Operator Theory – Spring 2010 – Summary

Week 2: Feb. 24-25

1.1 Examples of operators (continued)

Given an onb of $\mathcal{H}$, every bounded operator on $\mathcal{H}$ has a matrix. The converse fails!

Integral operators on $L^2(X, \mu)$: given a "nice" function $k : X \times X \to \mathbb{C}$, let $f \to A_k f$ where $A_k f(x) = \int k(x,y) f(y) d\mu(y)$.

2 Bounded Operators

2.1

The adjoint: if $T \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$, the formula

$$\langle T^* y, x \rangle_1 = \langle y, Tx \rangle_2, \qquad x \in \mathcal{H}_1, y \in \mathcal{H}_2$$

defines a unique $T^* \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$, and $\|T^*\| = \|T\|$.

Properties of the adjoint operation:

1. $A^{**} = A \qquad (A : \mathcal{H}_1 \to \mathcal{H}_2)$
2. $(A + \lambda B)^* = A^* + \bar{\lambda} B^* \qquad (A, B \in \mathcal{B}(\mathcal{H}_1 \mathcal{H}_2), \lambda \in \mathbb{C})$
3. $(AC)^* = C^* A^* \qquad (A : \mathcal{H}_1 \to \mathcal{H}_2,\ C : \mathcal{H}_2 \to \mathcal{H}_3)$
4. $\|A^* A\| = \|A\|^2 \qquad (A \in \mathcal{B}(\mathcal{H}_1 \mathcal{H}_2))$.

Generalisation: A **sesquilinear form** is a map $\phi : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathbb{C}$ which is linear in the first variable and antilinear (or conjugate linear) in the second.

Theorem 1 (The BLT theorem) [1] *A sesquilinear form ϕ is bounded (i.e. there is $M < \infty$ s.t. $\phi(x,y) \leq M \|x\| \|y\|$ for all $x \in \mathcal{H}_1$, $y \in \mathcal{H}_2$) iff there exists $T \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ s.t.*

$$\phi(x,y) = \langle Tx, y \rangle, \qquad x \in \mathcal{H}_1,\ y \in \mathcal{H}_2.$$

This T is unique and the least bound M for ϕ is $\|T\|$.

Polarisation. Every sesquilinear $\phi : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ is uniquely determined by the associated "quadratic form": for all $x, y \in \mathcal{H}$

$$4\phi(x,y) = \phi(x+y, x+y) - \phi(x-y, x-y) + i\phi(x+iy, x+iy) - i\phi(x-iy, x-iy).$$

Hence, if $\langle Tx, x \rangle = \langle Sx, x \rangle$ for all $x \in \mathcal{H}$, then $S = T$. What happens in real Hilbert space?!

[1] Bacon, Lettuce, Tomato

2.2 Classes of operators

• An **isometry** $X : \mathcal{H}_1 \to \mathcal{H}_2$ is ... an isometry: $\|Xx\| = \|x\|$ for all $x \in \mathcal{H}_1$; equivalently, $\langle Xx, Xy\rangle = \langle x, y\rangle$ for all $x, y \in \mathcal{H}_1$ (polarise); X is isometric iff $X^*X = I_{\mathcal{H}_1}$.
• Special case: a **unitary** operator $Y : \mathcal{H}_1 \to \mathcal{H}_2$ is an onto isometry. Y is unitary iff $Y^*Y = I_{\mathcal{H}_1}$ and $YY^* = I_{\mathcal{H}_2}$; equivalently, if it is invertible and $Y^{-1} = Y^*$.

Example: The unilateral shift $S : e_n \to e_{n+1}$ on $\ell^2(\mathbb{Z}_+)$ is isometric, not unitary. Note $S^*(e_n) = e_{n-1}$ for $n > 0$ but $S^*(e_0) = 0$.

The bilateral shift $U : e_n \to e_{n+1}$ on $\ell^2(\mathbb{Z})$ is unitary: $U^*(e_n) = e_{n-1}$ for all $n \in \mathbb{Z}$.
• $A \in \mathcal{B}(\mathcal{H})$ is called **normal** iff $A^*A = AA^*$.

Example: the shift S isn't; any unitary $V : \mathcal{H} \to \mathcal{H}$ is.
• Special case: $B \in \mathcal{B}(\mathcal{H})$ is called **selfadjoint** if $B^* = B$; equivalently, if $\langle Bx, x\rangle \in \mathbb{R}$ for all $x \in \mathcal{H}$ (polarise).
• Special case: $C \in \mathcal{B}(\mathcal{H})$ is called **positive** if $\langle Cx, x\rangle \geq 0$ for all $x \in \mathcal{H}$.

Examples: All multilpication operators M_f ($f \in L^\infty(X, \mu)$) are normal, because $M_f^* = M_{\bar{f}}$. M_f is selfadjoint iff $f(t) \in \mathbb{R}$ for μ-almost all $t \in X$; it is positive iff $f(t) \geq 0$ for μ-almost all $t \in X$; it is isometric iff it is unitary iff $|f(t)| = 1$ for μ-almost all $t \in X$.

Lemma 2 *If $S \geq 0$ then for all $x, y \in \mathcal{H}$*

$$\begin{aligned} &(i)\ |\langle Sx, y\rangle|^2 \leq \langle Sx, x\rangle \langle Sy, y\rangle \\ &(ii)\ \|S\| = \sup\{|\langle Sx, x\rangle| : \|x\| \leq 1\} \\ &(iii)\ \|Sx\|^2 \leq \|S\| \langle Sx, x\rangle . \end{aligned}$$

NB. (ii) also holds for S normal. Not generally (even for 2×2 matrices).
Proof (i) is just C-S. For (ii): If $a \equiv \sup\{|\langle Sx, x\rangle| : \|x\| \leq 1\}$ then $a \leq \|S\|$.

For the opposite inequality, apply (i) to $\langle Sx, y\rangle$:

$$|\langle Sx, y\rangle|^2 \leq \langle Sx, x\rangle \langle Sy, y\rangle \leq a^2.$$

Take sup over x, y in *ball*$\mathcal{H}$ to obtain $\|S\|^2 \leq a^2$.

For (iii): apply (ii) to $y = Sx$.

Proposition 3 *Let (B_n) be a monotone sequence of selfadjoint operators which is uniformly bounded, i.e. $\sup_n \|B_n\| < \infty$. Then there is a unique $B = B^* \in \mathcal{B}(\mathcal{H})$ such that $\|B_n x - Bx\| \to 0$ for all $x \in \mathcal{H}$.* We say $B_n \to B$ in the strong operator topology (SOT).

Projections If $M \subseteq \mathcal{H}$ is a closed subspace, define the *(orthogonal) projection* $P_M \in \mathcal{B}(\mathcal{H}))$.
• An operator $P \in \mathcal{B}(\mathcal{H}))$ is a projection iff $P = P^2 = P^*$ (write $P \in \mathcal{P}(\mathcal{B}(\mathcal{H}))$). Then $P = P_M$ where $M = P(\mathcal{H})$. Also $0 \leq P \leq I$.
• If $P, Q \in \mathcal{P}(\mathcal{B}(\mathcal{H})))$ then

$$P(\mathcal{H}) \subseteq Q(\mathcal{H}) \iff P \leq Q \iff PQ = P \iff QP = P.$$

• If $P_n \in \mathcal{P}(\mathcal{B}(\mathcal{H})))$ and $(P_n) \nearrow$ (resp. $(P_n) \searrow$) then $P_n \overset{SOT}{\to} P$ where $P = P_M$, $M = \overline{\cup_n P_n(\mathcal{H})}$ (resp. $M = \cap_n(P_n(\mathcal{H}))$). [Monotonicity cannot be omitted.]