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COMPACT OPERATORS AND THE GEOMETRIC STRUCTURE OF C^*-ALGEBRAS

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ABSTRACT. Given a C^*-algebra $\mathcal{A}$ and an element $A \in \mathcal{A}$, we give necessary and sufficient geometric conditions equivalent to the existence of a representation $(\phi, \mathcal{H})$ of $\mathcal{A}$ so that $\phi(A)$ is a compact or a finite-rank operator. The implications of these criteria on the geometric structure of C^*-algebras are also discussed.

Given a C^*-algebra $\mathcal{A}$ and an element A of $\mathcal{A}$, we give necessary and sufficient geometric conditions equivalent to the existence of a representation $(\phi, \mathcal{H})$ of $\mathcal{A}$ so that $\phi(A)$ is a compact operator. Our work goes into further detail; as we shall see, we can characterize when A can be represented as a finite-rank or a rank-one operator. In order to implement these characterizations we introduce a new Banach space-geometric notion, the geometric rank. Roughly speaking, elements of finite geometric rank lie at the opposite pole of extreme points in the intuitive sense that they are located on the "flat" parts of the boundary of the unit ball. It turns out that the finiteness of the geometric rank for a particular element $A \in \mathcal{A}$ guarantees that A can be represented as a finite-rank operator and vice versa.

There are many ways one can interpret the results of the present work. One can actually view them as necessary conditions for a Banach space $(\mathcal{X}, \| \ \|)$ to be isometrically isomorphic to a C^*-algebra. Indeed, the spatial structure of the compact operators translates into a purely geometric structure (Theorem 3.2). We consider this as one of the main accomplishments of our work.

There is some work which relates to the one in the present paper. Indeed, the problem of characterizing which elements of a C^*-algebra can be represented as rank-one operators has already attracted some attention (see [3]). However, the existing criteria are of algebraic (and not of geometric) nature.

1. NOTATION AND PRELIMINARIES

If $(\mathcal{X}, \| \ \|)$ is a Banach space, then $\mathcal{X}_a$ denotes the closed ball with center 0 and radius a. If $\mathcal{S}$ is any subset of $\mathcal{X}_1$, then the *contractive perturbations of* $\mathcal{S}$ is the set

$$\mathrm{cp}(\mathcal{S}) = \{x \in \mathcal{X} \mid \|x \pm s\| \leq 1, \ \forall s \in \mathcal{S}\}.$$

The *n-th contractive perturbations of* $\mathcal{S}$ are defined as $\mathrm{cp}^{(n)}(\mathcal{S}) = \mathrm{cp}(\mathrm{cp}^{(n-1)}(\mathcal{S}))$, $n = 2, 3, \dots$. It is clear that $\mathrm{cp}(\mathcal{S})$ is a norm-closed convex subset of $\mathcal{X}_1$. One can also verify that $\mathcal{S} \subseteq \mathrm{cp}^{(2)}(\mathcal{S})$; from this it follows that $\mathrm{cp}^{(3)}(\mathcal{S}) = \mathrm{cp}(\mathcal{S})$. Thus,

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the consideration of the n-th contractive perturbations, for $n \geq 3$, is not of any interest.

Definitions. Let $(\mathcal{X}, \|\cdot\|)$ be a Banach space and x a norm-one element of $\mathcal{X}$. The *geometric rank* of x, denoted as $r_g(x)$, is the dimension of the linear space generated by $\mathrm{cp}^{(2)}(\{x\})$. In particular, x is said to have geometric rank one (i.e. $r_g(x) = 1$) if $\mathrm{cp}^{(2)}(\{x\}) = \{\lambda x, |\lambda| \leq 1\}$.

In this paper we show that the finiteness of the geometric rank of an element x of a C^*-algebra is equivalent to the fact that x can be represented as a finite-rank operator (i.e., there exists a faithful representation $(\phi, \mathcal{H})$ of $\mathcal{A}$ so that the range of $\phi(x)$ is finite dimensional). The spatial rank of the representative for x and $r_g(x)$ do not necessarily coincide; one reason is that the (spatial) rank always depends on the particular representation of the operator algebra while the geometric rank does not. But, even in the case where $\mathcal{X} = B(\mathcal{H})$, the (spatial) rank of an element x and $r_g(x)$ do not coincide (the only exception is the case of rank-one operators).

The reader should pay special notice to the elements of geometric rank one. Indeed, it is easily seen that an element x of a Banach space $(\mathcal{X}, \|\ \|)$ is an extreme point of $\mathcal{X}_1$ iff $\mathrm{cp}^{(2)}(\{x\}) = \mathcal{X}_1$. Hence, the elements of geometric rank one lie at the opposite pole of extreme points. This conceptual distinction between extreme points and elements of geometric rank one becomes visibly evident in the Banach space $\mathbb{R}^3$, equipped with the maximum norm. In this space, the closed unit ball is the cube $\mathcal{Q}$, with vertices $(1,1,1)$, $(1,1,-1)$, etc. The extreme points are the vertices of $\mathcal{Q}$, while the elements of geometric rank one are the centroids of its two-dimensional faces.

Definition. Let $(\mathcal{X}, \|\ \|)$ be a Banach space and let x be a norm-one element of $\mathcal{X}$. Then x is said to be *geometrically compact* if the set $\mathrm{cp}^{(2)}(\{x\})$ is a norm-compact subset of $\mathcal{X}_1$.

The notion of geometric compactness is a mild generalization of the notion of finiteness for the geometric rank. The term "geometric compactness" for a particular "operator" x not only relates to the compactness of its second contractive perturbations but also reflects our expectations. We hope that for large classes of operator algebras, the two notions of compactness coincide. As we shall see, in C^*-algebras this is indeed the case: if $x \in \mathcal{A}$ is geometrically compact, then there exists a faithful representation $(\phi, \mathcal{H})$ of $\mathcal{A}$ so that $\phi(x)$ is a compact operator!

The reader may have already noticed that the second contractive perturbations play a central role in all of our definitions. In general, given a Banach space $(\mathcal{X}, \|\cdot\|)$ and $\mathcal{S} \subseteq \mathcal{X}_1$, a complete characterization of $\mathrm{cp}^{(2)}(\mathcal{S})$ seems to be elusive. In the case of operator algebras, the following theorem will be extremely useful for understanding the basic properties of $\mathrm{cp}^{(2)}(\mathcal{S})$. It was discovered by R. Moore and T. Trent [6] and, independently, by J. Rovnyak [7]. Its proof depends on the Heinz-Kato inequality.

Theorem 1.1. *Let $\mathcal{H}$ be a Hilbert space and let $A \in B(\mathcal{H})_1$. If X is any contraction, then*

$$\|A \pm (I - |A^*|)^{1/2} X (I - |A|)^{1/2}\| \leq 1.$$

In particular, if $\|X\| \leq 1/2$, then $\|A \pm (I - AA^*)^{1/2} X (I - A^*A)^{1/2}\| \leq 1$.

We now use the above theorem in order to produce non-trivial operators in the second contractive perturbations of certain sets.

Proposition 1.2. *Let $\mathcal{A}$ be any collection of operators acting on $\mathcal{H}$. Let $A \in \mathcal{A}_1$ and let $X \in B(\mathcal{H})$ such that $AXA \in \mathcal{A}$. If $\|X\| \le 1/2$, then $AXA \in \operatorname{cp}^{(2)}(\{A\})$.*

Proof. Let B be any element of $\operatorname{cp}(\{A\})$. Since $\|A \pm B\| \le 1$, it follows that

$$A^*A + B^*B + A^*B + B^*A \le I,$$
$$A^*A + B^*B - A^*B - B^*A \le I,$$

and so $A^*A \le I - B^*B$. Douglas' Majorization Theorem [2] shows that there exists a contraction $S \in B(\mathcal{H})$ such that $A = S(I - B^*B)^{1/2}$. In a similar fashion, there exists a contraction $T \in B(\mathcal{H})$ such that $A = (I - BB^*)^{1/2}T$. Thus, $AXA = (I - BB^*)^{1/2}TXS(I - B^*B)^{1/2}$ and since $\|X\| \le 1/2$, Theorem 1.1 shows that $\|B \pm AXA\| \le 1$. Since B is an arbitrary element of $\operatorname{cp}(\{A\})$, the conclusion follows. □

Another application of Theorem 1.1 is the following result; we omit its proof.

Proposition 1.3. *Let $A = \sum_{i=1}^{\infty} \lambda_i e_i \otimes f_i$ be a norm-one compact operator, acting on a Hilbert space $\mathcal{H}$, where $\{e_i\}_{i=1}^{\infty}, \{f_i\}_{i=1}^{\infty}$ are orthonormal sequences and $\{\lambda_i\}_{i=1}^{\infty}$ is a sequence of positive numbers, decreasing to 0. Let $E_k = [\{e_1, e_2, \ldots, e_k\}], F_k = [\{f_1, f_2, \ldots, f_k\}]$ and let $R_k = E_k \vee F_k$. If X is any contraction in $B(\mathcal{H})$, then*

$$\|A \pm (1 - \lambda_k) R_k^{\perp} X R_k^{\perp}\| \le 1.$$

2. Compact operators and C^*-algebras

We emphasize that, in general, a concrete C^*-algebra may not contain any compact operators. This fact should not be considered as a disadvantage in our investigation. Indeed, there is a variety of (non-isomorphic) C^*-algebras which contain an abundance of compact operators (for instance, the C^*-algebras generated by weighted shift operators). On the other hand, our emphasis is on the interplay between compactness and the geometry of the unit ball. We start with an easy lemma.

Lemma 2.1. *Let A be any operator and let a be a positive number. If the set $A(B(\mathcal{H}))_a A$ is norm precompact, then A is a compact operator.*

Proof. Let $\{f_n\}_{n=1}^{\infty}$ be any bounded sequence in $\mathcal{H}$. Without loss of generality we may assume that $\|f_n\| \le a$, for all $n = 1, 2, \ldots$. Let e be any unit vector in $(\operatorname{Ker} A^*)^{\perp}$. For every $n \in \mathbb{N}$, let $X_n = e \otimes f_n$. Then $AX_nA = A^*e \otimes Af_n$. Since $A(B(\mathcal{H}))_a A$ is a precompact set, the sequence $\{A^*e \otimes Af_n\}_{n=1}^{\infty}$ has a convergent subsequence. This is easily seen to imply that the sequence $\{Af_n\}_{n=1}^{\infty}$ has a norm-convergent subsequence. The conclusion now follows. □

Theorem 2.2. *Let $\mathcal{A}$ be a (not necessarily unital) C^*-algebra and let A be a norm-one element of $\mathcal{A}$. Then the following are equivalent:*

(i) *There exists a faithful representation $(\phi, \mathcal{H})$ of $\mathcal{A}$ so that $\phi(A)$ is a compact operator.*
(ii) *A is a geometrically compact element of $\mathcal{A}$.*

In particular, there exists a faithful representation $(\phi, \mathcal{H})$ of $\mathcal{A}$ so that $\phi(\mathcal{A})$ is a finite-rank operator if and only if $r_g(A) < \infty$.

Proof. Let $(\phi, \mathcal{H})$ be a faithful representation of A so that $\phi(A) \in B(\mathcal{H})$ is a compact operator. Let us identify $\mathcal{A}$ with $\phi(\mathcal{A})$.

Without loss of generality we may assume that $\mathcal{A}$ is non-degenerate. Thus, if $\{G_b\}_{b\in B}$ is an approximate unit for $\mathcal{A}$, then the net $\{G_b\}_{b\in B}$ converges strongly to $I \in B(\mathcal{H})$.

We will show that $\mathrm{cp}^{(2)}(\{A\})$ is totally bounded. Let $\varepsilon > 0$; it is enough to produce a finite-rank projection R so that $\max\{\|R^{\perp}X\|, \|XR^{\perp}\|\} \leq \varepsilon$, for all $X \in \mathrm{cp}^{(2)}(\{A\})$.

Let

$$A = \sum_{i=1}^{\infty} \lambda_i e_i \otimes f_i$$

be as in Proposition 1.3. Let $k \in \mathbb{N}$ so that $(1-(1-\lambda_k)^2)^{1/2} \leq \varepsilon$. Let X be an arbitrary element of $\mathrm{cp}^{(2)}(\{A\})$. Observe that both $E_k = [\{e_1, e_2, \ldots, e_k\}]$ and $F_k = [\{f_1, f_2, \ldots, f_k\}]$ belong to $\mathcal{A}$, and so $R_k = E_k \vee F_k$ also belongs to $\mathcal{A}$. Hence, Proposition 1.3 implies that $(1-\lambda_k)R_k^{\perp}G_bR_k^{\perp}$ belongs to $\mathrm{cp}(\{A\})$, for all $b \in \mathbb{B}$. Since $X \in \mathrm{cp}^{(2)}(\{A\})$, we conclude that $\|X \pm (1-\lambda_k)R_k^{\perp}G_bR_k^{\perp}\| \leq 1$, for all $b \in \mathbb{B}$. Taking strong limits in the previous inequality, it follows that $\|X \pm (1-\lambda_k)R_k^{\perp}\| \leq 1$. This implies that

$$X^*X \leq I - (1-\lambda_k)^2 R_k^{\perp},$$

and so

$$\begin{aligned} R_k^{\perp}X^*XR_k^{\perp} &\leq R_k^{\perp}[I-(1-\lambda_k)^2R_k^{\perp}]R_k^{\perp} \\ &= (1-(1-\lambda_k)^2)R_k^{\perp}. \end{aligned}$$

Hence, $\|XR_k^{\perp}\| \leq (1-(1-\lambda_k)^2)^{1/2} \leq \varepsilon$. Similar arguments show that $\|R_k^{\perp}X\| \leq \varepsilon$. Letting $R = R_k$, the conclusion follows.

Conversely, assume that $\mathrm{cp}^{(2)}(\{A\})$ is a norm-compact subset of $\mathcal{A}_1$. Let $\{(\phi_i, \mathcal{H}_i)\}_{i\in I}$ be a maximal family of pairwise inequivalent irreducible representations of $\mathcal{A}$ and let ϕ be the representation $(\sum_{i\in I} \oplus\phi_i, \sum_{i\in I} \oplus\mathcal{H}_i)$. Since all ϕ_i are pairwise disjoint, the w^*-closure of $\phi(\mathcal{A})$ equals $\sum_{i\in I} \oplus B(\mathcal{H}_i)$. In addition, ϕ is faithful since the family $\{(\phi_i, \mathcal{H}_i)\}_{i\in I}$ is maximal. (The representation $(\phi, \sum_{i\in I} \oplus\mathcal{H}_i)$ is the so-called reduced atomic representation of $\mathcal{A}$; see [5] for more details.) Kaplansky's Density Theorem shows now that $\phi(\mathcal{A}_1)$ is w^*-dense in $(\sum_{i\in I} \oplus B(\mathcal{H}_i))_1$ and so $\phi(A)\phi(\mathcal{A}_{1/2})\phi(A)$ is w^*-dense in

$$\sum_{i\in I} \oplus \phi_i(A)B(\mathcal{H}_i)_{1/2}\phi_i(A).$$

However, Proposition 1.2 shows that $\phi(A)\phi(\mathcal{A}_{1/2})\phi(A)$ is contained in the set $\mathrm{cp}^{(2)}(\{\phi(A)\})$, which is, by assumption, a norm-compact set. Thus

$$\left(\sum_{i\in I} \oplus \phi_i(A)B(\mathcal{H}_i)_{1/2}\phi_i(A)\right) \subseteq \mathrm{cp}^{(2)}(\{\phi(A)\})$$

and so $\sum_{i\in I} \oplus\phi_i(A)B(\mathcal{H}_i)_{1/2}\phi_i(A)$ is precompact. Lemma 2.1 shows now that each $\phi_i(A)$, $i \in I$, is a compact operator. In addition, given any $\varepsilon > 0$, one can easily verify that the precompactness of $\sum_{i\in I} \oplus\phi_i(A)B(\mathcal{H}_i)_{1/2}\phi(A)$ implies that $\|\phi_i(A)\| \leq \varepsilon$, for all but finitely many $i \in I$. This shows that $\phi(A)$ is a compact operator. $\square$

If $(\mathcal{X}, \| \, \|)$ is a Banach space and $\mathcal{M}$ a closed subspace of $\mathcal{X}$, then $\mathcal{M}$ is said to be an M-ideal iff $\mathcal{X}^* = \mathcal{M}^* \oplus_1 \mathcal{M}^{\perp}$. It is known that in any C^*-algebra, the M-ideals coincide with the two-sided norm closed ideals (see [4]). This fact, together with Theorem 2.2, can be used to give a purely geometrical characterization for the elements of a C^*-algebra which admit a representation as compact operators.

Corollary 2.3. *Let $\mathcal{A}$ be a C^*-algebra and let A be an element of $\mathcal{A}$. Then the following are equivalent*:

(i) *There is a representation $(\phi, \mathcal{H})$ of $\mathcal{A}$ such that $\phi(A)$ is a non-zero compact operator.*
(ii) *There is a proper M-ideal $\mathcal{J}$ of $\mathcal{A}$ such that $(A+\mathcal{J})/\|A+\mathcal{J}\|$ is a geometrically compact element of $\mathcal{A}/\mathcal{J}$.*

Another application of Theorem 2.2 is the following result, which gives the first indication that the theory of geometric compactness is susceptible to analysis in non-selfadjoint operator algebras as well. Its proof is similar to that of Theorem 2.2.

Corollary 2.4. *Let $\mathcal{A}$ be a norm-closed operator algebra which acts on a Hilbert space $\mathcal{H}$ and contains $C_\infty(\mathcal{H})$, the set of compact operators on $\mathcal{H}$. If $A \in \mathcal{A}$, then A is a compact operator if and only if $A/\|A\|$ is a geometrically compact element of $\mathcal{A}$.*

The result above applies to a variety of operator algebras. For instance, it applies to C^*-algebras generated by weighted shift operators, thus strengthening Theorem 2.2 in that case. It also applies to a variety of non-selfadjoint operator algebras, such as the quasitriangular algebras (see [1]) for a definition).

3. The behavior of the geometric rank

In this section we examine the behavior of the geometric rank over a C^*-algebra $\mathcal{A}$. This behavior is described in detail in Theorem 3.2. We show that the geometric rank is affiliated with the spatial rank (see Lemma 3.1 and the remarks following it) and so its behavior resembles that of the spatial rank.

Outside the class of C^*-algebras, the behavior of the geometric rank may be different from that of the spatial rank; this claim is justified by several examples which occupy the rest of the section.

We have already mentioned that the geometric rank of a finite-rank operator in $B(\mathcal{H})$ does not necessarily equal its spatial rank. We now compute.

Lemma 3.1. *If $A \in B(\mathcal{H})$ is a contraction whose spatial rank is equal to n, then $\dim[\mathrm{cp}^{(2)}(\{A\})] = n^2$.*

Proof. A simple application of the polar decomposition shows that

$$\dim[\mathrm{cp}^{(2)}(\{A\})] = \dim[\mathrm{cp}^{(2)}(\{|A|\})].$$

Thus, without loss of generality, we may assume that A is positive; let P be the range projection of A.

It is clear that $B(P^{\perp}(\mathcal{H})) \oplus 0 \subseteq [\mathrm{cp}(\{A\})]$ and so $[\mathrm{cp}^{(2)}(\{A\})] \subseteq 0 \oplus B(P(\mathcal{H}))$. It suffices to show the the previous inclusion is actually an equality.

Let $A = \sum_{i=1}^{n} \lambda_i e_i \otimes e_i$, where $\{e_i, e_2, \ldots, e_n\}$ is an orthonormal set, and let $X_{ij} = (1/2) e_i \otimes e_j$, $1 \leq i, j \leq n$. Proposition 1.2 shows now that $A X_{ij} A$ belongs to $\mathrm{cp}^{(2)}(\{A\})$. However, $A X_{ij} A = (\lambda_i \lambda_j /2) e_i \otimes e_j$, and the conclusion follows. ☐

Using the previous lemma, one can evaluate the geometric rank of any contraction in some finite-dimensional C^*-algebra. Such a computation should be based on the fact that if $\mathcal{A} = \bigoplus_{i=1}^k M_{n_i}(\mathbb{C})$ and $A = \bigoplus_{i=1}^k A_i \in \mathcal{A}$ then $r_g(A) = \sum_{i=1}^k \dim[\mathrm{cp}^{(2)}(\{A_i\})]$. Thus, one can see that Lemma 3.1 fails in an arbitrary C^*-algebra. The same computation is also used in the following theorem.

Theorem 3.2. *Let $\mathcal{A}$ be a C^*-algebra and let $F(\mathcal{A})$ be the set of all norm-one elements of $\mathcal{A}$ with finite geometric rank. Then*

(i) *If $A, B \in F(\mathcal{A})$, then $(A+B)/\|A+B\| \in F(\mathcal{A})$ and*

$$r_g((A+B)/\|A+B\|) \leq [(r_g(A))^{1/2} + (r_g(B))^{1/2}]^2.$$

(ii) *If $A \in F(\mathcal{A})$, then A is a linear combination of rank-one elements from $\mathcal{A}$.*

(iii) *If A, A_n, $n = 1, 2, \dots$, belong to $F(\mathcal{A})$ and $A = \lim A_n$, then $r_g(A) \leq \liminf r_g(A_n)$.*

Proof. We only prove (iii). First, we claim that if $P \in \mathcal{A}$ has finite geometric rank, then $r_g(PXP/\|PXP\|) \leq r_g(X)$, for any norm-one element $X \in F(\mathcal{A})$. Indeed, Lemma 3.1 shows that the claim is true for $\mathcal{A} = B(\mathcal{H})$, and the remarks succeeding that lemma prove the claim in any finite-dimensional C^*-algebra.

In general, let Q be the finite-rank projections generated by P, the range projection of X and $(\operatorname{Ker} X)^\perp$. Notice that Q belongs to $\mathcal{A}$. In addition, the geometric rank of X (resp. $PXP/\|PXP\|$) in $\mathcal{A}$ coincides with the geometric rank of X (resp. $PXP/\|PXP\|$) in $Q\mathcal{A}Q$. The proof of the claim follows now from the fact that $Q\mathcal{A}Q$ is of finite dimension.

For the proof, notice that the conclusion is satisfied when $\mathcal{A} = B(\mathcal{H})$ (Lemma 3.1) and so the same is true for any finite-dimensional C^*-algebra. In general, let P be the projection generated by the range projection of A and $(\operatorname{Ker} A)^\perp$. Then, using the fact that the algebra $P\mathcal{A}P$ is finite dimensional, we obtain

$$\begin{aligned}\liminf r_g(A_n) &\geq \liminf r_g(PA_nP/\|PA_nP\|)\\ &\geq r_g(PAP/\|PAP\|) = r_g(A),\end{aligned}$$

and the conclusion follows. □

The following examples show that in an arbitrary Banach space, the conclusions of Theorem 3.2 may fail.

Examples. *Failure of property* (i). Consider the space $(l^1, \|\ \|_1)$ and let $A = (1/2, 1/2, \dots)$ and $B = (1/2, -1/2, 0, 0, \dots)$. We claim that both A and B have finite geometric rank in $(l^1, \|\cdot\|_1)$ but $(A+B)/\|A+B\|$ does not.

Indeed, notice that $\mathrm{cp}(\{A\})$ contains B and so, if $X \in \mathrm{cp}^{(2)}(A)$, then $\|X \pm B\| \leq 1$. This forces X to be of the form $X = (x_1, x_2, 0, 0, \dots)$, i.e., $r_g(A) \leq 2$. A similar argument shows that $r_g(B) \leq 2$. However, $A + B/\|A+B\| = (1, 0, 0, \dots)$ which is an extreme point in $(l^1, \|\ \|_1)$; thus $r_g((A+B)/\|A+B\|) = \infty$.

The reader should notice that a finite-dimensional variation of the previous example produces a finite-dimensional Banach space for which the estimate of Theorem 3.2 (i) fails.

Failure of property (ii). Any finite-dimensional Hilbert space $\mathcal{H}, \dim \mathcal{H} \geq 2$, serves as an example.

Failure of property (iii). Let $\mathcal{H} = \mathbb{C}^3$ and let e_1, e_2, e_3 be an orthonormal basis in $\mathcal{H}$. Consider the Banach space of all operators on $\mathcal{H}$, whose matrices with respect to the basis $\{e_1, e_2, e_3\}$ are upper triangular, equipped with the usual operator norm.

Let $A = e_3 \otimes e_1$ and let $A_n = [\frac{1}{n}e_2 + \sqrt{1-(\frac{1}{n})^2}e_3] \otimes e_1$; notice that $A = \lim A_n$. Let us compute the geometric rank of A.

First we need to identify $\operatorname{cp}(\{A\})$. Let X be some element of $\operatorname{cp}(\{A\})$. Then $\|X \pm A\| \leq 1$, and so $XX^* \leq I - AA^*$ and $X^*X \leq I - A^*A$. Thus, an upper triangular matrix X belongs to $\operatorname{cp}(\{A\})$ if $\operatorname{ran} X = \operatorname{ran} XX^* \subseteq [\{e_2, e_3\}]$ and $(\operatorname{Ker} X)^\perp \subseteq [\{e_1, e_2\}]$; in particular $Xe_1 \in [\{e_1, e_2\}]$. However, the upper triangularity of X implies that $Xe_1 \in [\{e_1\}]$ and so $Xe_1 = 0$. For the same reason, $Xe_2 \in [\{e_1, e_2\}]$ and so $Xe_2 = \lambda e_2$, for some scalar λ. Thus,

$$\operatorname{cp}(\{A\}) = \{\lambda e_2 \otimes e_2, |\lambda| \leq 1\}.$$

We are in position now to describe $\operatorname{cp}^{(2)}(\{A\})$. Arguing as above, one can show that $X \in \operatorname{cp}^{(2)}(\{A\})$ if $(\operatorname{Ker} X)^\perp \subseteq [\{e_1, e_3\}]$ and $\operatorname{ran} X \subseteq [\{e_1, e_3\}]$. Since X is upper triangular, $\langle Xe_1, e_3 \rangle = 0$ and so

$$[\operatorname{cp}^{(2)}(\{A\})] = [\{e_1 \otimes e_1, e_1 \otimes e_3, e_3 \otimes e_3\}],$$

i.e., $r_g(A) = 3$.

We claim now that $r_g(A_n) \leq 2$, $n = 1, 2, \ldots$.

Indeed, it is obvious that $[\sqrt{1-(\frac{1}{n})^2}e_2 - \frac{1}{n}e_3] \otimes e_2$ belongs to $\operatorname{cp}(\{A_n\})$. Thus, arguing as above, every element of $\operatorname{cp}^{(2)}(\{A_n\})$ should satisfy

$$(\operatorname{Ker} X)^\perp \subseteq [\{e_1, (\frac{1}{n}e_2 + \sqrt{1-(\frac{1}{n})^2}e_3)\}]$$

and $\operatorname{ran} X \subseteq [\{e_1, e_3\}]$. From these inclusions, it follows that $\operatorname{ran} X \subseteq [\{e_1\}]$; indeed if not, then there exist scalars λ_1, λ_2 and λ_3, with $\lambda_3 \neq 0$, so that $Xe_2 = \lambda_1 e_2$ and $Xe_3 = \lambda_2 e_1 + \lambda_3 e_3$. But then $X(\sqrt{1-(\frac{1}{n})^2}e_2 - \frac{1}{n}e_3) \neq 0$, which contradicts $([\{e_1, (\frac{1}{n}e_2 + \sqrt{1-(\frac{1}{n})^2}e_3)\}])^\perp \subseteq \operatorname{Ker} X$. Thus, $r_g(A_n) \leq 2$ and we are done.

There are several ways one can use Theorem 3.2 in order to describe the geometric structure of certain C^*-algebras. For instance, using the fact that every maximal ideal of a C^*-algebra $\mathcal{A}$ is the kernel of some irreducible representation of $\mathcal{A}$, one has the following.

Corollary 3.3. *Let $\mathcal{A}$ be a GCR C^*-algebra and let $\mathcal{J}$ be a proper maximal M-ideal of $\mathcal{A}$. Then, $F(\mathcal{A}/\mathcal{J}) \neq \varnothing$ and the geometric rank over $\mathcal{A}/\mathcal{J}$ satisfies the properties described in Theorem* 3.2.

Similarly, one can obtain invariants for AF C^*-algebras, by noticing that these spaces are generated by finite-dimensional subspaces, on which the geometric rank behaves as in Theorem 3.2.

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