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A 'short' proof of the Riesz representation theorem

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1. *Introduction.* Textbook proofs of the Riesz theorem on the representation of linear functionals on $C(X)$ by measures tend to be self-contained, but consequently are rather long, and use *ad hoc* methods (see, for example (2, 4, 5)). The purpose of this note is to give a short proof by appealing to standard methods of modern analysis.

Inspection of the proof shows that it depends on the following: (i) the Hahn–Banach theorem, (ii) the Carathéodory procedure for extending a measure from an algebra of sets to the σ-algebra which it generates, (iii) the Stone–Čech compactification of a discrete space, (iv) Urysohn's lemma and (v) the regularity of Baire measures. As far as (iii) is concerned, we need to know that a discrete space S has a Stone–Čech compactification βS which is extremally disconnected ((3), problem 6 M) and which has the property that a mapping from S into a compact space X can be extended to a continuous mapping from βS into X ((3), theorem 6.5).

The form of the theorem which we establish is weak in the sense that we represent linear functionals by Baire measures rather than by regular Borel measures. Let us remark, however, first that we are only interested in integrating continuous functions, so that Baire measures are adequate and natural; secondly, that there is a standard procedure for extending a Baire measure on a compact space to a unique regular Borel measure ((4), chapter X) and thirdly, that the ideas which we employ can in fact be used to simplify this extension procedure a little, as the reader may care to verify.

Another short proof has been given by Varadarajan(6), and has been rediscovered recently by Báez–Duarte (1). This uses the fact that a compact space is the continuous image of a closed subset of a suitable product of discrete two-point sets; careful inspection shows that this proof and ours are somewhat similar, although superficially they appear to be very different.

2. *The Riesz representation theorem.* We prove this in the following form:

THEOREM. *If F is a positive linear functional on the space $C(X)$ of continuous real-valued functions on a compact Hausdorff space X, then there exists a unique Baire measure π on X such that $F(f) = \int_X f d\pi$ for each f in $C(X)$.*

Let S denote the space X with the discrete topology, let βS be the Stone–Čech compactification of S, and let $\phi: \beta S \to X$ be the continuous extension of the identity mapping from S to X. The mapping ϕ defines an isometric mapping T from $C(X)$ (with its usual supremum norm) into $C(\beta S)$: $(Tf)(h) = f(\phi(h))$. Since F is positive, F is continuous and $\|F\| = F(1)$. By the Hahn–Banach theorem there is a continuous linear functional G on $C(\beta S)$ such that $\|G\| = \|F\|$ and such that $G(T(f)) = F(f)$ for all f in $C(X)$. Since $G(1) = G(T(1)) = F(1) = \|F\| = \|G\|$, G is a positive linear functional on $C(\beta S)$.

Now let R denote the algebra of open-and-closed sets in βS. If $A \in R$, the characteristic function χ_A of A is continuous: let $\nu(A) = G(\chi_A)$. ν is clearly a finitely additive non-negative function on R. If (A_i) is a disjoint sequence of sets of R and if

$$\bigcup_{i=1}^{\infty} A_i = A \in R,$$

then only finitely many A_i are non-empty, since the A_i are open and A is compact. Thus ν is trivially σ-additive, and by the Carathéodory procedure ν can be extended to a measure μ on the σ-algebra $S(R)$ generated by R (cf. (4), sections 10–12 or (5), Theorem 4.2). Now if f is a continuous function on X,

$$\{x : f(x) \leqslant c\} = \bigcap_{n=1}^{\infty} \bar{C}_n,$$

where $C_n = \{x : f(x) < c + n^{-1}\}$. C_n is open, so that $\bar{C}_n \in R$, and $\{x : f(x) \leqslant c\} \in S(R)$. Thus f is $S(R)$-measurable, so that $S(R)$ contains the Baire sets; since clearly each set of $S(R)$ is a Baire set, $S(R)$ is the σ-algebra of Baire sets of βS. A similar argument shows that if $g \in C(\beta S)$, $G(g) = \int_{\beta S} g\, d\mu$. Now if B is a Baire set in X, $\phi^{-1}(B)$ is a Baire set in βS; let $\pi(B) = \mu(\phi^{-1}(B))$. Then π is a Baire measure on X and if $f \in C(X)$

$$F(f) = G(T(f)) = \int_{\beta S} T(f)\, d\mu = \int_X f\, d\pi.$$

The uniqueness of π follows by using the regularity of π (cf. (4), theorem 52G) and Urysohn's lemma in the usual way.

Finally we remark that the theorem can be extended in the usual way to represent arbitrary continuous linear functionals, and to the complex case.

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