

Specimen Exam - Operator Theory - 2025/26

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Instructions: (a) This is a 3 hour examination.

(b) Exercises 1–6 are mandatory.

(c) Choose one of Exercises 7 and 8 (if solutions for both are submitted, then the higher mark will be considered).

Exercise 1 (10 points). Let A be an abelian unital Banach algebra. Show that for any $a \in A$ there is a continuous function $f: \Omega(A) \rightarrow \sigma(a)$, where $\Omega(A)$ is endowed with the weak*-topology and $\sigma(a)$ with the usual topology of $\mathbb{C}$.

Exercise 2 (10 points). (i) Let A be a unital Banach algebra. Show that $r(a^n) = r(a)^n$ for any $a \in A$ and $n \in \mathbb{N}$.

(ii) Let A be an abelian unital Banach algebra. Show that the Gelfand representation is isometric if and only if $\|a^2\| = \|a\|^2$ for all $a \in A$.

Exercise 3 (10 points). Let $T \in \mathcal{B}(H)$ for a Hilbert space H . Show that if T is selfadjoint and bounded below then it is invertible. Give an example of a selfadjoint operator T that is not invertible but with $\ker T = (0)$.

Exercise 4 (10 points). (i) Let $H = \ell^2(\mathbb{N})$ be the Hilbert space with inner product

$$\langle x, y \rangle = \sum_{n=1}^{\infty} x_n \overline{y_n}.$$

Let $S \in \mathcal{B}(\ell^2(\mathbb{N}))$ be defined by $Se_n = e_{n+1}$, where $\{e_n\}_{n \in \mathbb{N}}$ is the usual orthonormal basis. Find the spectrum $\sigma(S)$.

(ii) Let $H = \ell^2(\mathbb{Z})$ be the Hilbert space with inner product

$$\langle x, y \rangle = \sum_{n=-\infty}^{\infty} x_n \overline{y_n}.$$

Let $U \in \mathcal{B}(\ell^2(\mathbb{Z}))$ be defined by $Ue_n = e_{n+1}$, where $\{e_n\}_{n \in \mathbb{Z}}$ is the usual orthonormal basis. Find the spectrum $\sigma(U)$.

Exercise 5 (10 points). (i) Let H be a Hilbert space. Show that the operator norm of $\mathcal{B}(H)$ satisfies the C*-identity.

(ii) Let $\phi: A \rightarrow B$ be a *-homomorphism between two C*-algebras A and B . Show that $\ker \phi$ is a closed ideal of A and $\text{ran } \phi$ is a C*-subalgebra of B .

Exercise 6 (10 points). Let $M_2(\mathbb{C})$ be the matrix C*-algebra of linear operators on $\mathbb{C}^{(2)}$. Consider the matrix

$$a = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

Diagonalise a , and find a_+ and a_- . What is the form of $f(a)$ for a continuous function f on $\sigma(a)$?

Exercise 7 (20 points). Let A be a unital C*-algebra and $\tau: A \rightarrow \mathbb{C}$ be a positive linear functional.

(i) Show that τ is bounded with $\|\tau\| = \tau(1_A)$.

(ii) Suppose in addition that $\tau(a) \neq 0$ for every $0 \neq a \in A_+$. Show that the form

$$\langle \cdot, \cdot \rangle: A \rightarrow A \rightarrow \mathbb{C}; \langle a, b \rangle = \tau(b^*a),$$

defines an inner product on A .

Exercise 8 (20 points). Let A be a unital C^* -algebra.

(i) Describe how $M_2(A)$ with the natural matrix operations becomes a C^* -algebra.

(ii) Let $\alpha: A \rightarrow A$ be a $*$ -homomorphism that satisfies $\alpha \circ \alpha = \text{id}_A$. Is α an isometric $*$ -automorphism of A ?

(iii) Let $B \subseteq M_2(A)$ given by

$$B := \left\{ \begin{bmatrix} a & b \\ \alpha(b) & \alpha(a) \end{bmatrix} : a, b \in A \right\}.$$

Is B a C^* -subalgebra of $M_2(A)$?

(iii) Let $\pi: A \rightarrow B$ be given by

$$\pi(a) = \begin{bmatrix} a & 0 \\ 0 & \alpha(a) \end{bmatrix}, \text{ for all } a \in A.$$

Is π an isometric $*$ -homomorphism?

(iv) Show that $B = \pi(A) + \pi(A)u$ for

$$u = \begin{bmatrix} 0 & 1_A \\ 1_A & 0 \end{bmatrix}.$$

Show that u is a selfadjoint and unitary operator in $M_2(A)$.

(v) Let C be a unital C^* -algebra, $v \in C$ is a selfadjoint unitary and $\rho: A \rightarrow C$ is a $*$ -homomorphism such that

$$\rho(\alpha(a)) = v\rho(a)v^*, \text{ for all } a \in A.$$

Show that there exists a $*$ -homomorphism $\rho': B \rightarrow C$ such that

$$\rho'(u) = v \quad \text{and} \quad \rho'(\pi(a)) = \rho(a), \text{ for all } a \in A.$$