On the Riesz-Markov Representation Theorem

1. Let X be a compact Hausdorff space. Each *positive* regular Borel measure μ on X defines a *positive* linear form

$$\varphi_\mu : C(X) \to \mathbb{C} : f \mapsto \int_X f d\mu .$$

(Recall that φ_μ is automatically $\|\cdot\|_\infty$-continuous and $\|\varphi_\mu\| = \varphi_\mu(1)$.)

2. Uniqueness The measure μ is uniquely determined by φ_μ. In other words,

Πρόταση 1. *If μ and ν are regular Borel measures on X and*

$$\int_X f d\mu = \int_X f d\nu \quad \textit{for all } f \in C(X)$$

then $\mu = \nu$.

Απόδειξη. Let $A \subseteq X$ be a Borel set. We show that $\mu(A) = \nu(A)$.

Given $\varepsilon > 0$, by regularity of μ there is a compact set K_μ and an open set V_μ such that $K_\mu \subseteq A \subseteq V_\mu$ and $\mu(V_\mu) - \mu(K_\mu) < \varepsilon$ (recall that μ is finite); and likewise for ν. Replacing K_μ and K_ν be their union K, and replacing V_μ and V_ν by their intersection V, we have a compact set K and an open set V such that

$$K \subseteq A \subseteq V \text{ and } \mu(V) - \mu(K) < \varepsilon \text{ and } \nu(V) - \nu(K) < \varepsilon.$$

By Urysohn, there is a continuous function $f : X \to [0,1]$ such that
- $f|_K = 1$ so $\chi_K \leq f$, and
- $f|_{V^c} = 0$ so $f \leq \chi_V$.

Since $\chi_K \leq f \leq \chi_V$ and the measures are positive, we get

$$\mu(K) = \int \chi_K d\mu \leq \int f d\mu \leq \int \chi_V d\mu = \mu(V) .$$

Combining with $\mu(K) \leq \mu(A) \leq \mu(V)$ yields

$$\left| \int f d\mu - \mu(A) \right| \leq \mu(V) - \mu(K) < \varepsilon$$

and similarly,

$$\left| \int f d\nu - \nu(A) \right| \leq \nu(V) - \nu(K) < \varepsilon .$$

But since $\int_X f d\mu = \int_X f d\nu$, these inequalities give

$$|\mu(A) - \nu(A)| < 2\varepsilon.$$

Αφου το ε ηταν τυχον, δειξαμε οτι $\mu(A) = \nu(A)$, οπως θελαμε.

We would like to prove the converse of **(1)**:

Θεώρημα 2. *If X is a compact Hausdorff space and $\varphi : C(X) \to \mathbb{C}$ a positive linear form, there is a (unique) positive regular Borel measure μ on X such that*

$$\varphi = \int_X f d\mu \quad \textit{for all } f \in C(X).$$

For convenience, henceforth we normalize φ (dividing by $\varphi(\mathbf{1})$ if needed) so that

$$\varphi(\mathbf{1}) = 1$$

and then the required μ should be a probability measure.

3. The case of a discrete X

Now $X = \{x_1, \ldots, x_n\}$ for some $n \in \mathbb{N}$ (X is compact and discrete). Every function on X is continuous, so $C(X) = \ell^\infty[n] = \mathbb{C}^n$. Thus every $f \in C(X)$ is determined by a finite sequence

$$f \rightsquigarrow (f(x_1), \ldots, f(x_n)) \in \ell^\infty_n$$

and φ is determined by its values on the usual basis of ℓ^∞_n

$$\varphi \rightsquigarrow (\varphi(e_1), \ldots, \varphi(e_n)) \in \ell^1_n$$

where $e_j(x) = 1$ when $x = x_j$ and $e_j(x) = 0$ otherwise (i.e. $e_j = \chi_{\{x_j\}}$). Indeed,

$$\varphi(f) = \varphi(\sum_j f(x_j)e_j) = \sum_j f(x_j)\varphi(e_j)\,.$$

Positivity of φ is equivalent to $\varphi(e_j) \geq 0$ for all j. If we define

$$\mu(\{x_j\}) = \varphi(e_j) \quad \text{for all } j,$$

equivalently,

$$\mu(A) = \sum\{\varphi(e_j) : x_j \in A\}$$

for every subset A of X, then we have

$$\varphi(f) = \sum_j f(x_j)\varphi(e_j) = \sum_j f(x_j)\mu(\{x_j\}) = \int f d\mu$$

for every $f \in C(X)$, as required. □

Remark The crucial point is that $C(X)$ contains 'enough' characteristic functions (they span $C(X)$ linearly).

4. The case $X = \mathbf{2}^{\mathbb{N}}$ [1]

The space
$$X = \{x : \mathbb{N} \to \{0,1\}\}$$
is the Cartesian product of a countable number of discrete spaces, hence a compact metrisable space with the product topology. This is the weakest topology on X making all the coordinate projections continuous; equivalently it is the weakest topology on X making all the projections
$$\pi_n : \mathbf{2}^{\mathbb{N}} \to \mathbf{2}^n : (x(k)) \mapsto (x(1), \dots, x(n))$$
continuous.

Define the algebra $\mathcal{A}$ of all *cylinder sets*
$$\mathcal{A} := \bigcup_{n\in\mathbb{N}} \{\pi_n^{-1}(E_n) : E_n \subseteq \mathbf{2}^n\}.$$
Note that since $\mathbf{2}^n$ is discrete, every $E_n \subseteq \mathbf{2}^n$ is open and closed (hence Borel). Clearly $\mathcal{A}$ is an algebra of sets (closed under finite unions, intersections and complements) since the power set of every $\mathbf{2}^n$ is an algebra of sets.

Since every $A \in \mathcal{A}$ is open and closed, its characteristic function is continuous: $\chi_A \in C(X)$, hence we may define
$$\mu_0(A) := \varphi(\chi_A), \quad A \in \mathcal{A}.$$
It is clear that the set function μ_0 is positive, finitely additive on $\mathcal{A}$ and $\mu_0(\emptyset) = \varphi(0) = 0$.

Claim The set function μ_0 is countably additive on $\mathcal{A}$.

Proof Let $A_n \in \mathcal{A}$, $n \in \mathbb{N}$ be pairwise disjoint and suppose that their union
$$A := \bigcup_{n=1}^{\infty} A_n$$
belongs to $\mathcal{A}$. Then A is a closed, hence a *compact* set, and $\{A_n : n \in \mathbb{N}\}$ is a cover of A by *open* sets (recall that $\mathcal{A}$ consists of *clopen* sets). Hence it must have a finite subcover: there exists $N \in \mathbb{N}$ so that
$$A = \bigcup_{n=1}^{N} A_n.$$
Hence
$$\mu_0(A) = \sum_{n=1}^{N} \mu_0(A_n)$$
by finite additivity of μ_0. But since the family $\{A_n : n \in \mathbb{N}\}$ is pairwise disjoint and its first N members already cover A, the remaining $\{A_n : n \geq N+1\}$ must all be empty and so $\mu_0(A_n) = 0$ for all $n \geq N+1$. Thus the last equality gives
$$\mu_0(A) = \sum_{n=1}^{N} \mu_0(A_n) = \sum_{n=1}^{\infty} \mu_0(A_n)$$

[1] not discrete, but totally disconnected

which proves countable additivity of μ_0 on $\mathcal{A}$.

Now apply *Caratheodory's Extension Theorem [Fol, Theorem 4.14]*: There exists a unique (recall that $\mu_0(X) < \infty$) positive countably additive measure μ defined on the Borel subsets of X which extends μ_0, i.e. satisfies

$$\mu(A) = \mu_0(A) \quad \text{for all} \quad A \in \mathcal{A}.$$

Regularity of μ is automatic: every Borel measure on a compact metric space is regular [KouNeg, Theorem 4.17].

Claim For all $f \in C(X)$,

$$\varphi(f) = \int_X f d\mu .$$

Proof The measure μ defines a positive linear functional φ_μ on $C(X)$ by integration. The given functional φ agrees with φ_μ on all characteristic functions of sets in $\mathcal{A}$ by the definition of μ_0:

$$\varphi(\chi_A) = \mu_0(A) = \mu(A) = \int_X \chi_A d\mu, \quad A \in \mathcal{A}.$$

Hence, by linearity, $\varphi(f) = \varphi_\mu(f)$ for all $f \in \text{span}(\mathcal{A})$. But

The space $\text{span}(\mathcal{A}) \subseteq C(X)$:

- is an algebra (since $\chi_A\chi_B = \chi_{A\cap B}$ and $\chi_A + \chi_B = \chi_A + \chi_B - \chi_A\chi_B$)
- contains constants (since $\mathbf{1} = \chi_X$ and $X \in \mathcal{A}$)
- is selfadjoint (since it is the linear span of the selfadjoint elements $\chi_A, A \in \mathcal{A}$)
- separates points of X (since if $x, y \in X$ are distinct, there is an $n \in \mathbb{N}$ such that $\pi_n(x) \neq \pi_n(y)$, so taking $A = \pi_n^{-1}(E_n)$ where $E_n = \{\pi_n(x)\}$ we have $\chi_A(x) = 1$ while $\chi_A(y) = 0$).

Therefore, by the *Stone - Weierstarss Theorem*, $\text{span}(\mathcal{A})$ is sup-norm dense in $C(X)$.

Since both φ and φ_μ are *continuous on* $C(X)$ and agree on the dense space $\text{span}(\mathcal{A})$, they must be equal, όπως θέλαμε. $\square$

Remark The crucial point is that $C(X)$ contains 'enough' characteristic functions (they span a dense subspace of $C(X)$).

5. The case of a compact *metric* space X

There exists a continuous surjection

$$p : \mathbf{2}^{\mathbb{N}} \to X$$

(see `cpctmetric.pdf`). In the sequel we write Y for $\mathbf{2}^{\mathbb{N}}$ for brevity.

The map p induces a map

$$p^* : C(X) \to C(Y) : f \mapsto f \circ p .$$

This is clearly a *-homomorphism, and it is 1-1, since p is onto (verifications are immediate).

Considering $C(X)$ as a C*-subalgebra of $C(Y)$ (via p^*), the map

$$\varphi : C(X) \to \mathbb{C}$$

has a linear Hahn-Banach extension

$$\widetilde{\varphi} : C(Y) \to \mathbb{C}$$

with the same norm: $\|\widetilde{\varphi}\| = \|\varphi\| = 1$. Thus $\widetilde{\varphi}(\mathbf{1}) = \varphi(\mathbf{1}) = 1$. As we know, [2] the equality $\|\widetilde{\varphi}\| = \widetilde{\varphi}(\mathbf{1})$ implies that the functional $\widetilde{\varphi}$ is *positive*.

Therefore, since $Y = \mathbf{2}^{\mathbb{N}}$, by *Case 4* there exists a Borel probability measure $\widetilde{\mu}$ on Y such that

$$\widetilde{\varphi}(g) = \int_Y g(y)d\widetilde{\mu}(y) \quad \text{for all } \; g \in C(Y).$$

Now for each $f \in C(X)$ we have (noting that we have identified $C(X)$ with its image $p^*(C(X))$ in $C(Y)$)

$$\begin{aligned} \varphi(f) = \widetilde{\varphi}(p^*(f)) &= \int_Y p^*(f)d\widetilde{\mu} \\ &= \int_Y (f \circ p)d\widetilde{\mu} \\ &= \int_X fd(\widetilde{\mu} \circ p^{-1}) \end{aligned}$$

where in the last line we have used the familiar 'change of variable' formula which is easily verified. [3]

Therefore if we define the Borel probability measure μ on X by

$$\mu(A) := \widetilde{\mu}(p^{-1}(A)), \quad A \subseteq X \; \text{ Borel}$$

we finally have the required equality

$$\varphi(f) = \int_X fd\mu \quad \text{for all } \; f \in C(X).$$

□

Αναφορές

[Fol] Gerald B. Folland. *Real analysis*. Pure and Applied Mathematics (New York). John Wiley & Sons, Inc., New York, second edition, 1999.

[KouNeg] George Koumoullis, Stelios Negrepontis, *Measure Theory*, Symmetria Publications, Athens 2005.

[2] see for example arvext23.pdf, Πρόταση 1

[3] It suffices to check the equality $\int_Y (f \circ p)d\widetilde{\mu} = \int_X fd(\widetilde{\mu} \circ p^{-1})$ when f is the characteristic function of a Borel subset of X, in which case it follows immediately from the definitions.