Operator Theory – Spring 2010 – Summary

Week 4: Mar. 10-11

3 The functional calculus (continued)

Fix $A = A^* \in \mathcal{B}(\mathcal{H})$. We wish to define $f(A)$ for appropriate f.

Recall that

$$\sigma(A) \subseteq [a, b] \subseteq [-\|A\|, \|A\|]$$

where $a = \inf\{\langle Ax, x\rangle : \|x\| = 1\}$ and $b = \sup\{\langle Ax, x\rangle : \|x\| = 1\}$.

3.2 Continuous Functions on $\sigma(A)$

If $p(t) = c_0 + c_1 t + \cdots + c_n t^n$ $(c_k \in \mathbb{C})$ is a poly of a real variable, then $p(A) = c_0 I + c_1 A + \cdots + c_n A^n$. The map $\Phi_0 : p \to p(A)$ from the algebra of polynomials into $\mathcal{B}(\mathcal{H})$ preserves the algebraic operations $+, \cdot, *$ where $p^*(t) = \bar{p}(t) = \bar{c}_0 + \bar{c}_1 t + \cdots + \bar{c}_n t^n$.

To extend to functions that are "limits" of polynomials, need some sort of "continuity" of Φ_0:

Lemma 6 (Spectral Mapping Lemma) $\sigma(p(A)) = p(\sigma(A)) = \{p(\lambda) : \lambda \in \sigma(A)\}$.

Proposition 7 $\|p(A)\|_{\mathcal{B}(\mathcal{H})} = \sup\{|p(\lambda)| : \lambda \in \sigma(A)\} \equiv \|p\|_{\sigma(A)}$.

For some purposes, it is sufficient to know a weaker estimate:

Lemma 8 (Nelson, p. 67) [1] *Let $p(t) = a_0 + a_1 t + \cdots + a_n t^n$ be a polynomial. For all $x \in \mathcal{H}$, $\|p(A)x\| \le \max\{|p(t)| : t \in [-\|A\|, \|A\|]\} \|x\|$. Hence*

$$\|p(A)\|_{\mathcal{B}(\mathcal{H})} \le \max\{|p(t)| : t \in [-\|A\|, \|A\|]\}. \tag{1}$$

Proof Let $M := span\{x, Ax, \ldots, A^n x\}$; this is a finite dimensional subspace of $\mathcal{H}$ (automatically closed). Let E be the orthogonal projection onto M. Then, since $A^k x \in M$ when $k = 0, \ldots, n$, we have $p(A)x \in M$ and

$$p(A)x = Ep(A)Ex = p(EAE)x$$

(verify!)[2] Since EAE is a selfadjoint operator on the finite-dimensional space M, we may apply the spectral theorem for finite dimensional spaces to get

$$EAE = \sum \lambda_k P_{\lambda_k} \quad \text{and} \quad I = \sum P_{\lambda_k}$$

where the $\lambda_k' s$ are the eigenvalues with associated projections P_{λ_k}. It follows that

$$p(A)x = p(\sum \lambda_k P_{\lambda_k})x = \left(\sum p(\lambda_k) P_{\lambda_k}\right) x$$

[1] E. Nelson, Topics in Dynamics I: Flows, Princeton Univ. Press and the University of Tokyo Press, 1969

[2] Note that M is not in general A-invariant.

and by Pythagoras' theorem,

$$\begin{aligned}\|p(A)x\|^2 &= \sum |p(\lambda_k)|^2 \|P_{\lambda_k}x\|^2 \\ &\leq \max_k |p(\lambda_k)|^2 \sum \|P_{\lambda_k}x\|^2 = \max_k |p(\lambda_k)|^2 \|x\|^2\end{aligned}$$

since $\sum \|P_{\lambda_k}x\|^2 = \|x\|^2$. But each λ_k satisfies $|\lambda_k| \leq \|EAE\| \leq \|A\|$, hence is in the interval $[-\|A\|, \|A\|]$; therefore

$$\|p(A)x\| \leq \max\{|p(\lambda)| : \lambda \in [-\|A\|, \|A\|]\} \|x\| . \qquad \square$$

Remark 9 In general, the inequality may be strict: for example suppose A is a nonzero orthogonal projection, so $\sigma(A) = \{0, 1\}$ and let $p(t) = t - t^2$. Then $p(A) = A - A^2 = 0$ while $\max\{|p(\lambda)| : \lambda \in [-\|A\|, \|A\|]\} = p(1/2) = 1/4$. However here $\max\{|p(\lambda)| : \lambda \in \sigma(A)\} = 0$ as in Proposition 7.

To prove Proposition 7, use

Proposition 10 *Let $A = A^* \in \mathcal{B}(\mathcal{H})$. Then one of the numbers $\|A\|$ or $-\|A\|$ must belong to $\sigma(A)$. In particular,*

$$\sup\{|\lambda| : \lambda \in \sigma(A)\} = \|A\|.$$

Proof We will prove that the number $\|A\|^2$ is in $\sigma(A^2)$. It will follow that the product $(A - \|A\|I)(A + \|A\|I) = (A^2 - \|A\|^2 I)$ cannot be invertible, and hence the operators $(A - \|A\|I)$ and $(A + \|A\|I)$ cannot both be invertible, as required.

For each $\lambda \in \mathbb{R}$ and each $x \in \mathcal{H}$, since $\langle A^2x, \lambda^2 x\rangle \in \mathbb{R}$, we have

$$\begin{aligned}\|A^2x - \lambda^2 x\|^2 &= \langle A^2x - \lambda^2x, A^2x - \lambda^2x\rangle = \|A^2x\|^2 - 2\langle A^2x, \lambda^2x\rangle + \|\lambda^2x\|^2 \\ &= \|A^2x\|^2 - 2\lambda^2\|Ax\|^2 + \lambda^4\|x\|^2.\end{aligned}$$

But since $\|A\| = \sup\{\|Ax\| : \|x\| = 1\}$, there is a sequence (x_n) with $\|x_n\| = 1$ and $\|Ax_n\| \to \|A\|$. Using the previous equality with $x = x_n$ and $\lambda = \|A\|$, we obtain

$$\begin{aligned}\|A^2x_n - \lambda^2x_n\|^2 &= \|A^2x_n\|^2 - 2\lambda^2\|Ax_n\|^2 + \lambda^4 \\ &\leq (\|A\|\|Ax_n\|)^2 - 2\lambda^2\|Ax_n\|^2 + \lambda^4 = \lambda^4 - \lambda^2\|Ax_n\|^2 \to 0.\end{aligned}$$

This shows that the operator $A^2 - \lambda^2 I$ cannot be invertible (why?) and hence $\lambda^2 = \|A\|^2 \in \sigma(A^2)$. $\square$

Proof of Proposition 7 The idea is to reduce to the selfadjoint case and use the C*-property for the norm: Observe that if $p(t) = \sum_{k=0}^n a_k t^k$, then

$$p(A)^*p(A) = (\sum_{k=0}^n a_k A^k)^* (\sum_{r=0}^n a_r A^r) = (\sum_{k=0}^n \bar{a}_k A^k)(\sum_{r=0}^n a_r A^r) = q(A)$$

(since $A = A^*$) where q is the polynomial $q(t) = \bar{p}(t)p(t)$. Now $q(A)$ is selfadjoint so by Proposition 10 we get

$$\|q(A)\| = \sup\{|\mu| : \mu \in \sigma(q(A))\}.$$

But $\sigma(q(A)) = q(\sigma(A)) = \{q(\lambda) : \lambda \in \sigma(A)\}$ by the spectral mapping lemma, and so

$$\|q(A)\| = \sup\{|q(\lambda)| : \lambda \in \sigma(A)\}.$$

But the C*-property gives $\|p(A)\|^2 = \|p(A)^*p(A)\| = \|q(A)\|$ and so

$$\begin{aligned}\|p(A)\|^2 &= \|q(A)\| = \sup\{|q(\lambda)| : \lambda \in \sigma(A)\} \\ &= \sup\{|\bar{p}(\lambda)p(\lambda)| : \lambda \in \sigma(A)\} = (\sup\{|p(\lambda)| : \lambda \in \sigma(A)\})^2.\end{aligned}$$

The proof is complete. □

Theorem 11 *The map Φ_0 extends uniquely to an isometric *-homomorphism*

$$\Phi_c : (C(\sigma(A)), \|\cdot\|_{\sigma(A)}) \to (\mathcal{B}(\mathcal{H}), \|\cdot\|)$$

For $f \in C(\sigma(A))$, we write $f(A)$ for $\Phi_c(f)$.

Thus $f(a) = \lim p_n(A)$ where (p_n) is *any* sequence of polynomials converging to f uniformly on $\sigma(A)$.

4 Unbounded operators

4.1 Definitions

$\mathcal{H}, \mathcal{K}$ are Hilbert (or Banach) spaces. An **operator** from $\mathcal{H}$ to $\mathcal{K}$ is a pair $(\mathcal{D}(T), T)$ where $D(T) \subseteq \mathcal{H}$ is a linear manifold and $T : \mathcal{D}(T) \to \mathcal{K}$ is a linear map. We say that T is **densely defined** if its domain $\mathcal{D}(T)$ is dense in $\mathcal{H}$. Note that if T is densely defined and *continuous*, it admits a unique extension to a map defined on $\mathcal{H}$, with the same norm; but if T is *not continuous*, it cannot be extended continuously to the whole of $\mathcal{H}$. If T, S are operators from $\mathcal{H}$ to $\mathcal{K}$, we say S **extends** T and we write $T \subset S$ if $\mathcal{D}(T) \subseteq \mathcal{D}(S)$ and $S|_{\mathcal{D}(T)} = T$.

Example 12 (The "position operator" of Quantum Mechanics)
Let $\mathcal{H} = L^2(\mathbb{R})$ (Lebesgue measure understood), $\mathcal{D}(Q) = \{f \in \mathcal{H} : t \to tf(t) \text{ is in } \mathcal{H}\}$ and define $Q : \mathcal{D}(Q) \to \mathcal{H}$ by $(Qf)(t) = tf(t)$, $f \in \mathcal{D}(Q)$. Then Q is unbounded, but its **graph** *is a closed subspace of $\mathcal{H} \oplus \mathcal{H}$.*

Definition 4.1 *The* **graph** *of a linear operator $T : \mathcal{D}(T) \to \mathcal{K}$ is the following subspace of $\mathcal{H} \oplus \mathcal{K}$:*

$$Gr(T) = \{x \oplus Tx : x \in \mathcal{D}(T)\}.$$

This is of course a linear manifold. We say T is a **closed operator** *when $Gr(T)$ is a closed subspace of $\mathcal{H} \oplus \mathcal{K}$. The set of all closed operators is denoted $\mathcal{C}(\mathcal{H}, \mathcal{K})$.*

We say T is **closable** *when the subspace $\overline{Gr(T)}$ is the graph of some linear operator. This operator (if it exists) is unique and is denoted $\overline{T}$. Clearly $T \subset \overline{T}$.*

Example 13 *If $\mathcal{D}(Q_o) = \{f \in \mathcal{H} : f \text{ has compact support }\}$ and $Q_o : \mathcal{D}(Q_o) \to \mathcal{H}$ is given by $(Q_of)(t) = tf(t)$, $f \in \mathcal{D}(Q_o)$, then Q_o is closable and its closure is Q.*

A closed, everywhere defined operator is necessarily bounded (closed graph theorem!); so being closed is a (useful) weakening of continuity.