A note on the spectrum

Let [1] $\mathcal{B}$ be a unital Banach algebra (for example, $\mathcal{B} = \mathcal{B}(\mathcal{H})$) and denote by $GL(\mathcal{B})$ the group of invertible elements of $\mathcal{B}$.

Basic Lemma *If $T \in \mathcal{B}$ satisfies $\|T\| < 1$ then $I - T \in GL(\mathcal{B})$ and*

$$\sum_{k=0}^{\infty} T^k = (I - T)^{-1}.$$

Proof. The geometric series on the left converges absolutely, hence (completeness of $\mathcal{B}$) converges. If $S_n = \sum_{k=0}^{n} T^k$, it is immediate that $S_n(I-T) = (I-T)S_n = I - T^{n+1}$. Hence $\lim_n S_n(I-T) = \lim_n (I-T)S_n = I$. $\square$

If $T \in \mathcal{B}$ we define the *spectrum of T* to be

$$\sigma(T) = \{\lambda \in \mathbb{C} : (\lambda I - T) \notin GL(\mathcal{B})\}.$$

Lemma 1 *$\sigma(T)$ is bounded by $\|T\|$, so*

$$\rho(T) := \sup\{|\lambda| : \lambda \in \sigma(T)\} \leq \|T\|$$

Proof. If $|\lambda| > \|T\|$ then $\left\|\frac{T}{\lambda}\right\| < 1$ and so the geometric series

$$\sum_{k=0}^{\infty} \left(\frac{T}{\lambda}\right)^k = \left(I - \frac{T}{\lambda}\right)^{-1}$$

is convergent, showing that $(\lambda - T) = \lambda\left(I - \frac{T}{\lambda}\right) \in GL(\mathcal{B})$, hence $\lambda \notin \sigma(T)$. $\square$

Lemma 2 *If $\lambda \notin \sigma(T)$ so that $R_\lambda = (\lambda I - T)^{-1}$ is defined, then for all $w \in \mathbb{C}$ with $|w - \lambda| < \frac{1}{\|R_\lambda\|}$ we have $w \notin \sigma(T)$ and in fact*

$$-\sum_{k=0}^{\infty} (-R_\lambda)^{k+1}(w - \lambda)^k = (wI - T)^{-1} = R_w.$$

Proof. Set $z = \lambda - w$; if $|w - \lambda| < \frac{1}{\|R_\lambda\|}$ then $\|zR_\lambda\| < 1$ so $I - zR_\lambda \in GL(\mathcal{B})$ and

$$\sum_{k=0}^{\infty} (zR_\lambda)^k = (I - zR_\lambda)^{-1}$$

hence

$$\sum_{k=0}^{\infty} R_\lambda^{k+1} z^k = R_\lambda (I - zR_\lambda)^{-1} = \left((I - zR_\lambda)R_\lambda^{-1}\right)^{-1}$$
$$= (R_\lambda^{-1} - z)^{-1} = (\lambda - T - z)^{-1} = (w - T)^{-1}.$$

Conclusion 1 *The set $\sigma(T)$ is closed, because $\mathbb{C} \setminus \sigma(T)$ is open; in fact if $\lambda \notin \sigma(T)$ then $B(\lambda, \frac{1}{\|R_\lambda\|}) \subseteq \mathbb{C} \setminus \sigma(T)$.*

[1] notes, 23 November 11

Conclusion 2 *The function*

$$w \to R_w : \mathbb{C} \setminus \sigma(T) \to \mathbb{B}$$

has a power series expansion around each $\lambda \notin \sigma(T)$, hence is continuous on $\mathbb{C} \setminus \sigma(T)$.

(A power series converges uniformly on compact subsets of its set of convergence, and its partial sums are polynomials and hence continuous functions.)

Remark 3 *In fact $w \to R_w$ is holomorphic (differentiable in norm). This is proved exactly as in the case of a complex - valued power series.*

Alternatively, it follows from the following "resolvent identity":

If $\lambda, \mu \notin \sigma(T)$ are distinct,

$$\frac{R_\lambda - R_\mu}{\lambda - \mu} = \frac{(\lambda - T)^{-1} - (\mu - T)^{-1}}{\lambda - \mu} = -(\lambda - T)^{-1}(\mu - T)^{-1} = -R_\lambda R_\mu.$$

so that, if $\lambda \to \mu$, then $\left\| \frac{R_\lambda - R_\mu}{\lambda - \mu} + R_\mu^2 \right\| \to 0$ since $\|R_\lambda - R_\mu\| \to 0$ (continuity of $\lambda \to R_\lambda$).

Proof of the resolvent identity.

$$(\lambda - T)((\lambda - T)^{-1} - (\mu - T)^{-1})(\mu - T) = (\mu - T) - (\lambda - T) = \mu - \lambda\,.$$

Lemma 4 *The spectrum $\sigma(T)$ is nonempty.*

Proof. Suppose $\sigma(T)$ is empty. Then $w \to R_w$ is defined on $\mathbb{C}$ and has a power series expansion around each complex number. Therefore for each continuous linear form $\phi : \mathcal{B} \to \mathbb{C}$ the function

$$w \to \phi(R_w) : \mathbb{C} \to \mathbb{C}$$

has a power series expansion around each point, hence is entire.

Now if $w \in \mathbb{C}$ is such that $|w| > \|T\|$ then $\|w^{-1}T\| < 1$ hence

$$\begin{aligned}\phi(R_w) &= \phi((w - T)^{-1}) = w^{-1}\phi((I - w^{-1}T)^{-1}) \\ &= \frac{1}{w}\phi\left(\sum_{k=0}^{\infty}(w^{-1}T)^k\right) = \frac{1}{w}\sum_{k=0}^{\infty}\phi((w^{-1}T)^k) = \frac{1}{w}\sum_{k=0}^{\infty}\phi(T^k)\frac{1}{w^k}.\end{aligned}$$

This series converges uniformly on compact subsets of $\{w \in \mathbb{C} : |w| > \|T\|\}$. Therefore, integrating on a circle $\gamma(t) = re^{is}$, $s \in [0, 2\pi]$ with radius $r > \|T\|$, we obtain

$$\begin{aligned}\int_\gamma \phi(R_w)dw &= \int_\gamma \sum_{k=0}^{\infty}\phi(T^k)\frac{1}{w^{k+1}}dw = \sum_{k=0}^{\infty}\phi(T^k)\int_\gamma \frac{1}{w^{k+1}}dw \\ \text{(by uniform convergence)} \qquad &= \phi(T^0)\int_\gamma \frac{1}{w}dw = 2\pi i\phi(T^0).\end{aligned}$$

Since the function $w \to \phi(R_w)$ is entire, the left hand side vanishes for all ϕ. This forces $\phi(I) = \phi(T^0) = 0$ for each continuous linear form ϕ and hence $I = 0$ which is absurd. $\square$