Operator Theory – Spring 2010 – Summary

Week 1: Feb. 17-18

1 Introduction – Reminder

Blanket assumption: All linear spaces are assumed complex, unless otherwise specified explicitly! **Warning:** Some of the stated results are FALSE for real spaces!

The general notion of a linear operator: continuous vs discontinuous.

Scalar product. The Cauchy-Schwarz inequality.

Norm of an operator: boundedness $\iff$ uniform continuity.

Lemma E, F Banach spaces, $D \subseteq E$ dense linear manifold, $T : D \to F$ linear. Then: T extends to a bounded $\bar{T} : E \to F$ iff T is continuous. In this case, $\bar{T}$ is unique and $\|\bar{T}\| = \|T\|$.

Polynomials of an operator. The functional calculus: functions of an operator. Example: $\exp(T)$.

Examples of Hilbert spaces: $\ell^2, L^2(X, \mu), H^2(\mathbb{T})$.

1.1 Hilbert space

Three basic results:

• If M is a closed subspace of $\mathcal{H}$, then $M \oplus M^{\perp} = \mathcal{H}$ and Pythagoras holds. Hence the orth. projection P_M is well-defined and of norm 1 if not 0.

Orthonormal families.

Such a family is an **o.n. basis** if it has dense linear span.

• Every Hilbert space has an onb; hence it is isometrically isomorphic to some $\ell^2(I)$ space. It is separable iff I is countable.

Example: If $e_k(t) = e^{ikt}$, the family $\{e_k : k \in \mathbb{Z}\}$ is an onb of $L^2(\mathbb{T})$.

• Every bounded linear functional f on a Hilbert space $\mathcal{H}$ is of the form $f(x) = \langle x, y \rangle$ for a unique $y \in \mathcal{H}$. The map $f \to y$ is an antilinear isometry from the dual of $\mathcal{H}$ onto $\mathcal{H}$.

1.2 Examples of operators

On ℓ^2: Diagonal operators D_a. These are everywhere defined iff they are bounded iff $a \in \ell^\infty$.

The (unilateral) shift, $e_n \to e_{n+1}$ on ℓ^2: it is isometric, hence 1-1, not onto.

On $L^2(X, \mu)$: Multiplication operators. These are bounded iff $f \in L^\infty(X, \mu)$ (definition of $L^\infty(X, \mu)$ and the essential sup norm).

The Fourier transform: $F : L^2(\mathbb{T}) \to \ell^2(\mathbb{Z}) : f \to \hat{f}$ is an onto isometry.