Μετρα με τιμες τελεστες,
Φασματικο Θεωρημα, Θεωρημα Διαστολης Naimark

1 Μέτρα με τιμες θετικους τελεστες

Ορισμός 1. *Έστω $(K, \mathcal{S})$ μετρήσιμος χώρος.* [1] *Μια οικογένεια $\{E(\Omega) : \Omega \in \mathcal{S}\}$ φραγμενων τελεστών σ' έναν χώρο Hilbert H λέγεται* ***μετρο με τιμες θετικους τελεστες (positive operator valued measure, POVM)*** *αν ικανοποιεί τις ιδιότητες*

1. *Για κάθε $x \in H$, η απεικόνιση $\mu_{xx} : \Omega \longrightarrow \langle E(\Omega)x, x\rangle$ είναι (σ-προσθετικό, θετικό) μέτρο ορισμένο στην $\mathcal{S}$.*
2. *$E(\Omega) \in \mathcal{B}(H)_+$ για καθε $\Omega \in \mathcal{S}$*
3. *$E(\varnothing) = 0$ και $E(K) = I$*

   Η $\{E(\Omega) : \Omega \in \mathcal{S}\}$ λέγεται ***μετρο με τιμες προβολες (projection valued measure, PVM)*** *αν ικανοποιεί επιπλεον την ιδιοτητα*
4. *$E(\Omega_1 \cap \Omega_2) = E(\Omega_1).E(\Omega_2)$ για καθε $\Omega_1, \Omega_2 \in \mathcal{S}$.*

Παρατηρήσεις 1. *(α) Η ιδιοτητα (1) ειναι ισοδυναμη με την ακολουθη:* Για κάθε $x, y \in H$, η απεικόνιση $\mu_{xy} : \Omega \longrightarrow \langle E(\Omega)x, y\rangle$ είναι μιγαδικό μέτρο ορισμένο στην $\mathcal{S}$.

(β) Απο την (4) επεται οτι καθε $E(\Omega)$ ειναι ορθη προβολη (αυτοσυζυγης και ταυτοδυναμη) και οτι καθε μετρο με τιμες προβολες ειναι μεταθετικη *οικογενεια τελεστων (πραγμα που δεν ισχυει εν γενει για μετρα με τιμες θετικους τελεστες).*

(γ) Συνηθως ενδιαφομαστε για την περιπτωση που ο K ειναι συμπαγης χωρος Hausdorff και η $\mathcal{S}$ ειναι η σ-αλγεβρα των Borel υποσυνολων του K. Τοτε απαιτουμε συνηθως το E να ειναι κανονικο *μετρο, δηλαδη (εξ ορισμου) το μετρο μ_{xx} να ειναι κανονικο (θετικο) μετρο Borel για καθε $x \in H$.*

Εστω $\mathcal{L}^\infty(K)$ η C* αλγεβρα ολων των φραγμενων μετρησιμων ***συναρτησεων*** $f : K \to \mathbb{C}$.

Πρόταση 1. *Για καθε POVM E σ΄ εναν μετρήσιμο χωρο $(K, \mathcal{S})$, η απεικονιση $\chi_\Omega \mapsto E(\Omega)$ επεκτεινεται σε μια θετικη μοναδιαια γραμμικη απεικονιση*

$$\Psi_E : \mathcal{L}^\infty(K) \to \mathcal{B}(H).$$

Απόδειξη. For a simple measurable function $f = \sum_i c_i \chi_{\Omega_i}$ (where $\{\Omega_i\}$ is a (finite) measurable partition of K) we let $\Psi_0(f) := \sum_i c_i E(\Omega_i)$. The map Ψ_0 is a unital positive linear map (if $f \geqslant 0$ then $c_i \geqslant 0$ for all i and so $\Psi_0(f) \geqslant 0$). We claim that $\|\Psi_0(f)\| \leqslant 2\|f\|_K = 2\max_i |c_i|$. [2]

Indeed, for each $x \in H$ we have

$$\begin{aligned}|\langle \Psi_0(f)x, x\rangle| &= |\sum_i c_i \langle E(\Omega_i)x, x\rangle| \\ &\leqslant (\max|c_i|)\sum_i |\langle E(\Omega_i)x, x\rangle| = (\max|c_i|)\sum_i \langle E(\Omega_i)x, x\rangle \\ &= (\max|c_i|)\langle E(\cup_i \Omega_i)x, x\rangle = (\max|c_i|)\langle x, x\rangle = \|f\|_K \|x\|^2\end{aligned}$$

since $E(\Omega)$ is a positive operator and $\sum_i E(\Omega_i) = E(\cup_i \Omega_i) = E(K) = I$.

[1] povm modifed 17 Ιουνίου 2025

[2] This crude estimate will be improved below

Now for $(x, y) \in \text{ball}(H) \times \text{ball}(H)$,

$$\langle \Psi_0(f)x, y \rangle = \sum_{n=0}^{3} i^n \langle \Psi_0(f)x_n, x_n \rangle \quad \text{where} \;\; x_n := \frac{x + i^n y}{2}$$

and so

$$|\langle \Psi_0(f)x, y \rangle| \leqslant \sum_{n=0}^{3} |\langle \Psi_0(f)x_n, x_n \rangle| \leqslant \|f\|_K \sum_{n=0}^{3} \|x_n\|^2 \leqslant 2\|f\|_K$$

since $\sum_{n=0}^{3} \|x_n\|^2 = \frac{1}{4}(\|x+y\|^2 + \|x-y\|^2 + \|x+iy\|^2 + \|x-iy\|^2 = \frac{1}{4}(2\|x\|^2 + 2\|y\|^2 + 2\|x\|^2 + 2\|iy\|^2) = \|x\|^2 + \|y\|^2 \leqslant 2$ (by the paralellogram law). Thus $\|\Psi_0(f)\| = \sup\{|\langle \Psi_0(f)x, y \rangle| : \|x\|, \|y\| \leqslant 1\} \leqslant 2\|f\|_K$ as claimed.

Since every $f \in \mathcal{L}^\infty(K)$ is a uniform limit of a sequence of simple functions, the map Ψ_0 extends to a map $\Psi_E : \mathcal{L}^\infty(K) \to \mathcal{B}(H)$ which is linear, unital and positive. □

Note Now that Ψ_E has been extended to a positive unital linear map defined on an *abelian* C*-algebra, we know (as proved by Stinespring) that Ψ_E is *completely* positive, and so $\|\Psi_E\| = \|\Psi_E(\mathbf{1})\| = 1$.

Παρατήρηση 1. *Για καθε $(x, y) \in H \times H$ και $f \in \mathcal{L}^\infty(K)$, εχουμε*

$$\langle \Psi_E(f)x, y \rangle = \int_K f d\mu_{xy}$$

οπου $\mu_{xy}(\Omega) = \langle E(\Omega)x, y \rangle$ και το ολοκληρωμα ως προς το μιγαδικο μετρο μ_{xy} μπορει να ορισθει (εναλλακτικα) ως ο γραμμικος συνδυασμος $\int f d\mu_{xy} = \sum_{n=0}^{3} i^n \int f d\mu_{x_n, x_n}$ οπου $x_n := \frac{x+i^n y}{2}$ και τα μ_{x_n, x_n} ειναι θετικα μετρα.

Στην αντιστροφη κατευθυνση:

Στο εξης συμβολιζουμε με $(K, \mathcal{S})$ εναν συμπαγη χωρο Hausdorff K με την σ-αλγεβρα Borel $\mathcal{S}$. Παρατηρουμε οτι η $\mathcal{L}^\infty(K)$ περιεχει την C*-υπαλγεβρα $C(K)$, με την ιδια μοναδα.

Πρόταση 2. *Για καθε θετικη μοναδιαια γραμμικη απεικονιση $\Phi : C(K) \to \mathcal{B}(H)$ υπαρχει μοναδικο κανονικο Borel POVM E_Φ ωστε, αν $\mu_{xy}(\Omega) := \langle E_\Phi(\Omega)x, y \rangle$ για καθε $(x, y) \in H \times H$, να εχουμε*

$$\langle \Phi(f)x, y \rangle = \int_K f d\mu_{xy} \qquad \text{για καθε } f \in C(K). \tag{*}$$

Απόδειξη. The map Φ is completely positive and unital, hence it is bounded with $\|\Phi\| = \|\Phi(\mathbf{1})\| = 1$. For any $(x, y) \in H \times H$ the map $f \mapsto \langle \Phi(f)x, y \rangle$ is a linear functional on $C(K)$, bounded (by $\|x\|\|y\|$), which is positive when $x = y$. By the Riesz Representation theorem, it defines a unique Borel regular complex measure μ_{xy} on K satisfying (*). [3]

Claim For each Borel $\Omega \subseteq K$ the map $(x, y) \mapsto \mu_{xy}(\Omega) : H \times H \to \mathbb{C}$ is sesquilinear and bounded.

Proof For $x, y_1, y_2 \in H$ and $\lambda \in \mathbb{C}$ we have, for each $f \in C(K)$,

$$\int_K f d\mu_{x, y_1 + \lambda y_2} = \langle \Phi(f)x, y_1 + \lambda y_2 \rangle = \langle \Phi(f)x, y_1 \rangle + \bar{\lambda}\langle \Phi(f)x, y_2 \rangle = \int_K f d\mu_{x, y_1} + \bar{\lambda} \int_K f d\mu_{x, y_2}.$$

Thus the regular Borel measures $\mu_{x, y_1 + \lambda y_2}$ and $\mu_{x, y_1} + \bar{\lambda}\mu_{x, y_2}$ define the same bounded linear form on $C(K)$, and therefore are equal (by the uniqueness part of the Riesz Representation theorem).

We have shown that $(x, y) \mapsto \mu_{xy}(\Omega)$ is conjugate linear in y; the proof of linearity in x is identical.

To show that the map $(x, y) \mapsto \mu_{xy}(\Omega)$ is bounded, one way is to recall that $|\mu_{xy}(\Omega)| \leqslant \|\mu_{xy}\|$ for each Borel $\Omega \subseteq K$ by the definition of the total variation norm of μ_{xy} [4] and so $|\mu_{xy}(\Omega)|\| \leqslant \|x\|\|y\|$.

[3] Alternatively, this map is a linear combination of four positive linear maps, of the form $f \mapsto \langle \Phi(f)\xi, \xi \rangle$, each of which defines a unique positive regular Borel measure on K and then the complex measure μ_{xy} can be defined as the same linear combination of these positive measures.

[4] see W. Rudin, Real and Complex Analysis, Chapter 6

Here is an alternative proof:

Since μ_{xy} is a linear combination of four measures of the form $\mu_{\xi\xi}$, it suffices to consider this case. Now the measure $\mu_{\xi\xi}$ is a positive regular Borel measure, so for every $\epsilon > 0$ there exist a compact set F and an open set U with $F \subseteq \Omega \subseteq U$ such that $\mu_{\xi\xi}(U) - \mu_{\xi\xi}(F) < \epsilon$. By Urysohn's lemma, there exists a continuous $f : K \to [0,1]$ such that $f(t) = 1$ for $t \in F$ and $f(t) = 0$ for $t \notin U$. Thus

$$\chi_F \leqslant f \leqslant \chi_U$$

$$\text{so} \quad \mu_{\xi\xi}(F) = \int \chi_F d\mu_{\xi\xi} \leqslant \int f d\mu_{\xi\xi} \leqslant \int \chi_U d\mu_{\xi\xi} = \mu_{\xi\xi}(U)$$

$$\text{also} \quad \mu_{\xi\xi}(F) \leqslant \mu_{\xi\xi}(\Omega) \leqslant \mu_{\xi\xi}(U) \quad \text{since} \quad F \subseteq \Omega \subseteq U$$

$$\text{hence} \quad \left| \int f d\mu_{\xi\xi} - \mu_{\xi\xi}(\Omega) \right| \leqslant \mu_{\xi\xi}(U) - \mu_{\xi\xi}(F) < \epsilon$$

which shows that

$$\mu_{\xi\xi}(\Omega) \leqslant \int f d\mu_{\xi\xi} + \epsilon \leqslant \|f\|_K \|\xi\|^2 + \epsilon \leqslant \|\xi\|^2 + \epsilon$$

and since ϵ was arbitrary, we obtain $\mu_{\xi\xi}(\Omega) \leqslant \|\xi\|^2$. Now the usual polarization argument (see the proof of Proposition 1) yields the estimate $|\mu_{xy}(\Omega)| \leqslant 2$ [5] for $(x,y) \in \text{ball}(H) \times \text{ball}(H)$.

This completes the proof of the Claim. Thus the map $(x,y) \mapsto \mu_{xy}(\Omega) : H \times H \to \mathbb{C}$ is sesquilinear and bounded.

Now, by the Riesz Theorem for bounded sesquilinear forms on Hilbert space, there is a unique bounded operator $E_\Phi(\Omega)$ such that $\mu_{xy}(\Omega) = \langle E_\Phi(\Omega)x, y\rangle$ for all $(x,y) \in H \times H$.

The fact that each μ_{xy} is a complex regular Borel measure which is positive for $x = y$ and $\mu_{xy}(K) = \langle x, y\rangle$ yields immediately that E_Φ is a regular Borel POVM. Uniqueness of E_Φ follws from the uniqueness of each μ_{xy} which is guarranteed by the Riesz Representation theorem. □

Πόρισμα 1. *Καθε θετικη μοναδιαια γραμμικη απεικονιση* $\Phi : C(K) \to \mathcal{B}(H)$ *επεκτεινεται σε μια θετικη μοναδιαια γραμμικη απεικονιση* $\Psi : \mathcal{L}^\infty(K) \to \mathcal{B}(H)$ *που ικανοποιει* $\Psi(\chi_\Omega) = E_\Phi(\Omega)$ *για καθε* $\Omega \in \mathcal{S}$.

Συνεπως $\Psi_{E_\Phi}|_{C(K)} = \Phi$.

Απόδειξη. The map Φ defines the POVM E_Φ as in Proposition 2. Apply Proposition 1 to E_Φ to obtain the map $\Psi_{E_\Phi} := \Psi$. The fact that Ψ extends Φ follows since from Remark 1 we have

$$\langle \Psi_{E_\Phi}(f)x, y\rangle = \int_K f d\mu_{xy} \quad \text{for all } f \in \mathcal{L}^\infty(K)$$

and by (*) of Proposition 2:

$$\langle \Phi(f)x, y\rangle = \int_K f d\mu_{xy} \quad \text{for all } f \in C(K)$$

which show that if $f \in C(K)$ then

$$\langle \Phi(f)x, y\rangle = \langle \Psi_{E_\Phi}(f)x, y\rangle$$

for all $(x,y) \in H \times H$, and thus $\Phi(f) = \Psi_{E_\Phi}(f)$. □

Συμβολισμος: Συμβολιζουμε τον περιορισμο της $\Psi_E : \mathcal{L}^\infty(K) \to \mathcal{B}(H)$ στην $C(K)$ με Φ_E.

Παρατήρηση 2. *Οι απεικονισεις* $\Phi \mapsto E_\Phi$ *και* $E \mapsto \Phi_E$ *ειναι αντιστροφες η μια της αλλης.*

[5] the bound 2 will be improved to 1 below, when we show that $0 \leqslant E(\Omega) \leqslant I$

Απόδειξη. The fact that given $\Phi : C(K) \to \mathcal{B}(H)$ we have $\Phi_{E_\Phi} = \Phi$ was shown in Corollary 1.

On the other hand, given a (Borel, regular) POVM $E(\cdot)$ on K, Proposition 1 defines a unital positive linear map $\Psi_E : \mathcal{L}^\infty(K) \to \mathcal{B}(H)$ which is uniquely determined by the condition

$$\langle \Psi_E(f)x, y\rangle = \int_K f d\mu_{xy} \tag{1}$$

for all $(x, y) \in H \times H$ and $f \in \mathcal{L}^\infty(K)$, where $\mu_{xy}(\Omega) = \langle E(\Omega)x, y\rangle$.

Applying Proposition 2 to the restriction $\Phi := \Phi_E$ of the map Ψ_E to $C(K)$ yields a POVM E_Φ such that, writing $\tilde{\mu}_{xy}(\Omega) := \langle E_\Phi(\Omega)x, y\rangle$, we have

$$\langle \Phi_E(g)x, y\rangle = \int_K g d\tilde{\mu}_{xy} \tag{2}$$

for all $(x, y) \in H \times H$ and $g \in C(K)$. Comparing (1) and (2), we have

$$\int_K g d\mu_{xy} = \int_K g d\tilde{\mu}_{xy}$$

for all $g \in C(K)$. By uniqueness in the Riesz Representation theorem, the scalar measures μ_{xy} and $\tilde{\mu}_{xy}$ are equal, for all $(x, y) \in H \times H$. This shows that the POVM's $E(\cdot)$ and $E_\Phi(\cdot)$ are equal. In other words, $E_{\Phi_E} = E$.

Πρόταση 3. *Στην αμφιμονοσημαντη αντιστοιχια $\Phi \leftrightarrow E$ που ορισαμε, η Φ ειναι *-μορφισμος αν και μονον αν το E ειναι μετρο με τιμες προβολες (PVM).*

Απόδειξη. Assume first that $E(\cdot)$ is a PVM. Then for $\Omega_i \subseteq K$ Borel $(i = 1, 2)$ we have

$$E(\Omega_1)E(\Omega_2) = E(\Omega_1 \cap \Omega_2)$$

hence

$$\Psi_E(\chi_{\Omega_1})\Psi_E(\chi_{\Omega_2}) = \Psi_E(\chi_{\Omega_1 \cap \Omega_2}) = \Psi_E(\chi_{\Omega_1}\chi_{\Omega_2})$$

so that Ψ_E is multiplicative on characteristic functions. By linearity and continuity it follows that Ψ_E is multiplicative on the closed linear span of characteristic functions, which is $\mathcal{L}^\infty(K)$.

Also, Ψ_E is a positive linear map, and so selfadjoint. Thus, it is a *-morphism. Hence, so is its restriction Φ_E to $C(K)$, as claimed.

The converse is more interesting:
We start with a *-morphism $\Phi : C(K) \to \mathcal{B}(H)$ and we wish to prove that the associated POVM $E(\cdot)$ is a PVM. *Equivalently,* we wish to prove that the extension $\Psi : \mathcal{L}^\infty(K) \to \mathcal{B}(H)$ of Φ associated to E as defined in Corollary 1 is multiplicative. [6]

We will achieve our goal in two steps: First we show that we have

$$\Psi(hg) = \Psi(h)\Psi(g) \;\text{ when } h \in \mathcal{L}^\infty(K) \text{ but } g \in C(K)$$

and then that

$$\Psi(hh') = \Psi(h)\Psi(h') \;\text{ for all } h, h' \in \mathcal{L}^\infty(K).$$

Fix $(x, y) \in H \times H$. If $g \in C(K)$ then, for all $f \in C(K)$,

$$\begin{aligned}\int_K fg d\mu_{xy} &= \langle \Phi(fg)x, y\rangle = \langle \Phi(f)(\Phi(g)x), y\rangle = \langle \Phi(f)x_g, y\rangle \\ &= \int_K f d\mu_{x_g,y}\end{aligned}$$

[6] The difficulty is that this extension was not constructed using some sort of continuity (it is not the case that bounded measurable functions are approximable by continuous ones in some topology); it was constructed as a two-step process via the family of measures $\{\mu_{xy} : (x, y) \in H \times H\}$ defined from Φ by duality.

where $x_g := \Phi(g)x$. Uniqueness in the Riesz Representation theorem shows that the measures $gd\mu_{xy}$ and $d\mu_{x_g,y}$ (more formally, the measures $\Omega \mapsto \int_\Omega gd\mu_{xy}$ and $\Omega \mapsto \mu_{x_g,y}(\Omega)$) are equal. It follows that for every $h \in \mathcal{L}^\infty(K)$ we have

$$\int_K hgd\mu_{xy} = \int_K hd\mu_{x_g,y} .$$

[7] But

$$\int_K hd\mu_{x_g,y} = \langle \Psi(h)(\Phi(g)x), y\rangle = \langle \Phi(g)x, \Psi(h)^* y\rangle = \langle \Phi(g)x, y_h\rangle = \int_K gd\mu_{x,y_h}$$

where $y_h = \Psi(h)^* y$ and so the previous displayed equality gives

$$\int_K ghd\mu_{xy} = \int_K gd\mu_{x,y_h}$$

for all $g \in C(K)$. This shows that the measures $hd\mu_{xy}$ and $d\mu_{x,y_h}$ are equal and so

$$\int_K h'hd\mu_{xy} = \int_K h'd\mu_{x,y_h}$$

for all $h' \in \mathcal{L}^\infty(K)$. Thus

$$\langle \Psi(h'h)x, y\rangle = \int_K h'hd\mu_{xy} = \int_K h'd\mu_{x,y_h} = \langle \Psi(h')x, y_h\rangle = \langle \Psi(h')x, \Psi(h)^* y\rangle = \langle \Psi(h)\Psi(h')x, y\rangle$$

and since the last equality holds for all $(x, y) \in H \times H$ we finally conclude that

$$\Psi(h'h) = \Psi(h)\Psi(h')$$

holds for all $h, h' \in \mathcal{L}^\infty(K)$. Thus Ψ is multiplicative on the abelian algebra $\mathcal{L}^\infty(K)$. In particular, setting $h = \chi_{\Omega_1}$ and $h' = \chi_{\Omega_2}$ we obtain

$$E(\Omega_1 \cap \Omega_2) = \Psi(\chi_{\Omega_1 \cap \Omega_2}) = \Psi(\chi_{\Omega_1}\chi_{\Omega_2}) = \Psi(h)\Psi(h') = E(\Omega_1)E(\Omega_2),$$

οπως θελαμε. □

2 Το Φασματικο Θεωρημα και το Θεωρημα Διαστολης του Naimark

Θεώρημα 1 (Το Φασματικο Θεωρημα). *Αν $A \in \mathcal{B}(H)$ ειναι φυσιολογικος τελεστης, υπαρχει μοναδικο κανονικο μετρο Borel με τιμες προβολες (PVM) E στο $\sigma(A)$ ωστε*

$$A = \int_{\sigma(A)} \lambda dE_\lambda$$

δηλαδη

$$\int_{\sigma(A)} f_1 d\mu_{xy} = \langle Ax, y\rangle \quad \text{για καθε} \;\; (x, y) \in H \times H$$

οπου $f_1(\lambda) = \lambda,\ \lambda \in \sigma(A)$.

Απόδειξη. Since A is normal, by the continuous functional calculus there exists a unique isometric unital *-morphism $\Phi : C(\sigma(A)) \to \mathcal{B}(H)$ such that $\Phi(f_1) = A$.

If $E := E_\Phi$ is the Borel regular POVM associated to Φ (Proposition 2), for every $(x, y) \in H \times H$ we have

$$\int_{\sigma(A)} fd\mu_{xy} = \langle \Phi(f)x, y\rangle \quad \text{for all} \;\; f \in C(K)$$

[7] It follows that $\Psi(hg) = \Psi(h)\Psi(g)$, but we won't need this

and in particular

$$\int_{\sigma(A)} f_1 d\mu_{xy} = \langle \Phi(f_1)x, y\rangle = \langle Ax, y\rangle .$$

But by Proposition 3, since Φ is multiplicative on $C(K)$, the POVM E is in fact a PVM, οπως θελαμε. □

Θεώρημα 2 (Naimark's dilation theorem)**.** *Εστω $(K, \mathcal{S})$ συμπαγης χωρος Hausdorff K με την σ-αλγεβρα Borel $\mathcal{S}$. Εστω $\{E(\Omega) : \Omega \in \mathcal{S}\} \subseteq \mathcal{B}(H)_+$ κανονικο Borel μετρο με τιμες θετικους τελεστες (POVM). Το E δεχεται διαστολη σε ενα μετρο με τιμες προβολες (PVM) $\tilde{E}$ σ' εναν «μεγαλυτερο» χωρο Hilbert H': υπαρχει ενας χωρος Hilbert H', μια ισομετρια $V : H \to H'$ και ενα PVM $\{\tilde{E}(\Omega) : \Omega \in \mathcal{S}\} \subseteq \mathcal{B}(H')_+$ τετοιο ωστε*

$$E(\Omega) = V^*\tilde{E}(\Omega)V \quad \textit{για καθε} \;\; \Omega \in \mathcal{S} .$$

Απόδειξη. The POVM E defines a unital positive linear map $\Psi_E : \mathcal{L}^\infty(K) \to \mathcal{B}(H)$ (Proposition 1) which restricts to a map $\Phi_E : C(K) \to \mathcal{B}(H)$ such that

$$\langle \Phi_E(f)x, y\rangle_H = \int_K f d\mu_{xy} \quad \text{for all} \;\; f \in C(K) \tag{+}$$

for all $(x, y) \in H \times H$ (where μ_{xy} is the scalar measure associated to E). Since $C(K)$ is abelian, the map Φ_E is in fact completely positive.

Thus by Stinespring's theorem Φ_E dilates to a *-representation: there is a Hilbert space H', an isometry $V : H \to H'$ and a *-representation $\pi : C(K) \to \mathcal{B}(H')$ such that

$$\Phi_E(f) = V^*\pi(f)V \quad \text{for all} \;\; f \in C(K) .$$

By Proposition 2 the map π defines a unique POVM $\tilde{E} : \mathcal{S} \to \mathcal{B}(H')$ such that

$$\langle \pi(f)\xi, \eta\rangle_{H'} = \int_K f d\tilde{\mu}_{\xi\eta} \quad \text{for all} \;\; f \in C(K)$$

(where $\tilde{\mu}_{xy}$ is the scalar measure associated to $\tilde{E}$). Since the map π is a unital *-morphism, $\tilde{E}$ is in fact a PVM (Proposition 3).

Now for all for all $(x, y) \in H \times H$ we have

$$\langle \Phi_E(f)x, y\rangle_H = \langle V^*\pi(f)Vx, y\rangle_H = \langle \pi(f)(Vx), (Vy)\rangle_{H'} = \int_K f d\tilde{\mu}_{\xi\eta} \quad \text{for all} \;\; f \in C(K)$$

where $\xi := Vx$ and $\eta := Vy$. Comparing with (+), we obtain

$$\int_K f d\mu_{xy} = \int_K f d\tilde{\mu}_{\xi\eta} \quad \text{for all} \;\; f \in C(K)$$

and hence uniqueness in the Riesz representation theorem shows that the measures μ_{xy} and $\tilde{\mu}_{\xi\eta}$ are equal. This means that for all Borel sets $\Omega \subseteq K$ we have

$$\begin{aligned} \mu_{xy}(\Omega) &= \tilde{\mu}_{\xi\eta}(\Omega) \\ \text{i.e.} \;\; \langle E(\Omega)x, y\rangle_H &= \langle \tilde{E}(\Omega)Vx, Vy\rangle_{H'} = \langle V^*\tilde{E}(\Omega)Vx, y\rangle_H \end{aligned}$$

for all $(x, y) \in H \times H$, and so $E(\Omega) = V^*\tilde{E}(\Omega)V$, οπως θελαμε. □