von Neumann Algebras and Unbounded Operators

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1 Unbounded Operators and their adjoints

1.1 Unbounded operators

Let[1] H, K be Hilbert spaces. In the sequel, by an **operator** A from H to K we will mean a linear transformation defined on a linear manifold $\mathcal{D}(A) \subseteq H$ into K. If A is bounded, then it extends by continuity to the closure $\overline{\mathcal{D}(A)}$, and, defining A to be 0 on $\mathcal{D}(A)^{\perp}$, we may assume that A is everywhere defined. However when A is not bounded, then it is necessary to specify its domain of definition. Operators with different domains may have quite different properties, as we shall see (for example, the operator of differentiation defines different operators on various subspaces of $L^2(\mathbb{R})$).
Recall that the direct sum $H \oplus K$ becomes a Hilbert space with scalar product

$$\langle (x, y), (z, w) \rangle = \langle x, z \rangle + \langle y, w \rangle.$$

Definition 1.1 *The* **graph** *of an operator A is*

$$Gr(A) = \{(x, Ax) : x \in \mathcal{D}(A)\} \subseteq H \oplus K.$$

We say A **extends** *B and write $A \subseteq B$ when $Gr(A) \subseteq Gr(B)$.*

Suppose A is an operator defined on a **closed** subspace $\mathcal{D}(A)$. If A is bounded, it is easy to see that its graph is a closed subspace of $H \oplus K$. The **closed graph theorem** states that the converse also holds. Without the assumption that $\mathcal{D}(A)$ is complete, this is no longer true (see Example 1.1 below). Having a closed graph proves to be a useful substitute for continuity in many cases.

Definition 1.2 *A is a* **closed operator** *if $Gr(A)$ is closed in $H \oplus K$, i.e. if*
$[x_n \in \mathcal{D}(A), x_n \to x$ and $Ax_n \to y$ implies $x \in \mathcal{D}(A)$ and $Ax = y.]$
The set of closed, densely defined operators from H to K will be denoted $\mathcal{C}(H, K)$.

Example 1.1 *On $H = L^2(\mathbb{R})$, let $\mathcal{D}(Q) = \{f \in H : t \to tf(t)$ is in $L^2(\mathbb{R})\}$ and*
$(Qf)(t) = tf(t)$ $(f \in \mathcal{D}(Q))$. Then Q is (not continuous, but it is) closed.

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*Notes for the 1995 Samos Summer School - revised spring 2010
[1]samosf, draft 13 April 2010

Proof. To see that Q is unbounded, let $f_n = \chi_{[n,n+1]}$ and observe that $f_n \in \mathcal{D}(Q)$ and $\|f_n\| = 1$ while $\|Qf_n\| \to \infty$ as $n \to \infty$.
Suppose now that $(h_n, Qh_n) \in Gr(Q)$ and $(h_n, Qh_n) \to (h, g)$. To show $(h, g) \in Gr(Q)$, fix $k \in \mathbb{N}$ and let $P_k f = \chi_{[-k,k]} f$ $(f \in H)$. Note that for each $f \in H$, $P_k f$ vanishes outside a compact set, hence $P_k f \in \mathcal{D}(Q)$. Since $\lim_n Qh_n = g$ and P_k is a bounded operator, we have $\lim_n P_k Q h_n = P_k g$. If Q_k is defined by $(Q_k f)(t) = t\chi_{[-k,k]}(t) f(t)$ $(f \in H)$, then Q_k is an everywhere defined bounded operator (verify!) which extends $P_k Q$. Hence $\lim_n P_k Q h_n = \lim_n Q_k h_n = Q_k h$. Thus $Q_k h = P_k g$, so $\|Q_k h\| = \|P_k g\| \leq \|g\|$. In other words
$$\int_{-k}^{k} |th(t)|^2 dt \leq \|g\|^2$$
for all $k \in \mathbb{N}$, so that $\int_{\mathbb{R}} |th(t)|^2 dt < \infty$, i.e. $h \in \mathcal{D}(Q)$. Thus $P_k Q h = Q_k h = P_k g$ for all k. Thus Qh and g agree a.e. on each $[-k, k]$, and so $Qh = g$, i.e. $(h, g) \in Gr(Q)$. □

A small modification of this example yields a non-closed operator:

Example 1.2 *If $\mathcal{D}(Q_o) = \{f \in H : f$ vanishes a.e. outside a compact set$\}$ and $(Q_o f)(t) = tf(t)$ $(f \in \mathcal{D}(Q_o))$, then clearly Q extends Q_o and an easy modification of the above argument shows that $\overline{Gr(Q_o)} = Gr(Q)$. Thus the operator Q_o is not closed, but it has a closed extension, and in fact Q is the minimal closed extension of Q_o.*

Definition 1.3 *An operator A is* **closable** *iff there exists a closed operator extending A. In this case A has a minimal closed extension denoted $\overline{A}$ which is defined by $Gr(\overline{A}) = \overline{Gr(A)}$.*

Remark 1.3 *A is closable iff $\overline{Gr(A)}$ is a graph, i.e. iff $(0, y) \in \overline{Gr(A)}$ implies $y = 0$.*

Proof: Exercise.

Remark 1.4 *Any bounded operator is closable.*

Indeed if $(0, y) \in \overline{Gr(A)}$ there exists a sequence (x_n, Ax_n) in $Gr(A)$ such that $x_n \to 0$ and $Ax_n \to y$; since A is continuous, $Ax_n \to 0$ and so $y = 0$.

Remark 1.5 *If A is closable, then of course $Gr(A) \subseteq Gr(\overline{A}) = \overline{Gr(A)}$, so $\mathcal{D}(A) \subseteq \mathcal{D}(\overline{A}) \subseteq \overline{\mathcal{D}(A)}$. It is not however generally the case that $\mathcal{D}(\overline{A})$ is closed. This happens iff A is bounded.*

Indeed if A is bounded then so is $\overline{A}$ (why?); replacing A by $\overline{A}$, we may therefore assume that A is closed. For each $x \in \overline{\mathcal{D}(A)}$ there is a sequence x_n in $\mathcal{D}(A)$ such that $x_n \to x$. By continuity, (Ax_n) also converges, say to y. Thus $(x, y) = \lim(x_n, Ax_n) \in \overline{Gr(A)} = Gr(A)$ and so $x \in \mathcal{D}(A)$.
Conversely if $\mathcal{D}(\overline{A})$ is closed then, by the closed graph theorem, $\overline{A}$ is bounded and hence so is its restriction A.

Example 1.6 *On $H = \ell^2(\mathbb{N})$, let $e_\infty = \sum \frac{1}{n} e_n$. If $\mathcal{D}(A) = [e_\infty, e_1, e_2, ...]$ (lin. span) and $A(ae_\infty + \sum_{k=1}^{n} a_k e_k) = ae_\infty$, then A is not closable.*

For the proof, observe that both (e_∞, e_∞) and $(e_\infty, 0)$ belong to $\overline{Gr(A)}$.

1.2 Adjoints

We wish to define the adjoint of an unbounded operator.
Let $A \in \mathcal{B}(H, K)$, the set of all bounded (everywhere defined) operators $A : H \to K$, where H, K are Hilbert spaces. Recall the definition of the **adjoint** A^* of A:
For each $x \in K$, the map $y \to \langle Ay, x\rangle : H \to \mathbb{C}$ is linear and continuous (indeed $|\langle Ay, x\rangle| \leq (\|A\|.\|x\|)\|y\|$); thus by the Riesz representation theorem there exists a unique vector $A^*x \in H$ such that $\langle y, A^*x\rangle = \langle Ay, x\rangle$ for all $y \in H$. One checks that the map $x \to A^*x$ is linear (and continuous).
If A is not bounded, the above argument may fail for some vectors x; the domain $\mathcal{D}(A^*)$ of A^* will be defined to be the set of all $x \in K$ for which the argument works, namely the set of all $x \in K$ such that the map $y \to \langle Ay, x\rangle$ is continuous on $\mathcal{D}(A)$. For such x, this map has a continuous (Hahn-Banach) extension, say ϕ_x, from $\mathcal{D}(A)$ to all of H; by the Riesz representation theorem there exists a vector $\xi_x \in H$ such that $\langle \xi_x, y\rangle = \phi_x(y)$ for all $y \in H$. Thus $\langle \xi_x, y\rangle = \langle x, Ay\rangle$ for all $y \in \mathcal{D}(A)$.
Now *if* $\mathcal{D}(A)$ is dense in H, then this last relation defines the vector ξ_x uniquely; thus we may write $\xi_x = A^*x$. It is easy to check that $\mathcal{D}(A^*)$ is a linear manifold and that the map $A^* : \mathcal{D}(A^*) \to H$ is linear.
However if $\mathcal{D}(A)$ is not dense, ξ_x will depend not only on x and A, but also on the choice of extension of the mapping $y \to \langle Ay, x\rangle$ from $\mathcal{D}(A)$ to H. For this reason, the adjoint A^* of A is only defined for densely defined operators:

Definition 1.4 *If $A : \mathcal{D}(A) \to K$ is a densely defined operator, then its* **adjoint** A^* *is defined on the linear manifold*

$$\mathcal{D}(A^*) = \{x \in K : \mathcal{D}(A) \ni y \to \langle Ay, x\rangle \text{ is continuous}\}$$

by the formula
$$\langle A^*x, y\rangle = \langle x, Ay\rangle, \qquad y \in \mathcal{D}(A).$$
The operator $A^* : \mathcal{D}(A^*) \to H$ is linear.

Remark 1.7 *(i) Note that $x \in \mathcal{D}(A^*)$ iff there exists $K_x < +\infty$ s.t. $|\langle Ay, x\rangle| \leq K_x\|y\|$ for all $y \in \mathcal{D}(A)$.*
(ii) Also, $(x, u) \in Gr(A^*)$ iff $\langle u, y\rangle = \langle x, Ay\rangle$ for all $y \in \mathcal{D}(A)$.

Remark 1.8 *The adjoint A^* need not be densely defined. In fact, $\mathcal{D}(A^*)$ may be $\{0\}$.*

Example. Let $H = \ell^2(\mathbb{N} \times \mathbb{N})$ with orthonormal basis $\{e_{n,m} : n, m \in \mathbb{N}\}$ and let $K = \ell^2(\mathbb{N})$ with its usual basis $\{e_n : n \in \mathbb{N}\}$. Define $\mathcal{D}(A) = [e_{n,m} : n, m \in \mathbb{N}]$ (the linear span of $\{e_{n,m} : n, m \in \mathbb{N}\}$); let $Ae_{n,m} = e_n$ for all m, n and extend linearly [2] to $\mathcal{D}(A)$.
Suppose $y \in \mathcal{D}(A^*)$. Then we claim that $y = 0$. Indeed for each fixed $n \in \mathbb{N}$, since $\{\mathrm{e}_{n,m} : m \in \mathbb{N}\}$ is an orthonormal set we have

$$\sum_{m=1}^{\infty} |\langle A^*y, e_{n,m}\rangle|^2 \leq \|A^*y\|^2.$$

[2] Thus for $x = \sum_{n=1}^{N}\sum_{m=1}^{M} x_{n,m}e_{n,m} \in \mathcal{D}(A)$ we have $Ax = \sum_{n=1}^{N}(\sum_{m=1}^{M} x_{n,m})e_n$, right?

But, for all m, $\langle A^*y, e_{n,m}\rangle = \langle y, Ae_{n,m}\rangle = \langle y, e_n\rangle$ so for the above sum to be finite we must have $\langle y, e_n\rangle = 0$. This holds for all $n \in \mathbb{N}$; since $\{e_n : n \in \mathbb{N}\}$ is a basis of H, it yields $y = 0$. $\Box$

Let
$$V : H \oplus K \to K \oplus H : (x, y) \to (-y, x).$$
It is easy to see that V is a linear isometry satisfying $V^2 = -I$, hence is onto. Thus V is unitary, so if $E \subseteq H \oplus K$ is a subspace, then $(V(E))^\perp = V(E^\perp)$.

Lemma 1.9 *If A is densely defined, then (i) $Gr(A^*) = V(Gr(A))^\perp$ and (ii) $V(Gr(A^*)) = Gr(A)^\perp$.*

Proof. We have $(x, y) \in V(Gr(A))^\perp$ iff $\langle (x, y), (z, w)\rangle = 0$ for all $(z, w) \in V(Gr(A))$ i.e. $\langle (x, y), (-Au, u)\rangle = 0$ for all $u \in \mathcal{D}(A)$. This holds iff $\langle x, Au\rangle = \langle y, u\rangle$ for all $u \in \mathcal{D}(A)$, which is equivalent to $(x, y) \in Gr(A^*)$ (see Remark 1.7).
The other assertion follows since $V^2 = -I$ and $Gr(A) = -Gr(A)$. $\Box$

Proposition 1.10 *Suppose $\mathcal{D}(A)$ is dense in H. Then (i) A^* is closed.
(ii) A is closable iff $\mathcal{D}(A^*)$ is dense. If so $\overline{A} = A^{**}$.*

Proof. (i) If $(x_n, A^*x_n) \in Gr(A^*)$ and $(x_n, A^*x_n) \to (x, y)$, then for all $z \in \mathcal{D}(A)$ we have
$$\langle Az, x\rangle = \lim\langle Az, x_n\rangle = \lim\langle z, A^*x_n\rangle = \langle z, y\rangle$$
so that $z \to \langle Az, x\rangle = \langle z, y\rangle$ is continuous on $\mathcal{D}(A)$. This shows that $x \in \mathcal{D}(A^*)$ and
$$\langle z, A^*x\rangle = \langle Az, x\rangle = \langle z, y\rangle$$
for all z in the dense set $\mathcal{D}(A)$, hence $A^*x = y$.
Alternatively: by the Lemma we have $Gr(A^*) = V(Gr(A))^\perp$. Since V is isometric, it maps closed subspaces to closed subspaces. The result now follows since $(Gr(A))^\perp$ is closed.

(ii) Note that $\overline{Gr(A)} = ((Gr(A))^\perp)^\perp$ since $Gr(A)$ is a linear manifold in $H \oplus K$. By the Lemma, $V(Gr(A^*)) = (Gr(A))^\perp$ so $\overline{Gr(A)} = (V(Gr(A^*)))^\perp$.
If A^* is densely defined, applying the Lemma to A^* we have $Gr(A^{**}) = V(Gr(A^*))^\perp$ and so $\overline{Gr(A)} = Gr(A^{**})$, which shows that A is closable and $\overline{A} = A^{**}$.
Conversely suppose that $\mathcal{D}(A^*)$ is not dense in K and let $y \neq 0$ be in $(\mathcal{D}(A^*))^\perp$. For all $(x, A^*x) \in Gr(A^*)$ we have $\langle (y, 0), (x, A^*x)\rangle = 0$ so $(y, 0) \in (Gr(A^*))^\perp$. Thus $(0, -y) \in V(Gr(A^*))^\perp = \overline{Gr(A)}$, so that $\overline{Gr(A)}$ is not a graph. $\Box$

Remark 1.11 *If $\mathcal{D}(A^*) = K$ then A is bounded.*

Proof. The operator A^* is everywhere defined and closed, hence bounded by the closed graph theorem. But then A^{**} is also everywhere defined and bounded (see the comments preceding Definition 1.4). However, by Proposition 1.10, A^{**} extends A, so A must be bounded.

1.3 Symmetric and selfadjoint operators

Definition 1.5 *A densely defined operator A from H to H is said to be* **symmetric** *or* **Hermitian** *if*

$$\langle Ax, y\rangle = \langle x, Ay\rangle \quad \forall x, y \in \mathcal{D}(A),$$

equivalently iff $A \subseteq A^*$, i.e. iff $\mathcal{D}(A) \subseteq \mathcal{D}(A^*)$ and $Ax = A^*x$ for all $x \in \mathcal{D}(A)$.
A is said to be **selfadjoint** *if $A = A^*$, i.e. if it is symmetric and $\mathcal{D}(A) = \mathcal{D}(A^*)$.*
A is **essentially selfadjoint** *if it is closable and $\overline{A}$ is selfadjoint.*
If A is closed, a linear manifold $\mathcal{D} \subseteq \mathcal{D}(A)$ is called a **core** *for A if A is the closure of $A|_{\mathcal{D}}$.*

Example 1.12 *The operator Q defined in Example 1.1 is selfadjoint, and $\mathcal{D}(Q_{\mathbf{o}})$ is a core.*

Proof. Exercise.

Remark 1.13 *A symmetric unbounded operator cannot be everywhere defined.*

This follows immediately from 1.11.

Remark 1.14 *A symmetric operator may have many, or no, selfadjoint extensions (see Examples 1.27, 1.28, 1.29 below). An operator A has a* unique *selfadjoint extension iff it is essentially selfadjoint. Thus A is essentially selfadjoint iff $A^{**} = A^*$.*

We leave the proofs as exercises.

1.4 Examples

Reminder: Absolutely continuous functions

Definition 1.6 *If $J \subseteq \mathbb{R}$ is an interval, a function $f : J \to \mathbb{C}$ is said to be* **absolutely continuous** *if given $\epsilon > 0$ there exists $\delta > 0$ such that:*
if $[s_i, t_i] \subseteq [a, b]$ are pairwise disjoint intervals with total length $\sum_{i=1}^n |t_i - s_i| < \delta$, then $\sum_{i=1}^n |f(t_i) - f(s_i)| < \epsilon$.
We write $f \in AC(J)$.

For example, if $f \in L^1([a, b])$ then its indefinite integral $F(x) = \int_a^x f(t)dt$ (with respect to Lebesgue measure) is absolutely continuous.

Theorem 1.15 *A function $f : [a, b] \to \mathbb{C}$ is absolutely continuous if and only if it is almost everywhere differentiable* [3]*, its derivative f' is in $L^1([a, b])$ and, for all $x \in [a, b]$,*

$$f(x) - f(a) = \int_a^x f'(t)dt.$$

[3]that is, for almost all $x \in [a, b]$ the limit $\lim\limits_{h\to 0} \frac{f(x+h)-f(x)}{h} \equiv f'(x)$ exists

Example 1.16 Let $L^2([0,1])$ and define three operators as follows:

$$\begin{aligned}
\mathcal{D}(T_1) &= \{f \in H : f \in AC, f' \in L^2\} \\
\mathcal{D}(T_2) &= \{f \in D(T_1) : f(0) = f(1)\} \\
\mathcal{D}(T_3) &= \{f \in \mathcal{D}(T_2) : f(0) = f(1) = 0\} \\
T_k f &= if, \quad f \in \mathcal{D}(T_k),\ k = 1,2,3
\end{aligned}$$

(thus $\quad T_3 \subset T_2 \subset T_1$). Then $\quad T_1^* = T_3, \quad T_2^* = T_2, \quad T_3^* = T_1$.
Hence T_3 is symmetric, non-selfadjoint, and has selfadjoint extensions (for example, T_2), T_2 is selfdjoint and its extension T_1 is not symmetric.

For the proof, see Rudin, Functional Analysis, Example 13.4.

1.5 The inevitability of unboundedness

One form of *Heisenberg's uncertainty principle* states that the observables P and Q representing momentum (P) and position (Q) of a quantum mechanical particle correspond to 'matrices', or operators, which satisfy the famous "canonical commutation relation"

$$(PQ - QP)f = if \quad \text{for all } f \text{ in the common domain of } P \text{ and } Q \qquad \text{(CCR)}$$

One possible representation of this relation is the following:
Let $H = L^2(\mathbb{R})$ and $\mathcal{D} = C_c^\infty(\mathbb{R})$, the infinitely differentiable functions of compact support. Define P and Q on $\mathcal{D}$ by

$$Pf = if', \qquad (Qf)(t) = tf(t), \qquad f \in \mathcal{D}.$$

Note that P and Q are well defined on the linear manifold $\mathcal{D}$, which is invariant under P and Q, and so one can form polynomials in P and Q (such as $PQ - QP$). It is trivial to verify that (CCR) holds for all $f \in \mathcal{D}$.
But the misfortune is that these operators are not bounded!
The question arises:

Is it possible to represent (CCR) by a pair of bounded operators?

The answer is NO:

Proposition 1.17 *If x, y are elements of a normed algebra with identity e (such as $\mathcal{B}(H)$) then*

$$xy - yx \neq e.$$

Proof (Wielandt) Suppose, by way of contradiction, that $xy - yx = e$. It then follows that, for all $n \in \mathbb{N}$, we have

$$x^n y - yx^n = nx^{n-1} \quad \text{and} \quad x^{n-1} \neq 0.$$

Indeed, for $n = 1$ the first statement is just the hypothesis and the second is obvious. If these statements are assumed to hold for some n, then the first equality gives $x^n \neq 0$ and

$$\begin{aligned}
x^{n+1}y - yx^{n+1} &= x^n(xy - yx) + (x^n y - yx^n)x \\
&= x^n e + nx^{n-1}x = (n+1)x^n
\end{aligned}$$

which is the induction step. It follows that

$$0 \neq \left\|nx^{n-1}\right\| = \|x^n y - yx^n\| \leq 2\|x^n\|\|y\| \leq 2\left\|x^{n-1}\right\|\|x\|\|y\|$$

which implies that $n \leq 2\|x\|\|y\|$ for all $n \in \mathbb{N}$, an absurdity.

1.6 The spectrum of an unbounded operator

The inverse of an unbounded operator Let S be a densely defined operator from H to K. Suppose first that S is closed, and bijective as a map $S : \mathcal{D}(S) \to K$. Consider the usual inverse $T : K \to \mathcal{D}(S)$; this is defined everywhere on K by

$$T : y \to x \quad \text{where} \quad x \in \mathcal{D}(S) \text{ and } y = Sx.$$

In other words, $(y, x) \in G(T)$ iff $(x, y) \in G(S)$, that is, $G(T) = \Gamma(G(S))$ where $\Gamma : H \oplus K \to K \oplus H$ is the 'flip' $(x, y) \to (y, x)$. But note that, since $G(S)$ is closed in $H \oplus K$ and Γ is obviously unitary, the subspace $G(T) = \Gamma(G(S))$ is closed in $K \oplus H$. Thus T is an everywhere defined map between Hilbert spaces and its graph is closed; the closed graph theorem shows that T must be a bounded operator. The relation

$$x = Ty \quad \text{iff} \quad x \in \mathcal{D}(S) \text{ and } y = Sx$$

means

$$ST = I_K \quad \text{and} \quad TS \subset I_H.$$

These remarks motivate the following

Definition 1.7 *We say that a linear map $S : \mathcal{D}(S) \to K$* **has an inverse** *if there is a* **bounded** *operator $T : K \to H$ such that*

$$ST = I_K \quad \textit{and} \quad TS \subset I_H.$$

The operator T is uniquely determined by this relation; it is called **the inverse** *of S and is denoted S^{-1}.*

We have shown above that if $S : \mathcal{D}(S) \to K$ is bijective and closed, then it has an inverse - which is bounded. Conversely if there is $T \in \mathcal{B}(K, H)$ satisfying the conditions of the definition, then the first relation gives that T must be 1-1, S must be onto and $\operatorname{ran}(T) \subseteq \mathcal{D}(S)$ (here $\operatorname{ran}(T)$ is the range of T), while the second forces $\operatorname{ran}(T) = \mathcal{D}(S)$. [4] Finally since $G(T)$ is closed (because T is bounded and everywhere defined) $G(S) = \Gamma(G(T))$ must be closed.
Thus only closed operators can have inverses.

Definition 1.8 (Resolvent and Spectrum) *If S is densely defined on $\mathcal{H}$,*

the **resolvent set** *of S:* $\rho(S) = \{\lambda \in \mathbb{C} : \lambda - S = \lambda I - S \textit{ has an inverse}\}$
the **spectrum** *of S:* $\sigma(S) = \mathbb{C} \setminus \rho(S)$.

(Note that $\mathcal{D}(\lambda - S) = \mathcal{D}(S)$.)

Recall that $\mathcal{C}(H)$ denotes the set of closed, densely defined operators on H.

Proposition 1.18 *If $S \in \mathcal{C}(H)$ then $\rho(S)$ is open in $\mathbb{C}$ and the map*

$$\rho(S) \to \mathcal{B}(H) : \lambda \to (\lambda - S)^{-1}$$

is holomorphic: it has a (norm-convergent) power series expansion around each $\lambda_o \in \rho(S)$. Moreover if $S \in \mathcal{B}(H)$ then in fact $\sigma(S)$ is nonempty and compact.

[4] If $x \in \mathcal{D}(S)$ then $TSx = x$ so $x \in \operatorname{ran}(T)$.

Proof. Claim (i) If $S \in \mathcal{B}(H)$ then $\sigma(S)$ is bounded, and in fact $\sigma(S) \subseteq \text{ball}(0, \|\text{S}\|)$. Indeed if $|\lambda| > \|S\|$ then

$$(\lambda - S)^{-1} = \sum_{n=0}^{\infty} \frac{1}{\lambda^{n+1}} S^n \qquad \text{(norm convergence)}$$

Proof: The partial sums of $\sum_n \left(\frac{S}{\lambda}\right)^n$ are norm-Cauchy since $\|S/\lambda\| < 1$.

Claim (ii) If $S \in \mathcal{C}(H)$ then $\rho(S)$ is open.
Proof of Claim. Fix $\lambda \in \rho(S)$ and write $T = (\lambda - S)^{-1}$. For $z \in \mathbb{C}$ with $|z| < \|T\|^{-1}$ (recall T is bounded) note that

$$\lambda - S + z = \lambda - S + z(\lambda - S)(\lambda - S)^{-1} = (\lambda - S)(I + zT).$$

But part (i) shows that $I + zT$ is invertible and

$$(I + zT)^{-1} = \sum_{n=0}^{\infty} (-zT)^n$$

$$\text{so} \quad (\lambda + z - S)^{-1} = (I + zT)^{-1}(\lambda - S)^{-1} = \sum_{n=0}^{\infty} (-1)^n T^{n+1} z^n$$

which is a convergent power series in z.
This shows that $\text{ball}(\lambda, \|\text{T}\|^{-1}) \subseteq \rho(\text{S})$ and the map $u \to (u - S)^{-1}$ has a power series expansion around λ.

Claim (iii) Suppose $S \in \mathcal{B}(H)$. Then $\sigma(S)$ cannot be empty.
Indeed if it were, then by (ii) for each $x, y \in H$ the complex function

$$f : \lambda \to \left\langle (\lambda - S)^{-1} x, y \right\rangle$$

would be entire. Also, when $|\lambda| > \|S\|$ by (i) we would have

$$f(\lambda) = \left\langle \sum_{n=0}^{\infty} \frac{1}{\lambda^{n+1}} S^n x, y \right\rangle = \frac{1}{\lambda} \sum_{n=0}^{\infty} \frac{1}{\lambda^n} \langle S^n x, y \rangle$$

$$\text{so} \quad |f(\lambda)| \leq \frac{1}{|\lambda|} \sum_{n=0}^{\infty} \frac{1}{|\lambda|^n} \|S\|^n \|x\| \|y\| = \frac{1}{|\lambda|} \left(1 - \frac{\|S\|}{|\lambda|}\right)^{-1} \|x\| \|y\|$$

so that $\lim_{|\lambda| \to \infty} |f(\lambda)| = 0$. Hence $f = 0$ by Liouville's theorem.
But since $x, y \in H$ are arbitrary, this would imply that $(\lambda - S)^{-1} = 0$ which is absurd. □

For $z \in \mathbb{C} \setminus \{\lambda, \mu\}$ we have

$$\frac{1}{\lambda - z} - \frac{1}{\mu - z} = \frac{\mu - \lambda}{(\lambda - z)(\mu - z)}.$$

This is also true for operators:

Definition 1.9 *Suppose $S \in \mathcal{C}(H)$ and let $\lambda, \mu \in \rho(S)$. Then*

$$(\lambda - S)^{-1} - (\mu - S)^{-1} = (\mu - \lambda)(\lambda - S)^{-1}(\mu - S)^{-1}.$$

This is called **the resolvent equality**. *Sometimes we write*

$$r_\lambda(S) \equiv (\lambda - S)^{-1}$$

and call the bounded operator-valued function $\lambda \to r_\lambda(S)$ the **resolvent** *of S.*

The proof is the following:

$$\begin{aligned}(\lambda - S)^{-1} - (\mu - S)^{-1} &= (\lambda - S)^{-1}(I - (\lambda - S)(\mu - S)^{-1}) \\ &= (\lambda - S)^{-1}((\mu - S) - (\lambda - S))(\mu - S)^{-1} \\ &= (\lambda - S)^{-1}(\mu - \lambda)(\mu - S)^{-1}.\end{aligned}$$

Corollary 1.19 *Let $S \in \mathcal{C}(H)$. Then*
(i) The map $\lambda \to r_\lambda(S) : \rho(S) \to \mathcal{B}(H)$ is continuous.
(ii) If $\lambda \in \rho(S)$ then $\mu \to r_\mu(S)$ is norm-differentiable at λ, that is, the (norm)-limit

$$\lim_{\mu \to \lambda} \frac{(\mu - S)^{-1} - (\lambda - S)^{-1}}{\mu - \lambda} \quad \textit{exists and equals} \quad -(\lambda - S)^{-2}.$$

Proof Both statements follow from the fact that the map $\lambda \to r_\lambda(S)$ has a local power series; the proof is the same as in elementary complex analysis.
Another proof of part (ii) follows from the resolvent equality: If $\lambda, \mu \in \rho(S)$ and $\lambda \neq \mu$ then

$$\frac{(\mu - S)^{-1} - (\lambda - S)^{-1}}{\mu - \lambda} = -(\lambda - S)^{-1}(\mu - S)^{-1}.$$

But by part (i) we have that $\lim_{\mu \to \lambda} \|(\mu - S)^{-1} - (\lambda - S)^{-1}\| = 0$ and thus when $\mu \to \lambda$ the right hand side converges (in norm) and in fact the limit is $-(\lambda - S)^{-2}$. $\square$

While the spectrum of a *bounded* operator is always bounded and nonempty, this need not be true for unbounded operators:
We show that the spectrum of an unbounded operator can be the whole complex plane or, with a 'small' change in the operator, it can be empty!

Example 1.20 *Let $H = L^2([0, 1]$ and define two operators as follows:*

$$\begin{aligned}\mathcal{D}(S_1) &= \{f \in H : f \in AC\} \\ \mathcal{D}(S_2) &= \mathcal{D}(S_1) \cap \{f \in AC : f(0) = 0)\} \\ S_k f &= if', \quad f \in \mathcal{D}(S_k),\ k = 1, 2\end{aligned}$$

Then $\quad \sigma(S_1) = \mathbb{C} \quad$ *and* $\quad \sigma(S_2) = \emptyset$.

Proof It is an exercise for the reader to prove that these operators are closed and densely defined.
Claim 1: $\sigma(S_1) = \mathbb{C}$; in fact, every $\lambda \in \mathbb{C}$ is an eigenvalue for S_1.

Proof of the Claim Given an arbitrary $\lambda \in \mathbb{C}$, we need to find a nonzero function $f \in \mathcal{D}(S_1)$ such that $(\lambda - S_1)f = 0$, equivalently,

$$\lambda f = if'.$$

But it is trivial to solve this differential equation: separating variables, we find that $f(t) = e^{-i\lambda t}$ is a solution and we observe that f is absolutely continuous, being continuously differentiable.
This shows that $\lambda - S_1$ is not even 1-1. $\square$

Claim 2: $\sigma(S_2) = \emptyset$; *that is, for every* $\lambda \in \mathbb{C}$, *not only is* $\lambda - S_2$ *injective, it also maps* $\mathcal{D}(S_2)$ *onto* H.
Proof of the Claim Given an arbitrary $\lambda \in \mathbb{C}$, we claim that the operator $S_\lambda : H \to H$ defined by

$$(S_\lambda g)(x) = ie^{-i\lambda x}\int_0^x e^{i\lambda s}g(s)ds \qquad g \in L^2([0,1])$$

is the inverse of $(\lambda - S_2)$: that is, we will prove that $(\lambda - S_2)S_\lambda = I$ and $S_\lambda(\lambda - S_2) \subset I$.
Let $g \in L^2([0,1]$ be arbitrary; observe that since $g \in L^1$ and $x \to e^{-i\lambda x}$ is C^1, the function $S_\lambda g$ is absolutely continuous and vanishes at 0, so it is in $\mathcal{D}(S_2)$.
Setting $f = S_\lambda g$ we have $f' = ig - i\lambda f$ so that $(\lambda - S_2)f = \lambda f - if' = g$. Thus

$$(\lambda - S_2)S_\lambda = I.$$

For the second claim, let $h \in \mathcal{D}(S_2)$; then $h \in AC$ and also $h(0) = 0$ so we obtain

$$\begin{aligned}
&(S_\lambda(\lambda - S_2)h)(x) = (S_\lambda(\lambda h - ih'))(x) = ie^{-i\lambda x}\int_0^x e^{i\lambda s}(\lambda h(s) - ih'(s))ds \\
&= i\lambda e^{-i\lambda x}\int_0^x e^{i\lambda s}h(s)ds + e^{-i\lambda x}\int_0^x e^{i\lambda s}h'(s)ds \\
&= i\lambda e^{-i\lambda x}\int_0^x e^{i\lambda s}h(s)ds + e^{-i\lambda x}[e^{i\lambda s}h(s)]_{s=0}^{s=x} - e^{-i\lambda x}\int_0^x (i\lambda)e^{i\lambda s}h(s)ds \\
&= h(x) - e^{i\lambda x}h(0) = h(x)
\end{aligned}$$

Thus $\quad S_\lambda(\lambda - S_2)h = h$ for all $h \in \mathcal{D}(S_2)$, which completes the proof. $\square$

The Cayley transform The mapping

$$\phi : \mathbb{R} \to \mathbb{T} \setminus \{1\} : \ t \to \frac{t-i}{t+i}$$

is a homeomorphic bijection between $\mathbb{R}$ and $\mathbb{T} \setminus \{1\}$. Thus if $T \in \mathcal{B}(H)$ is a (bounded) selfadjoint operator, hence with $\sigma(T) \subseteq \mathbb{R}$, by the Continuous Functional Calculus we may define a bounded operator

$$U = \phi(T) = (T - iI)(T + iI)^{-1}$$

(note that $-i \notin \sigma(T)$ and so $T + iI$ is invertible). Since $\phi(t)^{-1} = \overline{\phi(t)}$, the operator U is unitary: $U^* = \phi(T)^* = \phi(T)^{-1} = U^{-1}$; also $\sigma(U) = \{\phi(\lambda) : \lambda \in \sigma(T)\} \subseteq \mathbb{T} \setminus \{1\}$ by the spectral mapping theorem.

Conversely, every unitary operator U is of the form $\phi(T)$ for some $T = T^* \in \mathcal{B}(H)$, *provided* that $\sigma(U)$ does not contain the point 1; indeed, it suffices to observe that the function $\psi = \phi^{-1} : \mathbb{T} \setminus \{1\} \to \mathbb{R}$ is continuous and apply the Continuous Functional Calculus (for normal operators) to find the required $T = \psi(U)$.

We wish to extend the correspondence $T \longleftrightarrow U$, if possible, to unbounded operators.

Definition 1.10 *For the purposes of these notes, a linear map U from $\mathcal{D}(U) \subseteq H$ to K will be called a* **partial isometry** *iff its restriction to*

$$M \equiv \ker(U)^{\perp} \cap \mathcal{D}(U) = \{x \in \mathcal{D}(U) : [Uy = 0 \Rightarrow \langle x, y\rangle = 0]\}$$

is an isometry. The linear space M is called the **initial space** *and $U(M) = \mathrm{ran}(U)$ is called the* **final space** *of U. If $M = H$ then U is said to be an* **isometry**, *while if $\mathrm{ran}(U) = K$ then U is called a* **coisometry**. *Thus a partial isometry is unitary iff it is both an isometry and a coisometry.*

Note that a partial isometry is necessarily bounded, with norm equal to 1 (if nonzero). Thus U is closable, and its closure $\bar{U}$ is still a partial isometry, with $\mathcal{D}(\bar{U}) = \overline{\mathcal{D}(U)}$. However a partial isometry may have several distinct extensions. For example if $\mathcal{D}(U)$ and $\mathrm{ran}(U)$ are not dense, then any choice of a pair of unit vectors $e \in \mathcal{D}(U)^{\perp}$ and $f \in \mathrm{ran}(U)^{\perp}$ will give a partially isometric extension V of U defined on $\mathcal{D}(V) = [\mathcal{D}(U) \cup \{e\}]$ by $Vx = Ux$ if $x \in \mathcal{D}(U)$ and $Ve = f$.

Proposition 1.21 *If T is a symmetric operator on H, there exists a partial isometry*

$$U(T) : \mathrm{ran}(T + iI) \to \mathrm{ran}(T - iI)$$

given by [5] $$Tx + ix \to Tx - ix, \qquad x \in \mathcal{D}(T).$$

The operator $U(T)$ is called the **Cayley transform** *of T.*
In fact $I - U(T)$ is 1-1.
Also, $\mathcal{D}(T) = \mathrm{ran}(I - U(T))$ and

$$T(I - U(T)) = i(I + U(T)).$$

Thus T can be reconstructed from its Cayley transform.

Proof. Let $x \in \mathcal{D}(T)$. Then

$$\|Tx \pm ix\|^2 = \|Tx\|^2 + \|x\|^2 \pm \langle Tx, ix\rangle \pm \langle ix, Tx\rangle = \|Tx\|^2 + \|x\|^2$$

since $\langle Tx, x\rangle = \langle x, Tx\rangle$ because T is symmetric. Thus $\|Tx + ix\| = \|Tx - ix\|$ and therefore the map

$$U(T) : \mathrm{ran}(T + iI) \to \mathrm{ran}(T - iI) : Tx + ix \to Tx - ix$$

is a well-defined linear isometric bijection.

[5] Thus $U(T)$ coincides on $\mathrm{ran}(T + iI)$ with the linear map $(T - iI)(T + iI)^{-1}$; we avoid this notation because generally $T + iI$ does not have a bounded inverse (see later). The same remarks apply to the relation $T(I - U(T)) = i(I + U(T))$.

To prove the second statement, note that by definition a vector x belongs to $\mathcal{D}(T)$ if and only if $z = Tx + ix$ belongs to $\mathcal{D}(U(T))$, and then $U(T)z = Tx - ix$. Adding and subtracting,

$$(I - U(T))z = 2ix, \qquad (I + U(T))z = 2Tx.$$

Since the correspondence $z \to x$ is bijective, this shows that $I - U(T)$ is 1-1 on $\mathcal{D}(U(T))$, and that $\mathrm{ran}(I - U(T)) = \mathcal{D}(T)$. Thus, for all $z \in \mathcal{D}(U(T))$,

$$T(I - U(T))z = 2iTx = i(I + U(T))z. \qquad \square$$

Remark 1.22 *We have seen that the operator $I - U(T)$ is 1-1, i.e. the number 1 is not an eigenvalue of $U(T)$. In case T is a bounded selfadjoint operator, a stronger property holds: 1 is not in the spectrum of $U(T)$ (as we have seen). However when T is not bounded then $I - U(T)$ cannot be invertible: If $1 \notin \sigma(U(T))$, then T is bounded.*

Indeed if $1 \notin \sigma(U(T))$ and $M = \|(I - U(T))^{-1}\|$, the relation $(I - U(T))(Tx + ix) = 2ix$ for $x \in \mathcal{D}(T)$ gives $\|Tx + ix\| \leq 2M \|x\|$ and so $T + iI$, and hence also T, is bounded.

Lemma 1.23 *If T is selfadjoint, then the operator $Q \equiv I + T^2$ maps $\mathcal{D}(Q) = \mathcal{D}(T^2)$ bijectively onto H, and its inverse $(I + T^2)^{-1}$ is continuous.*

Proof. We have shown in Lemma 1.9 that $Gr(T^*) = V(Gr(T))^{\perp}$. Since $T = T^*$ is a closed operator, this shows that the sets $Gr(T) = \{(x, Tx) : x \in \mathcal{D}(T)\}$ and $V(Gr(T)) = \{(-Tx, x) : x \in \mathcal{D}(T)\}$ are orthogonal closed subspaces of $H \oplus H$ whose sum is $H \oplus H$. Thus for any $y \in H$ there are $x, z \in \mathcal{D}(T)$ such that

$$(0, y) = (z, Tz) + (-Tx, x).$$

Thus $z = Tx$ (so $x \in \mathcal{D}(T^2)$) [6] and $y = Tz + x = T^2x + x$. This shows that Q is onto. Also, for all $x \in \mathcal{D}(T^2)$ we have

$$\|Qx\|^2 = \|x + T^2x\|^2 = \|x\|^2 + \|T^2x\|^2 + 2\langle x, T^2x\rangle = \|x\|^2 + \|T^2x\|^2 + 2\langle Tx, Tx\rangle \geq \|x\|^2.$$

Thus Q is 1-1, and if $y = Qx$ we have $\|Q^{-1}y\| = \|x\| \leq \|Qx\| = \|y\|$ and hence Q^{-1} is continuous. $\square$

Lemma 1.24 *Let T be a symmetric operator. Then T is selfadjoint if and only if its Cayley transform $U(T)$ is unitary.*

Proof. Suppose that T is selfadjoint. For any $x \in \mathcal{D}(T^2)$, we have

$$(T + iI)(T - iI)x = (I + T^2)x = (T - iI)(T + iI)x.$$

It follows that $\mathrm{ran}(T + iI) = \mathrm{ran}(I + T^2) = \mathrm{ran}(T - iI)$. But $\mathrm{ran}(I + T^2) = H$ (Lemma 1.23), and hence $\mathcal{D}(U(T)) = \mathrm{ran}(T + iI) = H$ and $\mathrm{ran}(U(T)) = \mathrm{ran}(T - iI) = H$. Thus $U(T)$ is everywhere defined and onto. Since it is isometric, it is unitary.

Suppose conversely that $U = U(T)$ is unitary. First note that $\mathcal{D}(T) = \mathrm{ran}(I - U)$ is dense in H. Indeed, if $x \perp \mathrm{ran}(I - U)$, then

$$\|(I - U)x\|^2 = \langle (I - U^*)(I - U)x, x\rangle = \langle (I - U)(I - U^*)x, x\rangle = 0.$$

[6] Note that $\mathcal{D}(AB) = \{x \in \mathcal{D}(B) : Bx \in \mathcal{D}(A)\}$.

Since $I-U$ is 1-1 (Proposition 1.21), this shows that $x=0$. Thus T is densely defined, so T^* exists, and a short calculation[7] shows that $T \subseteq T^*$.
Let $y \in \mathcal{D}(T^*)$. We need to show that $y \in \mathcal{D}(T)$. Since $\operatorname{ran}(T+iI)=H$, there exists $x \in \mathcal{D}(T)$ such that $(T^*+iI)y=(T+iI)x$. But $T \subseteq T^*$, hence $(T+iI)x=(T^*+iI)x$. It follows that $(T^*+iI)(x-y)=0$, and hence, for all $z \in \mathcal{D}(T)$,

$$\langle (T-iI)z, x-y\rangle = \langle z, (T^*+iI)(x-y)\rangle = 0.$$

Thus $x-y$ is orthogonal to $\operatorname{ran}(T-iI)=H$, and hence is 0. Therefore $y=x\in\mathcal{D}(T)$, as required. $\square$

It is not hard to verify that, if T and S are symmetric operators, $T \subseteq S$ if and only if $U(T) \subseteq U(S)$. Therefore it follows from the last Lemma that if a densely defined symmetric operator T has a selfadjoint extension S then its Cayley transform $U(T)$ has a unitary extension $U(S)$. The converse also holds. For if U is a unitary extending $U(T)$, then, since $\operatorname{ran}(I-U(T))=\mathcal{D}(T)$ is dense, it follows that $\operatorname{ran}(I-U)$ must also be dense. A calculation similar to the one in the proof of the above Lemma shows that, since $I-U$ is normal, $\ker(I-U)=\ker(I-U^*)=(\operatorname{ran}(I-U))^\perp$ so that $I-U$ must be one to one. Thus the formula $S(I-U)x=i(I+U)x$ defines an operator on the dense domain $\mathcal{D}(S)=\operatorname{ran}(I-U)$. Clearly S extends T. It is easy to check that S is symmetric and, since $U(S)=U$, the previous Lemma shows S is selfadjoint.

Conclusion: *a densely defined symmetric operator T admits selfadjoint extensions if and only if its Cayley transform $U(T)$ admits unitary extensions.*

Now since $U(T)$ maps $\operatorname{ran}(T+iI)$ isometrically onto $\operatorname{ran}(T-iI)$, any unitary extension U of $U(T)$ must map $\operatorname{ran}(T+iI)$ isometrically onto $\operatorname{ran}(T-iI)$. But since U is unitary, it must also map $(\operatorname{ran}(T+iI))^\perp$ isometrically onto $(\operatorname{ran}(T-iI))^\perp$. It follows that the dimensions of the last two closed subspaces must be equal. Conversely: if these dimensions are equal, let $V:(\operatorname{ran}(T+iI))^\perp \to (\operatorname{ran}(T-iI))^\perp$ be *any* onto isometry. Then the map

$$U = V \oplus U(T) : (\operatorname{ran}(T+iI))^\perp \oplus \operatorname{ran}(T+iI) \to H,$$

given by $U(x+y)=Vx+U(T)y$ for $x \in (\operatorname{ran}(T+iI))^\perp$ and $y \in \operatorname{ran}(T+iI)$, extends to a unitary operator on H which extends $U(T)$.

Conclusion: *T admits selfadjoint extensions if and only the closed subspaces $(\operatorname{ran}(T+iI))^\perp$ and $(\operatorname{ran}(T-iI))^\perp$ have the same dimension.*

Note also that T is essentially selfadjoint if and only if it has a *unique* selfadjoint extension, that is, if and only if its Cayley transform $U(T)$ has a unique unitary extension. This clearly happens if and only if the linear manifolds $\operatorname{ran}(T+iI)$ and $\operatorname{ran}(T-iI)$ are dense in H. Noting that $(\operatorname{ran}(T\pm iI))^\perp = \ker(T^*\mp iI)$ (proof: exercise), we conclude that T is essentially selfadjoint if and only if $\ker(T^*+iI)=\ker(T^*-iI)=\{0\}$.
This discussion proves the following

Theorem 1.25 *Let T be a densely defined symmetric operator on H. Then*
(a) T is selfadjoint if and only if $\operatorname{ran}(T+iI)=\operatorname{ran}(T-iI)=H$.

[7] If $x=(I-U)z$ is in $\mathcal{D}(T)$, then $\langle x, Tx\rangle = \langle (I-U)z, T(I-U)z\rangle = \langle (I-U)z, i(I+U)z\rangle = \langle z, i(I-U^*)(I+U)z\rangle = \langle z, i(U-U^*)z\rangle \in \mathbb{R}$ since $i(U-U^*)$ is selfadjoint. Thus $\langle x, Tx\rangle = \langle Tx, x\rangle$. Polarisation now shows that $\langle x, Ty\rangle = \langle Tx, y\rangle$ for all $x, y \in \mathcal{D}(T)$.

(b) T has selfadjoint extensions if and only if its **deficiency indices** *$n_\pm$, defined by $n_\pm = \dim(\mathrm{ran}(T \pm iI))^\perp$, are equal.*
(c) T is essentially selfadjoint if and only if the operators $T^* + iI$ and $T^* - iI$ are injective.

Corollary 1.26 *If T is a densely defined symmetric operator on H, the following are equivalent:*
(a) T is selfadjoint;
(b) T is closed and $\ker(T^* - iI) = \ker(T^* + iI) = \{0\}$;
(c) $\mathrm{ran}(T + iI) = \mathrm{ran}(T - iI) = H$.

The following examples illustrate the strong dependence of an unbounded symmetric operator on its domain. In all three, the formula defining the operator is the same: $f \to if'$. However, on one domain the operator admits no selfadjoint extensions, on another it admits uncountably many, and on a third it is selfadjoint!

Example 1.27 *A symmetric operator with no selfadjoint extensions:*

Let $H = L^2((0, \infty))$. Define A by the formula $Af = if'$ on the set $\mathcal{D}(A)$ of all functions of the form $e \cdot p$, where $e(t) = e^{-t}$ and p is a polynomial without constant term (in particular $f(0) = 0$ for $f \in \mathcal{D}(A)$). It is left as an exercise to show that $\mathcal{D}(A)$ is dense in H. An integration by parts shows that A is symmetric. We claim that e is orthogonal to $\mathrm{ran}(A - iI)$. Indeed, for any $f \in \mathcal{D}(A)$ we have

$$\langle Af, e\rangle = \int_0^\infty if'(t)e^{-t}dt = i[f'(t)e^{-t}]_0^{+\infty} + \int_0^\infty f(t)e^{-t}dt = \langle if, e\rangle$$

so that $\langle Af - if, e\rangle = 0$ for each $f \in \mathcal{D}(A)$. Thus $\mathrm{ran}(A - iI)$ is not dense in H.
We show that $\mathrm{ran}(A + iI)$ is dense in H; it will follow from Theorem 1.25 (b) that A cannot have selfadjoint extensions.
Let $g \in \mathcal{D}(A)$. Define a function f on $[0, \infty)$ by

$$f(x) = -ie^{-x}\int_0^x e^t g(t)dt.$$

It is clear that $f \in \mathcal{D}(A)$ and it is easy to show that $(A + iI)f = i(f' + f) = g$. This shows that $\mathrm{ran}(A + iI)$ contains $\mathcal{D}(A)$, hence is dense in H. □

Example 1.28 *An operator with many selfadjoint extensions:*

Let $H = L^2([0, 1])$. Recall the operator $B = T_3$ of Example 1.16:

$$\mathcal{D}(B) = \{f \in AC([0, 1]) : f' \in L^2,\ f(0) = 0 = f(1)\}, \qquad Bf = if' \quad (f \in \mathcal{D}(A)).$$

Then B is symmetric and B^* is defined on $\mathcal{D}(B^*) = \{f \in AC([0, 1]) : f' \in L^2\}$ by $B^*f = if'$ (so $B \neq B^*$). Note that B^* is not symmetric. The self adjoint extensions of B are precisely the operators B_c ($|c| = 1$) defined on

$$\mathcal{D}(B_c) = \{f \in AC([0, 1]) : f' \in L^2,\ f(0) = cf(1)\}$$

by $B_c f = if'$ (for the proofs, see Reed & Simon, Functional Analysis, Section VII.2).
Thus B has uncountably many different selfadjoint extensions B_c and $B \subseteq B_c \subseteq B^*$.

Example 1.29 *Consider any of the selfadjoint extensions of the operator in the previous example: For instance, on $H = L^2([0, 1])$, let*

$$\mathcal{D}(P) = \{f \in AC([0, 1]) : f' \in L^2,\ f(0) = f(1)\}$$

and $(Pf)(t) = -if'(t)$ $(f \in \mathcal{D}(P))$. Then P is selfadjoint.