EXERCISES

OPERATOR THEORY MODULE, DECEMBER 2025

3. Ideals and positive functionals

Exercise 3.1. Let A be a C*-algebra and $x \in A_{sa}$. Recall that $x = x_+ - x_-$ with respect to its positive and negative part. Show that

$$\inf\{\|a + b\| : b \in A_+\} = \|x_+\|.$$

Deduce that $\operatorname{dist}(x, A_+) = \|x_-\|$.

Exercise 3.2. Let τ be a positive linear functional on a C*-algebra A and let (H_τ, ϕ_τ) be the GNS-representation.

(i) If I is a closed ideal in A, show that $I \subseteq \ker \tau$ if and only if $I \subseteq \ker \phi_\tau$.

(ii) We say that τ is faithful if $\tau(a) = 0$ implies $a = 0$ for all $a \in A_+$. Show that if τ is faithful then ϕ_τ is faithful.

(iii) Suppose that α is an automorphism of A such that $\tau(\alpha(a)) = \tau(a)$ for all $a \in A$. Define a unitary $u \in H_\tau$ by setting

$$u(a + N_\tau) = \alpha(a) + N_\tau.$$

Show that $\phi_\tau(\alpha(a)) = u\phi_\tau(a)u^*$ for all $a \in A$.

Exercise 3.3. Let A and B be C*-algebras. Show that if $\psi\colon A \to B$ is a positive linear map then it is necessarily bounded.

Exercise 3.4. (i) Let $a \in M_2(\mathbb{C})$. Show that if $a \geq 0$ and $a_{11} = 0$, then $a_{12} = a_{21} = 0$ and $a_{12} \geq 0$.

(ii) Show that there is **no** positive functional $\varphi\colon M_2(\mathbb{C}) \to \mathbb{C}$ satisfying $\varphi(E_{11}) = 0$ and $\varphi(E_{12}) = -1$.

(iii) Show that the functional

$$\varphi\colon M_n(\mathbb{C}) \to \mathbb{C}; \varphi(a) = a_{ii}$$

is a state on $M_n(\mathbb{C})$ for all $i = 1, \dots, n$.

(iv) Show that there is a state φ on $M_n(\mathbb{C})$ such that

$$\varphi(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}) = -1.$$

(v) Show that there is **no** state φ on $M_n(\mathbb{C})$ such that

$$\varphi(\begin{bmatrix} 1 & 0 \\ 0 & -1/2 \end{bmatrix}) = -1.$$

Exercise 3.5. (i) Let $f \in C(\Omega)$ for a compact Hausdorff space Ω. Show that f is a projection in $C(\Omega)$ if and only if there is a clopen $K \subseteq \Omega$ such that $f = \chi_K$ (the characteristic function on K).

(ii) Describe the projections in $M_n(\mathbb{C})$.

(iii) Suppose that $p, q \in M_n(\mathbb{C})$ are projections such that $\operatorname{Tr}(p) \leq \operatorname{Tr}(q)$. Find a partial isometry $v \in M_n(\mathbb{C})$ such that $v^*v = p$ and $vv^* = q$.

Exercise 3.6. Let $a, b \in A$ in a C*-algebra A with $\|a\| \leq 1$ and $\|b\| \leq 1$. Show that if $a, b \in A_+$ then $\|a - b\| \leq 1$. Does this still hold if you assume only that $a, b \in A_{sa}$?