Single Period Procurement Under Uncertainty
The Newsvendor Problem

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In these notes we will consider the production/procurement problem of a retailer who sells a product under random demand without keeping inventory. There are many situations in practice where keeping a product in inventory for future use is either impossible or impractical. This is the case for products such as newspapers and perishable food, e.g., milk. However a similar situation arises, although it is less obvious, when an apparel retailer makes orders in the beginning of the season for a fashion item. Such orders are made for one season only, and any unsold items are not kept in inventory to be sold next year. They are rather sold at deep discounts at the end of the season.

A typical decision problem in this category is the so called newsvendor or newsboy problem. It is a simple mathematical model that captures the basic issues of procurement under uncertainty in demand. The general description of this model is given next.

Consider a retailer who places an order for a product to his own supplier at the beginning of each period (day/month/season). The quantity procured is used solely to satisfy demand during the current period. No inventory is kept from one period to the next. The demand for this product during the current period is not known in advance. Instead, it is represented by a nonnegative random variable X. The cumulative distribution function of X is F, i.e.,

$$P(X \leq x) = F(x)$$

We assume that the distribution is continuous and the probability density function is $f(x)$.

The retailer must determine the order quantity *Q* which minimizes the expected cost at the end of the period. We thus need to discuss the modelling of costs.

There are two cost components:

C_0 = cost per unit of product left unsold at the end of the period (overstock cost).

C_u = cost per unit of unsatisfied demand (understock cost).

The two cost factors are conflicting; therefore a tradeoff exists between ordering a lot in order to reduce the understock cost and ordering little to reduce overstock cost. The objective of the problem is to find the quantity *Q* that reconciles the two cost factors, i.e., minimizes their sum.

This is done in the following three steps.

1. Compute an expression for the total cost as a function of *X*, *Q*.
2. Compute the expected value of the cost expression with respect to the demand probability distribution.
3. Find the quantity *Q* that minimizes the expected cost.

Computation of the expected cost function

Let $G(Q,X)$ denote the total cost at the end of the sales period, when a quantity *Q* has been ordered at the start of the period and the realized demand is equal to *X*. Then the quantity that remains unsold equals $\max\{Q - X, 0\}$. Similarly, the unsatisfied demand equals $\max\{X - Q, 0\}$. Thus, we find $G(Q,X)=C_0 max(0,Q-X)+C_U max(0,X-Q)$.

Since the demand is not known when the order is placed, the decision on the order quantity cannot be based on $G(Q,X)$, bur rather on the expected one-period cost, $G(Q)=E(G(Q,X))$.To compute this quantity we have

$$G(Q) = C_0 \int_0^\infty \max(0, Q - x) f(x)dx + C_U \int_0^\infty \max(0, x - Q) f(x)dx =$$

$$C_0 \int_0^Q (Q - x) f(x)dx + C_U \int_Q^\infty (x - Q) f(x)dx$$

The Optimal Ordering Policy

We must find the value of *Q* for which the expected cost *G(Q)* is minimized*.* To do this, we must first examine the properties of *G(Q)* a little further.

Differentiating the expected cost formula with respect to *Q* we obtain

$$\frac{dG(Q)}{dQ} = C_0 \int_0^Q 1 f(x)dx + C_u \int_Q^\infty (-1) f(x)dx = C_0 F(Q) - C_U (1 - F(Q))$$

Taking the second derivative,

$$\frac{d^2G(Q)}{dQ^2} = (C_0 + C_U) f(Q) \geq 0 \;\; \forall Q \geq 0.$$

Since the second derivative is always nonnegative, *G(Q)* is a convex function of *Q*. In addition, the value of the first derivative for *Q=0* is equal to $\frac{dG(Q)}{dQ} = C_0 F(0) - C_U (1 - F(0)) = -C_U < 0$, since F(0)=0.

It follows that *G(Q)* is decreasing at *Q=0.* Therefore, the optimal solution Q^* can be found as the value for which the first derivative *G(Q)*=0. Thus,

$$G'(Q^*) = (C_0 + C_U) F(Q^*) - C_U = 0 \Leftrightarrow F(Q^*) = \frac{C_U}{C_0 + C_U}$$

Since C_0 and C_U are positive scalars, it follows that $0 \leq \frac{C_U}{C_0 + C_U} \leq 1$. Since in addition *F* is a continuous distribution, the equation above always has a solution.

The fraction $\frac{C_U}{C_0 + C_U}$ is referred to as **the critical ratio**.

From the equation that determines the optimal policy we observe that the order quantity Q^* must be such that the probability of satisfying demand is equal to the critical ratio. The probability that the demand is fully satisfied in a period is a common measure of performance of an ordering policy, related to the degree of customer satisfaction. It is usually referred to as **service level of type 1.** For example if the service level of a policy is 0.7, this means that on average in 70% of the periods the order quantity is enough to satisfy the entire customer demand. The formula for the ordering policy combines in a concise form the economic aspects with the customer service issues involved in the optimal policy. It suggests ordering such a quantity that the service level is set equal to the critical ratio.

We can make some interesting observations from the optimal policy formula. When the over and understock costs are equal, i.e., $C_0 = C_U$, then the critical ratio is ½. This means that Q^* equals the median of the demand distribution, i.e., at a value such there is a 50% probability for the demand to exceed it. (Note that when the demand distribution is symmetric, then the median is equal to the mean).

When the demand distribution is continuous then F(x) is a continuous function and the equation

$$F(Q^*) = \frac{C_U}{C_0 + C_U}$$

has a unique solution in *Q*.. However when the distribution is discrete (i.e., the demand can take a finite or countable number of values), then the above equation may not have a solution. It can be shown, following an analysis similar to the above, that the optimal order quantity for a general case of demand distribution is

$$Q = \min\left\{Q : F(Q) \geq \frac{C_U}{C_0 + C_U}\right\}.$$

Example 1. Assume C_O=2, C_U=6, and the demand follows exponential distribution with parameter λ=1. Then the critical ratio is equal to 6/(6+2) = 0.75. The demand distribution is continuous, and the cumulative distribution function is

$$F(x) = 1 - e^{-\lambda x} = 1 - e^{-x}.$$

In this case the equation for the optimal order quantity is

$$F(Q^*) = \frac{C_U}{C_0 + C_U} \Rightarrow 1 - e^{-Q^*} = 0.75 \Rightarrow e^{-Q^*} = 0.25 \Rightarrow Q^* = -\log(0.25) = 1.39,$$

thus it is optimal for the retailer to purchase 1.39 units of the product.

Example 2. Assume again C_O=2, C_U=6, thus the critical ratio is equal to 6/(6+2) = 0.75. Now the demand follows discrete distribution. In particular, *X* can take values 0,1,2,3, or 4, with probability 1/5 for each value. In this case the cumulative distribution function is equal to

$$F(0) = 0.2,\ F(1) = 0.4,\ F(2) = 0.6,\ F(3) = 0.8,\ F(4) = 1.$$

We see that the previous equation does not have a solution. However the general formula for the order quantity yields Q^*=3, since this is the smallest value for which *F*(*Q*) exceeds the critical ratio value of 0.75.

Before we proceed, we should discuss the issue of cost modeling. The definition of C_0 and C_U is sufficiently general to encompass several situations in practice. Depending on the actual conditions, the form of the overstocking and understocking costs may be different from one problem to another. For example, consider a situation where the retailer procures the product from a supplier at a cost of *c* per unit and sells it at a retail price *r*. Unsold quantities at the end of the season are bought by a stock house at a salvage price $s<c$. In this case, the overstocking cost, i.e., the net cost to the retailer for every unit of product procured and not sold during the regular season is equal to $C_0=c-s$. On the other hand, the understocking cost, i.e., the cost for each unit of demand that cannot be satisfied due to lack of product is equal to the foregone profit that could be made, had this unit been available, thus, $C_U=r-c$. In many situations, a shortage may have more severe consequences than just a foregone profit, in the sense that it causes a loss of customer goodwill that may affect future sales as well. If that is the case, then the understocking cost must be set higher than the foregone profit, by an amount per unit that represents the cost of loss of goodwill.