Ο $\mathbf{2}^{\mathbb{N}}$ ειναι «καθολικος» συμπαγης μετρικος χωρος

Πρόταση 1. *Καθε συμπαγης μετρικος χωρος X ειναι συνεχης εικονα του συνολου $\mathbf{2}^{\mathbb{N}}$: υπαρχει μια συνεχης και επι απεικονιση $p : \mathbf{2}^{\mathbb{N}} \to X$.*

Απόδειξη. Claim. There is a sequence $\{m_n : n \in \mathbb{N}\}$ of natural numbers and for each $n \in \mathbb{N}$ a family $\{M_\tau : \tau \in \{0,1\}^{m_n}\}$ of closed subsets of X such that

(i) $X = \bigcup\{M_\tau : \tau \in \{0,1\}^{m_1}\}$

(ii) For each $n \in \mathbb{N}$ and each $\tau = (\tau(1), \dots, \tau(n)) \in \{0,1\}^{m_n}$, we have

- $\operatorname{diam}(M_\tau) \le \dfrac{1}{n}$
- $M_\tau = \bigcup\{M_\sigma : \sigma \in \{0,1\}^{m_{n+1}}\}$

where $\sigma(i) = \tau(i)$ for $i \le m_n$ and $\sigma(i) = \tau(m_n)$ for $m_n < i < m_{n+1}$.

Proof of the Claim. For $n = 1$: The space X is totally bounded, so it can be covered by a finite number of open sets of diameter at most 1. Allowing repetitions if necessary, we may assume that the required number is 2^{m_1} for some $m_1 \in \mathbb{N}$; noting that 2^{m_1} is the number of points in $\{0,1\}^{m_1}$, we may index the closures M_τ of these sets by $\tau \in \{0,1\}^{m_1}$, so $\tau = (\sigma_1, \dots, \sigma_{m_1})$ with $\sigma_i \in \{0,1\}$.

For $n = 2$: Each M_τ is totally bounded, so it can be covered by finitely many closed sets $\{M_{\tau,\sigma} : \sigma \in [2^{k_1}]\}$, each with diameter at most 1/2. Allowing repetitions if necessary, we may assume that each M_τ is covered by the same number of sets. So now
$(\tau, \sigma) \in \{0,1\}^{m_1} \times \{0,1\}^{k_1} = \{0,1\}^{m_2}$ where $m_2 = m_1 + k_1$ and $(\tau, \sigma) = (\sigma_1, \dots, \sigma_{m_2})$ with $\sigma_i \in \{0,1\}$.

Induction step: having constructed $\{M_\tau : \tau \in \{0,1\}^{m_n}\}$ as in the claim, each M_τ is totally bounded, so it can be covered by finitely many closed sets $\{M_{\tau,\sigma} : \sigma \in [2^{k_n}]\}$, each with diameter at most $\frac{1}{n+1}$. Allowing repetitions if necessary, we may assume that each M_τ is covered by the same muber of sets, so we may write
$(\tau, \sigma) \in \{0,1\}^{m_n} \times \{0,1\}^{k_n} = \{0,1\}^{m_{n+1}}$ where $m_{n+1} = m_n + k_n$.

This proves the Claim. Now for any $\sigma = (\sigma_1, \sigma_2, \dots) \in \{0,1\}^{\mathbb{N}}$ we have a sequence (branch of a tree)

$$M_{(\sigma_1,\dots,\sigma_{m_1})} \supseteq M_{(\sigma_1,\dots,\sigma_{m_2})} \supseteq M_{(\sigma_1,\dots,\sigma_{m_n})} \supseteq \dots$$

of closed sets of diameter $\operatorname{diam}(M_{(\sigma_1,\dots,\sigma_{m_n})}) \le \frac{1}{n}$.

Since X is a compact metric space, by Cantor there is a unique $x_\sigma \in X$ such that

$$\bigcap_{n\in\mathbb{N}} M_{\sigma_1,\dots,\sigma_{m_n}} = \{x_\sigma\}.$$

Thus we have a well defined map

$$p : \mathbf{2}^{\mathbb{N}} \to X : \sigma \mapsto x_\sigma .$$

Θα δείξω ότι η p ειναι επί του X.

Let $x \in X$. Since $X = \bigcup\{M_\tau : \tau \in \{0,1\}^{m_1}\}$, there exists a (not necessarily unique) $\tau \in \{0,1\}^{m_1}$ s.t. $x \in M_\tau = M_{(\sigma_1,\dots,\sigma_{m_1})}$. Since $\{M_{\tau,\sigma} : \sigma \in [2^{k_1}]\}$ is a cover for this M_τ, there exists a $\sigma \in [2^{k_1}]$ so that $x \in M_{\tau,\sigma} = M_{(\sigma_1,\dots,\sigma_{m_2})}$.

Continuing inductively we see that there exists $\sigma = (\sigma_1, \dots, \sigma_n, \dots) \in \mathbf{2}^{\mathbb{N}}$ such that for every $n \in \mathbb{N}$ we have $x \in M_{(\sigma_1,\dots,\sigma_{m_n})}$ and so

$$x \in \bigcap_{n\in\mathbb{N}} M_{(\sigma_1,\dots,\sigma_{m_n})} = \{x_\sigma\} = \{p(\sigma)\}.$$

Thus $x = p(\sigma)$; p is a surjection.

Θα δείξω ότι η p ειναι συνεχης.

Suppose (σ^i) is a sequence of elements of $\mathbf{2}^{\mathbb{N}}$ which converges to $\sigma \in \mathbf{2}^{\mathbb{N}}$. This means equivalently, by definition of the product topology, that $|\sigma^i_n - \sigma_n| \to 0$ for all $n \in \mathbb{N}$.

We will show that

$$\lim_i d(p(\sigma^i), p(\sigma)) = 0$$

where d is the metric on X.

Since $|\sigma^i_n - \sigma_n| \to 0$ for all n, there is i_n such that $|\sigma^i_n - \sigma_n| < \frac{1}{2}$ when $i \geq i_n$ and hence $|\sigma^i_n - \sigma_n| = 0$ when $i \geq i_n$ (because $|\sigma^i_n - \sigma_n| \in \{0,1\}$).

Let $\varepsilon > 0$. Choose $k \in \mathbb{N}$ with $\frac{1}{k} < \varepsilon$ and let $j_k := \max\{i_n : n \leq m_k\}$. Thus we have

$$i \geq j_k \;\Rightarrow\; \sigma^i_n = \sigma_n \;\text{ for }\; n \leq m_k\,.$$

But then

$$M_{\sigma^i_1,\sigma^i_2,\dots,\sigma^i_{m_k}} = M_{\sigma_1,\sigma_2,\dots,\sigma_{m_k}}$$

and therefore, by the definition of the function p,

$$p(\sigma^i) \in M_{\sigma_1,\sigma_2,\dots,\sigma_{m_k}}$$

for all $i \geq j_k$. Since both $p(\sigma^i)$ and $p(\sigma)$ belong to $M_{\sigma_1,\sigma_2,\dots,\sigma_{m_k}}$, it follows that

$$d(p(\sigma^i), p(\sigma)) \leq \operatorname{diam}(M_{\sigma_1,\sigma_2,\dots,\sigma_{m_k}}) \leq \frac{1}{k} < \varepsilon.$$

We have shown that given $\varepsilon > 0$ there exists $j_k \in \mathbb{N}$ (depending on ε) such that

$$i \geq j_k \;\Rightarrow\; d(p(\sigma^i), p(\sigma)) < \varepsilon,$$

όπως θέλαμε. $\square$