## Γενικά Έστω

$$z_p = \max f(\underline{x})$$
$$g_i(\underline{x}) \le 0 \quad i=1,\dots,m$$
$$\underline{x} \in S \subseteq \mathbb{R}^n$$

Ορίζω $\boxed{L(\underline{x},\underline{w}) = f(\underline{x}) - \sum_{i=1}^{m} w_i g_i(\underline{x})}$ Lagrangean

$\underline{x} \in \mathbb{R}^n$, $\underline{w} \in \mathbb{R}^m$, $\underline{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_m \end{pmatrix}$

$$Z_L(\underline{w}) = \max_{\text{υ.π. } \underline{x} \in S} L(\underline{x},\underline{w}) = \max_{\underline{x}} f(\underline{x}) - \sum_{i=1}^{m} w_i g_i(\underline{x}), \quad \underline{x} \in S$$

$w_i$ : Πολλαπλασιαστές Lagrange.

$z_L(\underline{w})$ : Λαγκρανζιανή χαλάρωση (Lagrangean relaxation problem)

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**Θεώρημα**: $z_p \le z_L(\underline{w}) \quad \forall \underline{w} \ge 0$

**Απόδειξη** $z_L(w) = \max\{L(x,\underline{w}) ; \underline{x} \in S\}$

$$Z_L(\underline{w}) = \max\{L(\underline{x},\underline{w}) : \underline{x} \in S\} \ge \max\{L(\underline{x},\underline{w}) : \underline{x} \in S,\ g_i(\underline{x}) \le 0\ \forall i\}$$

Για $\underline{w} \ge 0$ κ' $\underline{x} \in S$, $g_i(\underline{x}) \le 0\ \forall i$

$$L(\underline{x},\underline{w}) = f(\underline{x}) - \sum w_i g_i(\underline{x}) \ge f(\underline{x})$$
$$w_i \ge 0$$
$$g_i(\underline{x}) \le 0$$

$$\Rightarrow \max\{L(\underline{x},\underline{w}) : \underline{x} \in S, g_i(\underline{x}) \le 0\}$$
$$\ge \underbrace{\max\{f(\underline{x}) : \underline{x} \in S,\ g_i(\underline{x}) \le 0\}}_{= Z_p} \quad (\text{για } \underline{w} \ge 0)$$

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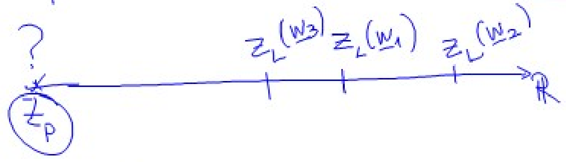
$$Z_p \le Z_L(\underline{w}) \quad \forall\, \underline{w} \ge 0 \Rightarrow z_p \le \inf\{z_L(\underline{w}), \underline{w} \ge 0\}$$

δυϊκό πρόβλημα




## Δυϊκό πρόβλημα

$$z_D = \inf\{z_L(\underline{w}) : \underline{w} \ge 0\}$$

$$\Rightarrow \quad z_p \le z_D \quad \left(\text{Ασθενές θεώρημα δυϊκότητας}\right)$$

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## Ειδική Περίπτωση Γραμμικός Προγρ.

Έστω $P = HK(1, A, \underline{b}, \underline{c})$

$$z_p = \max \underline{c}'\cdot\underline{x} \quad \text{υ.π. } A\underline{x} \le \underline{b},\ \underline{x} \ge 0 \qquad = \max \underline{c}'\cdot\underline{x},\ \ A\underline{x} - \underline{b} \le 0,\ \ \underline{x} \ge 0$$

$g_i(x) \le 0, i=1,\dots,m$

$\underline{x} \in S$

Τότε $g_i(\underline{x}) \le 0$ : $\underbrace{\underline{a}_i' x - b_i}_{g_i(\underline{x})} \le 0$, $\underline{a}_i'$: i-γραμμή του $A$

$$L(\underline{x},\underline{w}) = f(\underline{x}) - \sum_{i=1}^{m} w_i g_i(\underline{x})$$
$$= \underline{c}'\underline{x} - \sum_{i=1}^{m} w_i(\underline{a}_i' \underline{x} - b_i) = \sum_{i=1}^{m} w_i b_i + \underline{c}'\underline{x} - \underline{w}' A \underline{x}$$
$$= \underline{w}'\cdot\underline{b} + \underline{c}'\cdot\underline{x} - \underline{w}'\cdot A\cdot\underline{x} = \underline{w}'\cdot\underline{b} + (\underline{c}' - \underline{w}'A)\cdot\underline{x}$$

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$$\Rightarrow Z_L(\underline{w}) = \max\{L(\underline{x},\underline{w}) : \underline{x} \in S\}$$
$$= \max_{\underline{x}}\{\underline{w}'\underline{b} + (\underline{c}' - \underline{w}'A)\underline{x} : \underline{x} \ge 0\}$$
$$= \underline{w}'\cdot\underline{b} + \underbrace{\max\{(\underline{c}' - \underline{w}'A)\underline{x} : \underline{x} \ge 0\}}_{f_L(\underline{w})}$$

# Παρένθεση

Έστω $f : \mathbb{R}^n \to \mathbb{R}$ , $F_1 \subseteq F_2 \subseteq \mathbb{R}^n$

Τότε $\max\left\{ f(\underline{x}),\ x \in F_1 \right\} \leq \max\left\{ f(\underline{x}) : x \in F_2 \right\}$

$$f_L(\underline{w}) = \max\left\{\underbrace{(\underline{c}' - \underline{w}'A)}_{\underline{\varphi}'}\underline{x} : \underline{x} \geq 0\right\}$$

$$= \max\left\{\underline{\varphi}' \cdot \underline{x} : \underline{x} \geq 0\right\}, \quad \underline{\varphi} = \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_n \end{pmatrix}$$

$$\max_{x_1, x_2 \geq 0} (2x_1 - 3x_2) = +\infty \qquad \max_{x_1, x_2 \geq 0} 2x_1 + 3x_2 = +\infty$$

$$f_L(\underline{w}) = \begin{cases} +\infty & \text{αν } \exists j : \varphi_j > 0 \\ 0 & \text{αν } \varphi_j \leq 0 \ \forall j \end{cases}$$

$$Z_D = \inf\left\{Z_L(\underline{w}) : \underline{w} \geq 0\right\} = \inf\left\{\underline{w}' \cdot \underline{b} + f_L(\underline{w})\right\}$$

$$= \inf\left\{\underline{w}' \cdot \underline{b} + f_L(\underline{w}) : \underline{w} \geq 0, f_L(\underline{w}) < \infty\right\}$$

Όμως $f_L(\underline{w}) < \infty \Leftrightarrow \underline{\varphi}' \leq 0 \Rightarrow \underline{c}' - \underline{w}'A \leq 0$

και τότε $f_L(\underline{w}) = 0$

$$\Rightarrow Z_D = \inf\left\{\underline{w}' \cdot \underline{b} + 0 : \underline{w} \geq 0, \underline{c}' - \underline{w}'A \leq 0\right\}$$

$$\Rightarrow \boxed{\begin{aligned} Z_D = \inf \ & \underline{w}' \cdot \underline{b} \\ & A'\underline{w} \geq c \\ & \underline{w} \geq 0 \end{aligned}}$$

$$Z_P \leq Z_D$$