)\xi &= P(\chi_E\xi) = \lim_n P(f_n\xi) = \lim_n P\pi(f_n)\xi \\ &= \lim_n \pi(f_n)(P\xi) = \lim_n f_n P(\xi) = \chi_E P(\xi) = \pi(\chi_E)P\xi \end{aligned}$$

and so $P\pi(\chi_E) = \pi(\chi_E)P$.

Now let $g = P(\mathbf{1}) \in L^2(\Omega, \mu)$. Note that for every Borel set $E \subseteq \Omega$,

$$g\chi_E = \chi_E g = \pi(\chi_E)P\mathbf{1} = P\pi(\chi_E)\mathbf{1} = P(\chi_E\mathbf{1}) = P(\chi_E). \tag{2}$$

[5] thanks, George

We claim that $|g| \leq 1$ a.e. For this, we suppose that the set $E = \{\omega : |g(\omega)| > r\}$ has positive measure and we show that $r \leq 1$; it will follow that $\mu(\{\omega : |g(\omega)| > r\}) = 0$ for all $r > 1$ as claimed. Indeed, since $0 \leq r\chi_E \leq |g|\chi_E$ by definition of E we have

$$r\|\chi_E\|_2 \leq \|g\chi_E\|_2 = \|P(\chi_E)\|_2 \leq \|\chi_E\|_2$$

using (2) and the fact that $\|P\| = 1$. This shows that $|g| \leq 1$ a.e. Thus g defines a bounded multiplication operator M_g and now (2) shows that $M_g\chi_E = P\chi_E$ for every Borel set $E \subseteq \Omega$, hence $M_g f = Pf$ for all $f \in L^2(\Omega, \mu)$.

Thus P is a multiplication operator; since it is a projection, it corresponds to multiplication by the characteristic function of some Borel set $F \subseteq \Omega$. Since P is also invariant under all U_g, F must be invariant under the action of G, so that F is null or co-null, that is, $P = 0$ or 1. $\square$

Existence of covariant representations Let $(\mathcal{C}, \alpha, G)$ be a C*-dynamical system. Given any representation π_o of $\mathcal{C}$ on a Hilbert space H_o, we will construct a covariant representation (π, Λ) of $(\mathcal{C}, \alpha, G)$ on a *different* (larger) Hilbert space H.

$$\text{Define} \quad H = \ell^2(G, H_o) = \left\{ \xi : G \to H_o : \|\xi\|_2^2 = \sum_{s \in G} \|\xi(s)\|_{H_o} < \infty \right\}.$$

We may think of H as the Hilbert space direct sum $\oplus_{s\in G} H_o$ of $|G|$ copies of H_o. It is the completion of the linear space

$$c_{oo}(G, H_o) = \{\xi : G \to H_o : \operatorname{supp}\xi \ \text{ finite}\}$$

with respect to the scalar product

$$\langle \xi, \eta \rangle = \sum_s \langle \xi(s), \eta(s) \rangle_{H_o}.$$

Elements of H will be written

$$\xi = \sum_s \xi(s)\delta_s \qquad \left(\sum_{s \in G} \|\xi(s)\|_{H_o}^2 < \infty \right)$$

where, for $x \in H_o$, the 'monomial' $x\delta_s$ is the function $G \to H_o$

$$x\delta_s(t) = \begin{cases} x, & t = s \\ 0, & t \neq s \end{cases}$$

Thus, a bounded operator $A \in B(H)$ defines a '$|G| \times |G|$ matrix' $A = [A_{s,t}]$ where each $A_{s,t} \in B(H_o)$. The correspondence is given by the formula[6]

$$\langle A_{t,s}x, y \rangle_{H_o} = \langle A(x\delta_s), (y\delta_t) \rangle_H \qquad (x, y \in H_o, \ s, t \in G).$$

[6] recall the formula $\langle A\delta_n, \delta_m \rangle = a_{m,n}$ for operators on ℓ^2.

Define a representation π of $\mathcal{C}$ on H by

$$(\pi(a)\xi)(s) = \pi_o(\alpha_s^{-1}(a))(\xi(s)) \qquad (a \in \mathcal{C}, \xi \in \ell^2(G, H_o))$$

(for each $s \in G$, $\xi(s)$ is an element of H_o, hence so is $\pi_o(\alpha_s^{-1}(a))(\xi(s)))$

$$\text{equivalently} \;\; \pi(a)x\delta_s = (\pi_o(\alpha_s^{-1}(a))x)\delta_s \qquad (a \in \mathcal{C}, x \in H_o, s \in G).$$

Observe that

$$\langle \pi(a)(x\delta_s), (y\delta_t)\rangle_H = \begin{cases} \langle \pi_o(\alpha_s^{-1}(a))x, y\rangle_{H_o}, & s = t \\ 0, & s \neq t \end{cases}$$

so that

$$\pi(a) = \operatorname{diag}(\pi_o(\alpha_s^{-1}(a))).$$

It is easy to check that π is a *-representation of $\mathcal{C}$ which is faithful when π_o is faithful. Note, however, that π is *never* irreducible (when G is non-trivial): each copy of H_o in $H = \oplus_s H_o$ is invariant (in fact, reducing) for π.

Define a unitary representation Λ of G on H by[7]

$$(\Lambda_t\xi)(s) = \xi(t^{-1}s) \qquad (t \in G, \xi \in \ell^2(G, H_o)).$$

Equivalently

$$\Lambda_t(x\delta_r) = x\delta_{tr} \qquad (t \in G, x \in H_o, r \in G)$$

(because $(\Lambda_t x\delta_r)(s) = x\delta_r(t^{-1}s) = x\delta_{tr}(s)$ for all $s \in G$). Thus Λ is the left regular representation of G of 'multiplicity' equal to $\dim H_o$.

We verify the covariance condition:

$$\begin{aligned}(\Lambda_t\pi(a)\Lambda_t^*\xi)(s) &= (\pi(a)\Lambda_t^*\xi)(t^{-1}s) = \pi_o(\alpha_{t^{-1}s}^{-1}(a))((\Lambda_t^*\xi)(t^{-1}s)) \\ &= \pi_o(\alpha_{t^{-1}s}^{-1}(a))(\xi(s)) = \pi_o(\alpha_s^{-1}(\alpha_t(a)))(\xi(s)) = (\pi(\alpha_t(a))\xi)(s)\end{aligned}$$

for all $\xi \in H$, so

$$\Lambda_t\pi(a)\Lambda_t^* = \pi(\alpha_t(a)).$$

It is instructive to verify this in the case $G = \mathbb{Z}$ or even $G = \mathbb{Z}_3$. In the latter case, G is generated by $\bar{1}$ and we have:

$$\Lambda_{\bar{1}} = \begin{bmatrix} 0 & 0 & I \\ I & 0 & 0 \\ 0 & I & 0 \end{bmatrix} \qquad \pi(a) = \begin{bmatrix} \pi_o(\alpha(a)) & 0 & 0 \\ 0 & \pi_o(a) & 0 \\ 0 & 0 & \pi_o(\alpha^{-1}(a)) \end{bmatrix}$$

[7] Letting $\xi_t(s) = \xi(t^{-1}s)$ we have $\Lambda_t(\Lambda_s\xi) = \Lambda_t\xi_s = (\xi_s)_t$ where $(\xi_s)_t(r) = \xi_s(t^{-1}r) = \xi(s^{-1}t^{-1}r)$ so that $(\xi_s)_t = \xi_{ts} = \Lambda_{ts}\xi$, so that Λ is indeed a group morphism; the fact that $\|\Lambda_t\xi\|_2 = \|\xi\|_2$ is obvious, so Λ_t is an invertible isometry, hence a unitary.

where I is the identity operator on H_o and $\alpha \equiv \alpha_{\bar{1}}$. We have

$$\Lambda_{\bar{1}}\pi(a)\Lambda_{\bar{1}}^* = \begin{bmatrix} 0 & 0 & I \\ I & 0 & 0 \\ 0 & I & 0 \end{bmatrix} \begin{bmatrix} \pi_o(\alpha(a)) & 0 & 0 \\ 0 & \pi_o(a) & 0 \\ 0 & 0 & \pi_o(\alpha^{-1}(a)) \end{bmatrix} \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \\ I & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \pi_o(\alpha^{-1}(a)) & 0 & 0 \\ 0 & \pi_o(\alpha(a)) & 0 \\ 0 & 0 & \pi_o(a) \end{bmatrix}$$

and

$$\pi(\alpha(a)) = \begin{bmatrix} \pi_o(\alpha^2(a)) & 0 & 0 \\ 0 & \pi_o(\alpha(a)) & 0 \\ 0 & 0 & \pi_o(a) \end{bmatrix} = \begin{bmatrix} \pi_o(\alpha^{-1}(a)) & 0 & 0 \\ 0 & \pi_o(\alpha(a)) & 0 \\ 0 & 0 & \pi_o(a) \end{bmatrix}.$$

The twisted convolution algebra The idea is to 'enlarge' $\mathcal{C}$ by 'adding' unitary elements δ_t, $(t \in G)$ which will implement the action α of G on $\mathcal{C}$.

So we form the linear space[8]

$$c_{oo}(G, \mathcal{C}) = \{f : G \to \mathcal{C} : \operatorname{supp} f \text{ finite } \}$$

and we write its elements

$$f = \sum_{t \in G} f(t)\delta_t$$

where $f(t)$ in $\mathcal{C}$.

[In the case $G = \mathbb{Z}$, every $f = \sum_{n\in\mathbb{Z}} f(n)\delta_n = \sum_{n\in\mathbb{Z}} f(n)\delta_1^n \in c_{oo}(\mathbb{Z}, \mathcal{C})$ is a $\mathcal{C}$-valued polynomial in the generator δ_1.]

Note that $c_{oo}(G, \mathcal{C})$ can be considered to contain a copy of $\mathcal{C}$, by identifying $a \in \mathcal{C}$ with[9] $a\delta_e \in c_{oo}(G, \mathcal{C})$.

We define a 'twisted convolution' so that the unitaries implement the action. That is, we want to have $\alpha_s(a) = \delta_s a \delta_s^*$ or equivalently $\delta_s a = \alpha_s(a)\delta_s$ for $a \in \mathcal{C}$ and $s \in \mathcal{G}$.

This gives the motivation for the definition of the convolution: we multiply monomials according to the rule

$$(a\delta_s) * (b\delta_t) = a\alpha_s(b)\delta_s\delta_t = a\alpha_s(b)\delta_{st}$$

and define the involution by

$$(a\delta_t)^* = \delta_t^* a^* = \delta_{t^{-1}} a^* = \alpha_{t^{-1}}(a^*)\delta_{t^{-1}} = \alpha_t^{-1}(a)^*\delta_{t^{-1}}.$$

[8]If one uses the theory of tensor products, one may identify $c_{oo}(G, \mathcal{C})$ with the *algebraic* tensor product $\mathcal{C} \otimes c_{oo}(G)$: its elements can be written as (finite) sums $f = \sum_t f(t) \otimes \delta_t$, where $\{\delta_t\}$ is the usual (Hamel) basis of $c_{oo}(G)$.

[9]$e \in G$ is of course the identity element

Thus the definitions are

$$f * g = \left(\sum_s f(s)\delta_s\right) * \left(\sum_t g(t)\delta_t\right) = \sum_{s,t} f(s)\alpha_s(g(t))\delta_{st}$$

and

$$f^* = \left(\sum_s f(s)\delta_s\right)^* = \sum_s \alpha_s^{-1}(f(s))^*\delta_{s^{-1}}.$$

In other words (setting $r = st$)

$$f * g = \sum_r \left(\sum_s f(s)\alpha_s(g(s^{-1}r))\right)\delta_r$$

and (changing s to $r = s^{-1}$)

$$f^* = \sum_r \alpha_r(f(r^{-1}))^*\delta_r.$$

Thus

$$\begin{aligned} (f * g)(r) &= \sum_s f(s)\alpha_s(g(s^{-1}r)) \\ \text{and} \quad (f^*)(r) &= \alpha_r(f(r^{-1}))^* \qquad (r \in G). \end{aligned} \tag{3}$$

We summarize:

Definition 5 *The twisted convolution algebra $c_{oo}(G, \alpha, \mathcal{C})$ is the space of all functions $f : G \to \mathcal{C}$ with finite support, equipped with the convolution and involution defined by (3). The completion with respect to the ℓ^1 norm*

$$\|f\|_1 = \sum_t \|f(t)\|_{\mathcal{C}}$$

*is a Banach *-algebra with isometric involution, denoted $\ell^1(G, \alpha, \mathcal{C})$.*

The reduced crossed product In order to equip $c_{oo}(G, \alpha, \mathcal{C})$ with a suitable C*-norm, we construct a *-representation on a Hilbert space. In fact we can work directly with $\ell^1(G, \alpha, \mathcal{C})$.

If $\mathcal{C}$ consists of operators on a Hilbert space H_o, we consider the covariant pair (π, Λ) acting on $H = \ell^2(G, H_o)$ which we constructed above (with π_o the identity representation) and we define

$$\tilde{\pi}\left(\sum_s f(s)\delta_s\right) = \sum_s \pi(f(s))\Lambda_s \quad \left(f = \sum_s f(s)\delta_s \in \ell^1(G, \alpha, \mathcal{C})\right).$$

Since $\sum_t \|f(t)\|_{\mathcal{C}} < \infty$, the sum $\sum_t \pi(f(t))\Lambda_t$ converges (absolutely) in the norm of $B(\ell^2(G, H_o))$. Thus $\sum_t \pi(f(t))\Lambda_t \in B(H)$ and in fact

$$\left\|\sum_t \pi(f(t))\Lambda_t\right\|_{B(H)} \leq \sum_t \|f(t)\|_{\mathcal{C}} = \left\|\sum_t f(t)\delta_t\right\|_1.$$

The fact that $\tilde{\pi}$ is a *-representation follows immediately using the covariance condition (in fact the twisted convolution and the involution were defined precisely to ensure this).

We claim that $\tilde{\pi}$ is faithful. Indeed, if $f = \sum_t f(t)\delta_t \in \ell^1(G, \alpha, \mathcal{C})$ is nonzero then there exists $s \in G$ with $f(s) \neq 0$ hence $\alpha_s^{-1}(f(s)) \neq 0$. Choosing $x, y \in H_o$ such that $\langle \pi_o(\alpha_s^{-1}(f(s))x, y\rangle \neq 0$ we obtain

$$\begin{aligned} \langle \tilde{\pi}(f)x\delta_e, y\delta_s\rangle_H &= \left\langle \sum_t \pi(f(t))\Lambda_t(x\delta_e), y\delta_s \right\rangle_H \\ &= \sum_t \left\langle (\pi_o(\alpha_t^{-1}(f(t)))x)\delta_t, y\delta_s \right\rangle_H = \left\langle \pi_o(\alpha_s^{-1}(f(s)))x, y \right\rangle_{H_o} \neq 0 \end{aligned} \tag{4}$$

because $x\delta_t \perp y\delta_s$ in $\ell^2(G, H_o)$. Thus $\tilde{\pi}(f) \neq 0$.

Therefore if we define

$$\left\| \sum_t f(t)\delta_t \right\|_r = \left\| \sum_t \pi(f(t))\Lambda_t \right\|_{B(H)}$$

we obtain an algebra norm on $\ell^1(G, \alpha, \mathcal{C})$ satisfying the C*-condition.

Definition 6 *If $(\mathcal{C}, \alpha, G)$ is a C*-dynamical system, the* ***reduced crossed product*** *$\mathcal{C} \times_{\alpha r} G$ of $\mathcal{C}$ by α is the completion of $\ell^1(G, \alpha, \mathcal{C})$ (equivalently, of $c_{oo}(G, \alpha, \mathcal{C})$) with respect to the norm $\|\cdot\|_r$ just defined. Equivalently, $\mathcal{C} \times_{\alpha r} G$ is the concrete C*-algebra of operators on $\ell^2(G, H_o)$ generated (as a C*-algebra) by the family of operators*

$$\{\pi(a)\Lambda_t : a \in \mathcal{C},\ t \in G\}.$$

Remark 4 *We may identify $c_{oo}(G, \alpha, \mathcal{C})$ with its image in $B(\ell^2(G, \alpha, \mathcal{C}))$ and think of it as the (non-closed) *-subalgebra of $B(\ell^2(G, \alpha, \mathcal{C}))$ generated by $\{\pi(a)\Lambda_t : a \in \mathcal{C},\ t \in G\}$.*

Note that this family contains $\pi(\mathcal{C})$ (an image of $\mathcal{C}$): just set $t = e$ to get $\pi(a)\Lambda_e = \pi(a)$, but it does not contain the image $\{\Lambda_t : t \in G\}$, unless $\mathcal{C}$ is unital (proof?).

Remark 5 *The above argument works for any faithful representation (π_o, H_o) of the C*-algebra $\mathcal{C}$. Therefore:*

*For any faithful *-representation (π_o, H_o) of $\mathcal{C}$ the *-representation $\tilde{\pi}$ of $\ell^1(G, \alpha, \mathcal{C})$ on $\ell^2(G, H_o)$ is faithful, and extends by continuity to a faithful representation of $\mathcal{C} \times_{\alpha r} G$.*

Representations of $\ell^1(G, \alpha, \mathcal{C})$. We show that the algebraic structure of $\ell^1(G, \alpha, \mathcal{C})$ was constructed so as to 'absorb' all covariant representations of the C*-dynamical system $(\mathcal{C}, \alpha, G)$. More precisely, we generalize the construction of the previous paragraph to arbitrary covariant pairs:

Given a covariant representation (π, U) of $(\mathcal{C}, \alpha, G)$ on a Hilbert space H, we define a map $\pi \times U$ first on 'trigonometric polynomials' by the formula:

$$(\pi \times U)(\sum_t f(t)\delta_t) = \sum_t \pi(f(t))U_t, \quad f = \sum_t f(t)\delta_t \in c_{oo}(G, \mathcal{C}) \tag{5}$$

(thus the representation $\tilde{\pi}$ defined in Remark 5 is $\pi \times \Lambda$).

Proposition 6 *There exists a bijective correspondence between non-degenerate *-representations ρ of $\ell^1(G, \alpha, \mathcal{C})$ and covariant *-representations (π, U) of $(\mathcal{C}, \alpha, G)$ for which π is non-degenerate.*

Proof Given a covariant representation (π, U) let $\rho = \pi \times U$ be defined as above. Observe that ρ is a linear map and

$$\|\rho(\sum_t f(t)\delta_t)\| \leq \sum_t \|\pi(f(t))U_t\| = \sum_t \|\pi(f(t))\| \leq \sum_t \|f(t)\| = \|f\|_1$$

(we have used the fact that π is contractive). Thus ρ is $\|\cdot\|_1$-contractive and hence extends to $\ell^1(G, \alpha, \mathcal{C})$. For $x, y \in \mathcal{C}$ and $t, s \in G$, we have

$$\begin{aligned}
\rho((x\delta_t) * (y\delta_s)) &= \rho(x\alpha_t(y)\delta_{ts}) = \pi(x\alpha_t(y))U_{ts} \\
&= \pi(x)\pi(\alpha_t(y))U_t U_s = \pi(x)(U_t\pi(y)U_t^*)U_t U_s \\
&= \pi(x)U_t\pi(y)U_s = \rho(x\delta_t)\rho(y\delta_s) \\
\text{and} \quad \rho((x\delta_t)^*) &= \rho(\alpha_{t^{-1}}(x^*)\delta_{t^{-1}}) = \pi(\alpha_{t^{-1}}(x^*))U_{t^{-1}} \\
&= U_{t^{-1}}\pi(x^*)U_{t^{-1}}^* U_{t^{-1}} = U_t^*(\pi(x))^* \\
&= (\pi(x)U_t)^* = (\rho(x\delta_t))^*
\end{aligned}$$

showing that ρ is a *-representation. Since $\rho(a\delta_1)H = \pi(a)H$, if π is non-degenerate, so is ρ.

Conversely, let ρ be a non-degenerate contractive representation of the Banach *-algebra $\ell^1(G, \alpha, \mathcal{C})$ on some Hilbert space H.

Define a map

$$\pi : \mathcal{C} \to B(H) \text{ by } \pi(a) = \rho(a\delta_1).$$

Since $(a\delta_1) * (b\delta_1) = ab\delta_1$ we have $\pi(a)\pi(b) = \pi(ab)$, and since $(a\delta_1)^* = a^*\delta_1$ we have $\pi(a^*) = \pi(a)^*$, so that π is a *-representation.

To show that π is non-degenerate, observe that $\pi(\mathcal{C})H \supseteq \pi(\mathcal{C})H_1$, where H_1 is the linear manifold

$$H_1 = [\rho(b\delta_s)\eta : b \in \mathcal{C}, s \in G, \eta \in H].$$

Thus

$$\begin{aligned}
\pi(\mathcal{C})H &\supseteq [\rho(a\delta_1)\rho(b\delta_s)\eta : a, b \in \mathcal{C}, s \in G, \eta \in H] \\
&= [\rho(ab\delta_s)\eta : a, b \in \mathcal{C}, s \in G, \eta \in H].
\end{aligned}$$

Since ρ is assumed non-degenerate, this last linear manifold is dense in H (and hence so is H_1 which contains it).[10] Thus $\pi(\mathcal{C})H$ is dense in H, so π is non-degenerate.

Suppose for the moment that $\mathcal{C}$ is unital. Then we may define

$$U_t = \rho(\mathbf{1}\delta_t)$$

[10] For if $\langle \xi, \rho(ab\delta_s)\eta \rangle = 0$ for all $b \in \mathcal{C}$ and $\eta \in H$ then replacing a by e_λ for an approximate identity $\{e_\lambda\}$ and noting that for all $b \in \mathcal{C}$ we have $\|\rho((e_\lambda b - b)\delta_s)\| = \|(e_\lambda b - b)\delta_s\|_1 = \|e_\lambda b - b\|_{\mathcal{C}} \to 0$ we obtain $\langle \xi, \rho(b\delta_s)\eta \rangle = 0$ for all for all b, s and η. But the monomials $b\delta_s$ generate $\ell^1(G, \alpha, \mathcal{C})$ and so $\langle \xi, \rho(f)\eta \rangle = 0$ for all $f \in \ell^1(G, \alpha, \mathcal{C})$ and $\eta \in H$ hence $\xi = 0$ because ρ is non-degenerate.

and it is immediate that $U_tU_s = U_{ts}$ and $U_1 = I$. Also $\|U_t\| \le \|\mathbf{1}\delta_t\| = 1$ so that U_t and its inverse are contractions, and hence U_t is unitary. Note that

$$U_t(\rho(a\delta_s)\eta) = \rho(\mathbf{1}\delta_t)\rho(a\delta_s)\eta = \rho(\alpha_t(a)\delta_{ts})\eta \quad (a \in \mathcal{C}, s \in G, \eta \in H). \tag{6}$$

If $\mathcal{C}$ is not unital, the right hand side of (6) makes sense and can be used as the definition of U_t on H_1. So we define U_t initially on the dense subspace H_1 by

$$U_t(\rho(a\delta_s)\eta) = \rho(\alpha_t(a)\delta_{ts})\eta \quad (a \in \mathcal{C}, s \in G, \eta \in H). \tag{7}$$

It is clear that $U_tU_s = U_{ts}$.[11] Also, if $\xi' = \rho(a\delta_s)\xi$, $\eta' = \rho(b\delta_r)\eta$ are in H_1

$$\begin{aligned}
&\langle U_{t^{-1}}\xi', \eta'\rangle = \langle U_{t^{-1}}(\rho(a\delta_s)\xi), \rho(b\delta_r)\eta\rangle = \langle \rho(\alpha_{t^{-1}}(a)\delta_{t^{-1}s})\xi, \rho(b\delta_r)\eta\rangle \\
=&\langle \xi, (\rho(\alpha_{t^{-1}}(a)\delta_{t^{-1}s}))^*\rho(b\delta_r)\eta\rangle = \big\langle \xi, \rho(\alpha_{(t^{-1}s)^{-1}}(\alpha_{t^{-1}}(a^*))\delta_{(t^{-1}s)^{-1}})\rho(b\delta_r)\eta\big\rangle \\
=&\langle \xi, \rho(\alpha_{s^{-1}}(a^*)\delta_{s^{-1}t})\rho(b\delta_r)\eta\rangle \\
=&\langle \xi, \rho(\alpha_{s^{-1}}(a^*)\alpha_{s^{-1}t}(b)\delta_{s^{-1}tr})\eta\rangle = \langle \xi, \rho(\alpha_{s^{-1}}(a^*)\delta_{s^{-1}})\rho(\alpha_t(b)\delta_{tr})\eta\rangle \\
=&\langle \rho(a\delta_s)\xi, \rho(\alpha_t(b)\delta_{tr})\eta\rangle = \langle \rho(a\delta_s)\xi, U_t\rho(b\delta_r)\eta\rangle = \langle \xi', U_t\eta'\rangle.
\end{aligned}$$

By linearity, $\langle U_{t^{-1}}\xi', \eta\rangle = \langle \xi', U_t\eta'\rangle$ for all $\xi', \eta' \in H_1$. Now if $\xi' \in H_1$ then from the definition of U_t we see that $U_t\xi' \in H_1$. Thus $\langle U_t\xi', U_t\eta'\rangle = \langle U_{t^{-1}}U_t\xi', \eta'\rangle = \langle \xi', \eta'\rangle$ showing that U_t is an isometry on H_1. Hence it extends to an isometry on H, also to be denoted by U_t. The relation $\langle U_{t^{-1}}\xi, \eta\rangle = \langle \xi, U_t\eta\rangle$ is now valid for all $\xi, \eta \in H$, and shows that $U_t^* = U_{t^{-1}}$. Thus $t \to U_t$ is a unitary representation of G on H.

Alternative approach We choose a contractive approximate identity $\{e_\lambda\}$ for $\mathcal{C}$ and define

$$U_t\xi = \lim_\lambda \rho(e_\lambda\delta_t)\xi \qquad (\xi \in H). \tag{8}$$

We will prove that the limit exists when $\xi \in H_1$. For this, it suffices to consider ξ of the form $\xi = \rho(a\delta_s)\eta$. Then

$$\begin{aligned}
\|\rho(e_\lambda\delta_t)\xi - \rho(\alpha_t(a)\delta_{ts})\eta\|_H &= \|\rho(e_\lambda\delta_t)\rho(a\delta_s)\eta - \rho(\alpha_t(a)\delta_{ts})\eta\|_H \\
&= \|\rho(e_\lambda\alpha_t(a)\delta_{ts} - \alpha_t(a)\delta_{ts})\eta\|_H \\
&\le \|\rho(e_\lambda\alpha_t(a)\delta_{ts} - \alpha_t(a)\delta_{ts})\|\,\|\eta\|_H \\
&\le \|(e_\lambda\alpha_t(a) - \alpha_t(a))\delta_{ts}\|_1\,\|\eta\|_H \\
&= \|e_\lambda\alpha_t(a) - \alpha_t(a)\|_{\mathcal{C}}\,\|\eta\|_H \underset{\lambda}{\to} 0
\end{aligned}$$

because $\lim_\lambda \|e_\lambda b - b\|_{\mathcal{C}} \to 0$ for every $b \in \mathcal{C}$. Thus the limit exists and in fact[12]

$$\lim_\lambda \rho(e_\lambda\delta_t)\rho(a\delta_s)\eta = \rho(\alpha_t(a)\delta_{ts}\eta. \tag{9}$$

[11] Indeed,

$$U_tU_s(\rho(a\delta_r)\eta) = U_t\rho(\alpha_s(a)\delta_{sr})\eta = \rho(\alpha_t(\alpha_s(a))\delta_{tsr})\eta = \rho(\alpha_{ts}(a)\delta_{(ts)r})\eta = U_{ts}(\rho(a\delta_r)\eta).$$

[12] This relation incidentally shows that the definitions (7) and (8) are the same (on H_1).

Note that, given $\xi \in H_1$, the element $U_t\xi$ depends only on t and not on the approximate identity $\{e_\lambda\}$ used to define it. Indeed, this is clear from relation (9).

Now if $\xi \in H_1$

$$\|U_t\xi\| = \lim_\lambda \|\rho(e_\lambda\delta_t)\xi\| \leq \sup \|\rho(e_\lambda\delta_t)\| \, \|\xi\| \leq \sup \|e_\lambda\delta_t\|_1 \, \|\xi\| = \sup \|e_\lambda\|_{\mathcal{C}} \, \|\xi\| \leq \|\xi\|.$$

Hence U_t is a contraction on H_1 and therefore extends to a contraction on H. Also, we have

$$U_tU_s = U_{ts} \quad \text{for all } \; t, s \in G.$$

Indeed, for all $\xi \in H_1$,

$$U_tU_s\xi = U_t(\lim_\lambda \rho(e_\lambda\delta_s)\xi) = \lim_\lambda U_t(\rho(e_\lambda\delta_s)\xi) = \lim_\lambda \rho(\alpha_t(e_\lambda)\delta_{ts})\xi) = U_{ts}\xi$$

because for each fixed t, the net $\{\alpha_t(e_\lambda)\}_\lambda$ is a contractive approximate identity. Thus $U_tU_s - U_{ts}$ vanishes on H_1 and hence on H.

Since obviously $U_e = I$, we have that each U_t is invertible with inverse $U_{t^{-1}}$. Now U_t is a contraction and its inverse is also a contraction, so it must be an isometry, and so, being onto, it is unitary. We have shown that $t \to U_t$ is a unitary representation of G.

Having defined (π, U), we verify covariance:

$$\begin{aligned} U_t\pi(b)U_t^*(\rho(a\delta_s)\xi) &= U_t\pi(b)U_{t^{-1}}(\rho(a\delta_s)\xi) = U_t\rho(b\delta_e)\rho(\alpha_{t^{-1}}(a)\delta_{t^{-1}s})\xi \\ &= U_t\rho(b\alpha_{t^{-1}}(a)\delta_{t^{-1}s})\xi = \rho(\alpha_t(b\alpha_{t^{-1}}(a))\delta_{tt^{-1}s})\xi \\ &= \rho(\alpha_t(b)a\delta_s)\xi = \rho(\alpha_t(b)\delta_e)\rho(a\delta_s)\xi \\ &= \pi(\alpha_t(b))\rho(a\delta_s)\xi \end{aligned}$$

which shows (by linearity) that $U_t\pi(b)U_t^* = \pi(\alpha_t(b))$ on H_1, hence on H by continuity. Moreover, if $\eta = \rho(b\delta_s)\xi$,

$$\begin{aligned} (\pi \times U)(a\delta_t)\eta &= (\pi \times U)(a\delta_t)(\rho(b\delta_s)\xi) = \pi(a)U_t(\rho(b\delta_s)\xi) \\ &= \rho(a\delta_e)\rho(\alpha_t(b)\delta_{ts})\xi = \rho(a\alpha_t(b)\delta_{ts})\xi = \rho(a\delta_t)\rho(b\delta_s)\xi = \rho(a\delta_t)\eta \end{aligned}$$

showing that $\pi \times U = \rho$. $\square$

The (full) C*-crossed product $\mathcal{C} \times_\alpha G$. If $(\mathcal{C}, \alpha, G)$ is a C*-dynamical system, we define a norm $\|\cdot\|_*$ on the algebra $\ell^1(G, \alpha, \mathcal{C})$ as follows: Let Π be the set of all non-degenerate *-representations of $\ell^1(G, \alpha, \mathcal{C})$ on Hilbert space. Recall that these are automatically $\|\cdot\|_1$-contractive and are all of the form $\rho = \pi \times U$, where (π, U) is a covariant *-representation of $(\mathcal{C}, \alpha, G)$. For $a \in \ell^1(G, \alpha, \mathcal{C})$, we define

$$\|a\|_* = \sup\{\|\rho(a)\| : \rho \in \Pi\}.$$

Thus $\|a\|_* \leq \|a\|_1$. Since each $\|\rho(\cdot)\|$ is a C*-seminorm, clearly $\|\cdot\|_*$ is a C*-seminorm. Moreover, since $\mathcal{C}$ is a C*-algebra, it has a faithful *-representation, and hence Remark 5 shows that there exists a faithful *-representation of $\ell^1(G, \alpha, \mathcal{C})$. It follows that $\|\cdot\|_*$ is in fact a C*-norm on $\ell^1(G, \alpha, \mathcal{C})$.

Definition 7 *The C*-crossed product $\mathcal{C}\times_\alpha G$ is defined to be the completion of $\ell^1(G,\alpha,\mathcal{C})$ with respect to the C*-norm $\|\cdot\|_*$. It is an abstract C*-algebra.*

Example 7 If $G=\mathbb{Z}$, $\mathcal{C}=\mathbb{C}$ and α is the trivial action, then the unitary Λ in Remark 4 is just the bilateral shift on $\ell^2(\mathbb{Z})$, which is unitarily equivalent to multiplication by z on $L^2(\mathbb{T})$. If π_o is the identity representation of $\mathbb{C}$ as operators on $\mathbb{C}$, then (as we show below) the representation $\tilde{\pi}$ in Remark 5 extends to a faithful representation of $\mathbb{C}\times_\alpha\mathbb{Z}$ on $L^2(\mathbb{T})$. If $f=\sum f_k\delta_k$ is in $c_{oo}(\mathbb{Z})$, then $\tilde{\pi}(f)=\sum f_kU^k$ is the operator of multiplication by the function $\sum f_kz^k$, whose norm is precisely the supremum norm of the function. Since such functions are dense in $C(\mathbb{T})$, it follows that $\mathbb{C}\times_\alpha\mathbb{Z}$ is isometrically isomorphic to $C(\mathbb{T})$. The dense subalgebra $\ell^1(\mathbb{Z})$ of $\mathbb{C}\times_{id}\mathbb{Z}$ is mapped by $\tilde{\pi}$ to the *Wiener algebra*, that is the algebra of all $\phi\in C(\mathbb{T})$ whose Fourier series is absolutely convergent. $\square$

Each non-degenerate *-representation of $\ell^1(G,\alpha,\mathcal{C})$ is $\|\cdot\|_*$-contractive, by the very definition of the C*-norm. Thus it extends by continuity to a *-representation of the C*-algebra $\mathcal{C}\times_\alpha G$. Thus each covariant non-degenerate *-representation of $(\mathcal{C},\alpha,G)$ gives rise to a *-representation of the C*-algebra $\mathcal{C}\times_\alpha G$ which will also be denoted $\pi\times U$.

Proposition 8 *The correspondence $(\pi,U)\to\pi\times U$ is bijective between covariant *-representations of $(\mathcal{C},\alpha,G)$ for which π is non-degenerate and non-degenerate *-representations of $\mathcal{C}\times_\alpha G$.*

However it is not true in general that an injective *-representation of $\ell^1(G,\alpha,\mathcal{C})$ has an injective extension to $\mathcal{C}\times_\alpha G$. We will see below that this is true in special cases.

Fourier coefficients Suppose that π_o is a *-representation of $\mathcal{C}$ on H_o, and let $\tilde{\pi}$ be the corresponding *-representation of $\ell^1(G,\alpha,\mathcal{C})$ on $\ell^2(G,H_o)$ (Remark 5). Denote the extension of $\tilde{\pi}$ to $\mathcal{C}\times_\alpha G$ by the same symbol. Each $\phi\in\ell^1(G,\alpha,\mathcal{C})$ is an absolutely convergent sum $\phi=\sum_t\phi(t)\delta_t$ (where $\phi(t)\in\mathcal{C}$). We call $\phi(t)$ **the t-th Fourier coefficient** of ϕ. Note that $\langle\tilde{\pi}(\phi)x\delta_e,y\delta_t\rangle_H=\left\langle\pi_o(\alpha_t^{-1}(\phi(t)))x,y\right\rangle_{H_o}$ (see relation (4)) and so $\|\pi_o(\alpha_t^{-1}(\phi(t)))\|\le\|\tilde{\pi}(\phi)\|$ for each $t\in G$. Now

$$\begin{aligned}\|\phi(t)\|_{\mathcal{C}}=\|\alpha_t^{-1}(\phi(t))\|&=\sup\{\left\|\pi_o(\alpha_t^{-1}(\phi(t)))\right\|:\pi_o\text{ *-repr. of }\mathcal{C}\}\\&\le\sup\{\|\tilde{\pi}(\phi)\|:\tilde{\pi}\text{ *-repr. of }\ell^1(G,\alpha,\mathcal{C})\}=\|\phi\|_*.\end{aligned}$$

Thus for each $t\in G$ the map

$$E_t:\ell^1(G,\alpha,\mathcal{C})\to\mathcal{C}\quad\text{ given by }\ E_t(\phi)=\phi(t)$$

is (linear and) contractive with respect to the C*-norm. Therefore it extends to a contractive linear mapping $E_t:\mathcal{C}\times_\alpha G\to\mathcal{C}$. Thus for each $a\in\mathcal{C}\times_\alpha G$ we have a function $E(a):G\to\mathcal{C}$ given by $E(a)(t)=E_t(a)$. This function is bounded by $\|a\|_*$, so that $E(a) \in \ell^\infty(G, \mathcal{C})$ and $\|E(a)\|_\infty \leq \|a\|_*$. Moreover if $\phi \in c_{oo}(G, \mathcal{C})$ then $E(\phi)$ vanishes outside a finite subset of G, so that $E(\phi) \in c_{oo}(G, \mathcal{C})$. Since $\mathcal{C} \times_\alpha G$ is the closure of $c_{oo}(G, \mathcal{C})$ with respect to the C*-norm, it follows that $E(a) \in c_o(G, \mathcal{C})$ for each $a \in \mathcal{C} \times_\alpha G$.

Definition 8 *For $a \in \mathcal{C} \times_\alpha G$ and $t \in G$, the **t-th Fourier coefficient** of a is $E_t(a)$, where $E_t : \mathcal{C} \times_\alpha G \to \mathcal{C}$ is the contractive linear mapping defined above. The map*

$$E : \mathcal{C} \times_\alpha G \to c_o(G, \mathcal{C}) : a \to E(a)$$
$$\text{where } \; E(a)(t) = E_t(a) \;\; (t \in G)$$

*is a contractive linear mapping, called **the Fourier transform**.*

Note that if $\mathbb{C} \times_{id} \mathbb{Z}$ is identified with $C(\mathbb{T})$, the Fourier transform just defined coincides with the usual Fourier transform.

Suppose that π_o is a *faithful* (hence isometric) *-representation of $\mathcal{C}$. Then the inequality $\|\phi(t)\|_{\mathcal{C}} = \left\|\pi_o(\alpha_t^{-1}(\phi(t)))\right\| \leq \|\tilde{\pi}(\phi)\|$ extends from $\ell^1(G, \alpha, \mathcal{C})$ to $\mathcal{C} \times_\alpha G$; thus $\|E_t(a)\|_{\mathcal{C}} \leq \|\tilde{\pi}(a)\|$. It follows that if $a \in \mathcal{C} \times_\alpha G$ and $\tilde{\pi}(a) = 0$, then $E_t(a) = 0$ for each $t \in G$. Therefore, injectivity of $\tilde{\pi}$ will hold whenever the following condition holds:

Uniqueness : *If $a \in \mathcal{C} \times_\alpha G$ and $E_g(a) = 0$ for each $g \in G$, then $a = 0$.*

Abelian groups We will show that this condition is fulfilled when G is abelian. Let $\Gamma = \widehat{G}$ be the dual group. This is a compact abelian group, hence Haar measure is a probability measure. For each $\gamma \in \Gamma$, define a map $\theta_\gamma : \ell^1(G, \alpha, \mathcal{C}) \longrightarrow \ell^1(G, \alpha, \mathcal{C})$ by

$$\theta_\gamma(\sum_t \phi(t)\delta_t) = \sum_t \phi(t)\gamma(t)\delta_t$$

(recall that $|\gamma(t)| = 1$.)
Claim Each θ_γ extends to an isometric *-automorphism of $\mathcal{C} \times_\alpha G$.
Proof It is clear that θ_γ is a *-automorphism of $\ell^1(G, \alpha, \mathcal{C})$. Thus, for every *-representation π of $\ell^1(G, \alpha, \mathcal{C})$, the map $\pi \circ \theta_\gamma$ is a *-representation. By definition of the norm $\|\cdot\|_*$ of the crossed product, this means that $\|\pi(\theta_\gamma(a))\| \leq \|a\|_*$ for all $a \in \ell^1(G, \alpha, \mathcal{C})$. Since π is arbitrary, we therefore have $\|\theta_\gamma(a)\|_* = \sup_\pi \|\pi(\theta_\gamma(a))\| \leq \|a\|_*$ for all $a \in \ell^1(G, \alpha, \mathcal{C})$. Thus θ_γ is a contraction; since its inverse, $\theta_{\gamma^{-1}}$, is also a contraction, θ_γ is actually isometric. Hence it extends as claimed.

Thus θ defines an action of the group Γ on $\mathcal{C} \times_\alpha G$. The group $\{\theta_\gamma : \; \gamma \in \Gamma\}$ is called the **dual automorphism group**.

Now for $\phi = \sum_t \phi(t)\delta_t \in \ell^1(G, \alpha, \mathcal{C})$ and $s \in G$ we have

$$\int_\Gamma \theta_\gamma(\phi)\gamma(s^{-1})d\gamma = \sum_t \int_\Gamma \phi(t)\gamma(ts^{-1})\delta_t d\gamma = \phi_s\delta_s = E_s(\phi)\delta_s$$

since $\int \gamma(t)d\gamma = 1$ when $t = e$ (the identity of G) and 0 otherwise.[13]

For any $b \in \mathcal{C} \times_\alpha G$, let $b_i \in \ell^1(G, \alpha, \mathcal{C})$ be such that $\|b_i - b\|_* \to 0$ as $i \to \infty$. Since $\theta_\gamma(b_i) \to \theta_\gamma(b)$ as $i \to \infty$ uniformly in γ, it follows that $\int_\Gamma \theta_\gamma(b_i)\gamma(s^{-1})d\gamma \underset{i}{\longrightarrow} \int_\Gamma \theta_\gamma(b)\gamma(s^{-1})d\gamma$. This shows that

$$E_s(b)\delta_s = \int_\Gamma \theta_\gamma(b)\gamma(s^{-1})d\gamma$$

holds for all $b \in \mathcal{C} \times_\alpha G$ and $s \in G$.

Let ω be a continuous linear form on $\mathcal{C} \times_\alpha G$. Denote by $f : \Gamma \longrightarrow \mathbb{C}$ the (continuous) function $f(\gamma) = \omega(\theta_\gamma(b))$. Recall that $C(\Gamma) \simeq C^*(G)$; thus there is $\psi \in C^*(G)$ such that $\hat{\psi} = f$, and we have [14] $\psi(s) = \int_\Gamma f(\gamma)\gamma(s^{-1})d\gamma$. Thus

$$\psi(s) = \int_\Gamma \omega(\theta_\gamma(b))\gamma(s^{-1})d\gamma = \omega\left(\int_\Gamma \theta_\gamma(b)\gamma(s^{-1})d\gamma\right) = \omega(E_s(b)\delta_s)\,.$$

Thus if $\omega(E_s(b)\delta_s) = 0$ for each $s \in G$ then $\psi = 0$ and so $f = \hat{\psi} = 0$. Thus $\omega(\theta_\gamma(b)) = 0$ for each $\gamma \in \Gamma$ and hence $\omega(b) = 0$. The Hahn-Banach theorem now shows

Proposition 9 *If G is an abelian group, each $a \in \mathcal{C} \times_\alpha G$ belongs to the closed linear span of the set $\{E_t(a)\delta_t : g \in G\}$ of 'monomials'. In particular, if $E_t(a) = 0$ for each $t \in G$, then $a = 0$.*

Corollary 10 *Suppose G is abelian. Let $\mathcal{S} \subseteq \mathcal{C} \times_\alpha G$ be a closed subspace with the property that $E_s(a)\delta_s \in \mathcal{S}$ for each $a \in \mathcal{S}$ and $s \in G$. Then*

$$\mathcal{S} = \{a \in \mathcal{C} \times_\alpha G : E(a) \in E(\mathcal{S})\}.$$

Proof It is clear that $\mathcal{S}$ is contained in the right hand side. If conversely $a \in \mathcal{C} \times_\alpha G$ satisfies $E(a) \in E(\mathcal{S})$, then for each $t \in G$ there exists $a_t \in \mathcal{S}$ such that $E_t(a) = E_t(a_t)$. Thus each $E_t(a)\delta_t$ belongs to $\mathcal{S}$. It follows from the Proposition that $a \in \mathcal{S}$. $\square$

We can now prove

Proposition 11 *If G is abelian, the full crossed product $\mathcal{C} \times_\alpha G$ is isomorphic to the reduced crossed product $\mathcal{C} \times_{\alpha r} G$.*

---

[13] Recall that for $\psi = \sum_s \psi(s)\delta_s \in c_{oo}(G, \mathbb{C})$ we defined $\int_\Gamma \hat{\psi}(\gamma)d\gamma = \psi(e)$ where $\hat{\psi}(\gamma) = \sum_s \psi(s)\gamma(s)$. Apply this to $\psi = \delta_t$ (so $\hat{\delta}_t(\gamma) = \gamma(t)$) to obtain $\int_\Gamma \gamma(t)d\gamma = \delta_t(e)$.

[14] Recall that the set $\{\delta_s : s \in G\}$ is an orthonormal basis of $\ell^2(G)$. Since the Fourier transform is unitary, $\{\hat{\delta}_s : s \in G\}$ is an orthonormal basis of $L^2(\Gamma)$. Now $\hat{\delta}_s(\gamma) = \gamma(s)$ and hence, for $\psi = \sum_s \psi(s)\delta_s \in \ell^2(G)$,

$$\psi(s) = \langle \psi, \delta_s \rangle_{\ell^2(G)} = \langle F\psi, F\delta_s \rangle_{L^2(\Gamma)} = \left\langle \hat{\psi}, \hat{\delta}_s \right\rangle_{L^2(\Gamma)} = \int_\Gamma \hat{\psi}(\gamma)\overline{\gamma(s)}d\gamma = \int_\Gamma \hat{\psi}(\gamma)\gamma(s^{-1})d\gamma$$

because $\gamma : G \to \mathbb{T}$ is a group homomorphism.

Proof Recall that the reduced crossed product $\mathcal{C} \times_{\alpha r} G$ was defined to be the completion of $c_{oo}(G, \alpha, \mathcal{C})$ with respect to the norm

$$\left\| \sum_t f(t)\delta_t \right\|_r = \|\pi \times \Lambda\| = \left\| \sum_t \pi(f(t))\Lambda_t \right\|_{B(H)}$$

where $H = \ell^2(G, H_o)$ and H_o is the space where $\mathcal{C}$ acts.

Since $\|\cdot\|_r \le \|\cdot\|_*$, the representation $\tilde{\pi} = \pi \times \Lambda$ extends to a representation of the full crossed product $\mathcal{C} \times_\alpha G$ and we have to prove that it is faithful. Recall formula (4):

$$\langle \tilde{\pi}(f) x\delta_e, y\delta_s \rangle_H = \left\langle \pi_o(\alpha_s^{-1}(f(s)))x, y \right\rangle_{H_o}$$

valid for $f = \sum_s f(s)\delta_s \in c_{oo}(G, \alpha, \mathcal{C})$. Since $E_s(f) = f(s)$, we may rewrite this as

$$\langle \tilde{\pi}(f) x\delta_e, y\delta_s \rangle_H = \left\langle \pi_o(\alpha_s^{-1}(E_s(f)))x, y \right\rangle_{H_o}. \tag{10}$$

Now $c_{oo}(G, \alpha, \mathcal{C})$ is $\|\cdot\|_*$-dense in $\mathcal{C} \times_\alpha G$, and the maps $a \to \pi_o(\alpha_s^{-1}(E_s(a)))$ and $\tilde{\pi}$ are $\|\cdot\|_*$-continuous. Therefore relation (10) extends by continuity to $\mathcal{C} \times_\alpha G$:

$$\langle \tilde{\pi}(a) x\delta_e, y\delta_s \rangle_H = \left\langle \pi_o(\alpha_s^{-1}(E_s(a)))x, y \right\rangle_{H_o}, \qquad a \in \mathcal{C} \times_\alpha G.$$

This means that we can prove injectivity of $\tilde{\pi}$ as before: If $a \neq 0$ then, by Proposition 9, there exists $s \in G$ such that $E_s(a) \neq 0$, hence $\alpha_s^{-1}(E_s(a)) \neq 0$. Then we can find $x, y \in H_o$ such that $\langle \pi_o(\alpha_s^{-1}(E_s(a)))x, y \rangle_{H_o} \neq 0$ (π_o is the identity representation) and so $\tilde{\pi}(a) \neq 0$.

References

[1] H. Furstenberg, *Recurrence in Ergodic Theory and Combinatorial Number Theory*, Princeton Univ. Press, Princeton, N.J., 1981.