Absolute and Unconditional Convergence in Normed Linear Spaces

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ABSOLUTE AND UNCONDITIONAL CONVERGENCE IN NORMED LINEAR SPACES

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1. Let B be a real Banach space and denote by $||x||$ the norm of an element x of B. The series

$$\sum_{\nu=1}^{\infty} x_\nu, \ (x_\nu \in B, \ \nu = 1, 2, \ldots) \tag{1}$$

is called absolutely convergent if $\sum ||x_\nu|| < \infty$; it is called unconditionally convergent if the series $\sum y_\nu$ converges whenever the sequence $(y_\nu)_1^\infty$ is a rearrangement of the sequence $(x_\nu)_1^\infty$. An equivalent definition of unconditional convergence of (1) is obtained by requiring $\sum \pm x_\nu$ to be convergent for every choice of the signs. There are several other equivalent definitions; most of these have been discussed by T. H. Hildebrandt.[1]

It is clear that if B is of finite (linear) dimension then (1) is unconditionally convergent if and only if it is absolutely convergent. The problem of finding the spaces for which these two types of convergence are equivalent is mentioned by S. Banach.[2] The primary aim of this note is to settle this problem by proving the following result.

THEOREM 1. *The unconditionally convergent series coincide with the absolutely convergent series if and only if the space B is of finite dimension.*

Here the only non-trivial assertion is that, if B is of infinite dimension, there is a series (1), which is unconditionally but not absolutely convergent. It is easy to give examples of such series in Hilbert space and similar examples have been given[3] for all the usually encountered infinitely dimensional Banach spaces. Interesting partial results on the problem solved by Theorem 1 have been established by M. E. Munroe[4] and S. Karlin.[5] The two last mentioned papers treat also some related problems and give various consequences of Theorem 1.

Our method of proof yields not only Theorem 1 but also the following result.

THEOREM 2. *If B is of infinite dimension and $\sum c_\nu$ is any convergent series of positive terms, then there exists an unconditionally convergent series (1) satisfying $||x_\nu||^2 = c_\nu$ for $\nu = 1, 2, \ldots$.*

Applying this result with $c_\nu = \nu^{-1}[\log (1 + \nu)]^{-2}$ we obtain:

COROLLARY: *If B is of infinite dimension then there exists an unconditionally convergent series (1) having the property that $\sum ||x_\nu||^{2-\epsilon} = \infty$ for every $\epsilon > 0$.*

Theorem 1 is obviously an immediate consequence of this Corollary.

If B is a Hilbert space then $\sum\|x_\nu\|^2 < \infty$ for every unconditionally convergent series (1). Thus Theorem 2 and its Corollary are in a certain sense best possible results.

A result (Lemma 1) concerning convex bodies in Euclidean space is proved in section 2. In section 3 this lemma is used to prove Theorem 2, and remarks are made concerning its extension. In section 4 some geometrical properties of convex bodies are obtained from Lemma 1 and from the construction used to prove this lemma.

2. We consider the n-dimensional Euclidean space of points $U = (u_1, \ldots, u_n)$ and use the usual vector notation. We first prove our main lemma.

LEMMA 1. *Let C be a body[6] which is convex and has the origin O as center, and let r be an integer with $1 \leq r \leq n$. Then there are n points $A_1, \ldots, A_n$ on the boundary of C such that, if $\lambda_1, \ldots, \lambda_r$ are any r real numbers with $1 \leq r \leq n$, then the point $\lambda_1 A_1 + \lambda_2 A_2 + \ldots + \lambda_r A_r$ is in the body λC where*

$$\lambda^2 = \left[2 + \frac{r(r-1)}{n}\right](\lambda_1^2 + \lambda_2^2 + \ldots + \lambda_r^2). \qquad (2)$$

Proof: We inscribe in C an ellipsoid with O as center having the largest possible n-dimensional volume. Since it is enough to establish the lemma for any affine transform of C, we may assume that this ellipsoid is the sphere S of unit radius.

We first show that after a suitable orthogonal transformation has been applied there will be r points $A_1, \ldots, A_r$ of contact of C with S, satisfying for $\rho = 1, 2, \ldots, r$

$$\left.\begin{aligned} A_\rho &= (a_{\rho 1}, a_{\rho 2}, \ldots, a_{\rho\rho}, 0, \ldots, 0), \\ a_{\rho 1}^2 + \ldots + a_{\rho(\rho-1)}^2 &= 1 - a_{\rho\rho}^2 \leq \frac{\rho - 1}{n}. \end{aligned}\right\} \qquad (3)$$

For $r = 1$ this is clear; assuming it for $r = m - 1 < n$ we prove it for $r = m$. The ellipsoid

$$(1+\epsilon)^{n-m+1}(u_1^2 + \ldots + u_{m-1}^2) + (1 + \epsilon + \epsilon^2)^{-m+1}(u_m^2 + \ldots + u_n^2) \leq 1, \qquad (\epsilon > 0) \quad (4)$$

has a volume larger than that of S. Hence there is a point $A = A(\epsilon) = (a_1, \ldots . a_n)$ on the boundary of C in the ellipsoid (4). But, since A being on the boundary of C is not inside the unit sphere, we have $a_1^2 + \ldots + a_n^2 \geq 1$. It follows that A satisfies

$$[(1+\epsilon)^{n-m+1} - 1](a_1^2 + \ldots + a_{m-1}^2) + [(1+\epsilon+\epsilon^2)^{-m+1} - 1](a_m^2 + \ldots + a_n^2) \leq 0. \quad (5)$$

If $\epsilon \to 0$ through a suitable sequence of positive numbers the corresponding sequence $A(\epsilon)$ will converge to a point A_m. It is clear from (4) that A_m is a point of contact of S and the boundary of C, while from (5) we have in the limit

$$(n - m + 1)(a_{m1}^2 + \ldots + a_{m(m-1)}^2) + (-m + 1)(a_{mm}^2 + \ldots + a_{mn}^2) \leq 0. \quad (6)$$

By a suitable orthogonal transformation of the variables $u_m, \ldots, u_n$, leaving the points $A_1, \ldots, A_{m-1}$ invariant we may make the last $n - m$ coordinates of A_m vanish. Then, using (6) and the equation $a_{m1}^2 + \ldots + a_{mn}^2 = 1$ we obtain (3) with $\rho = m$. Thus (3) is proved for $\rho = 1, 2, \ldots, n$

Let $\lambda_1, \ldots, \lambda_r$ be any real numbers. By (3) the square of the distance from O to the point $\lambda_1 A_1 + \ldots + \lambda_r A_r$ is

$$\sum_{\sigma=1}^{r}\left(\sum_{\rho=\sigma}^{r} \lambda_\rho a_{\rho\sigma}\right)^2 \leq \sum_{\sigma=1}^{r}\left[2\lambda_\sigma^2 a_{\sigma\sigma}^2 + 2\left(\sum_{\rho=\sigma+1}^{r} \lambda_\rho a_{\rho\sigma}\right)^2\right]$$
$$\leq \sum_{\sigma=1}^{r} 2\left[\lambda_\sigma^2 a_{\sigma\sigma}^2 + \left(\sum_{\rho=\sigma+1}^{r} \lambda_\rho^2\right)\left(\sum_{\tau=\sigma+1}^{r} a_{\tau\sigma}^2\right)\right]$$
$$= 2\sum_{\rho=1}^{r}\left[a_{\rho\rho}^2 + \sum_{\tau=1}^{r}\sum_{\sigma=1}^{\min.(\rho-1,\tau-1)} a_{\tau\sigma}^2\right]\lambda_\rho^2.$$

But by (3), the last expression is less than or equal to

$$2\sum_{\rho=1}^{r}\left(1 + \sum_{\tau=1}^{r}\frac{\tau - 1}{n}\right)\lambda_\rho^2 = \left[2 + \frac{r(r-1)}{n}\right]\sum_{\rho=1}^{r}\lambda_\rho^2 = \lambda^2.$$

Thus the point $\lambda_1 A_1 + \ldots + \lambda_r A_r$ is contained in the sphere λS and so is contained in the body λC. This proves the lemma.

3. Before we prove Theorem 2 it is convenient to obtain the following consequence of Lemma 1.

LEMMA 2. *Let B be a Banach space of infinite dimension and let $c_1, \ldots, c_r$ be any given positive numbers. Then there exist points $x_1, \ldots, x_r$ in B with $||x_\rho||^2 = c_\rho$ for $\rho = 1, \ldots, r$ and such that, if $\sum'$ denotes the sum over any subset of the numbers $1, \ldots, r$, then*

$$||\textstyle\sum' x_\rho||^2 \leq 3\sum' c_\rho. \quad (7)$$

Proof: Write $n = r(r - 1)$. As B is of infinite dimension we can choose n linearly independent elements $z_1, \ldots, z_n$. Then the points $U = (u_1, \ldots, u_n)$ with $||u_1 z_1 + \ldots + u_n z_n|| \leq 1$ form a convex body C with the origin as center in n-dimensional Euclidean space. Let $A_1, \ldots, A_r$ be the points given by Lemma 1. Writing $A_\rho = (a_{\rho 1}, \ldots, a_{\rho n})$, we put

$$x_\rho = c_\rho^{1/2}(a_{\rho 1} z_1 + \ldots + a_{\rho n} z_n), \qquad \rho = 1, \ldots, r.$$

Then, as $A_1, \ldots, A_r$ are on the boundary of C, we have $||x_\rho||^2 = c_\rho$, for $\rho = 1, \ldots, r$. Further, as the point $\sum' c_\rho^{1/2} A_\rho$ is in λC where $\lambda^2 = 3\sum' c_\rho$, it follows that (7) is satisfied. This proves the lemma.

Proof of Theorem 2: Choose a strictly increasing sequence $n_1 = 0, n_2, n_3, \ldots$ of integers such that the series

$$\sum_{r=1}^{\infty} \left(\sum_{\nu = n_r + 1}^{n_{r+1}} c_\nu \right)^{1/2}$$

is convergent. By Lemma 2 we can choose x_ν for $n_r < \nu \leq n_{r+1}$ so that $||x_\nu||^2 = c_\nu$ and $||\sum^{(r)} x_\nu||^2 \leq 3\sum^{(r)} c_\nu$, the sum $\sum^{(r)}$ being taken over any subset of the integers ν with $n_r < \nu \leq n_{r+1}$. Let $\sum y_\nu$ be any rearrangement of the series $\sum x_\nu$. Let $\epsilon > 0$ be given. Choose r so large that

$$\sum_{\rho = r}^{\infty} \left(\sum_{\nu = n_\rho + 1}^{n_{\rho+1}} c_\nu \right)^{1/2} < \frac{\epsilon}{2}.$$

Choose p so large that the sum $\sum_{\nu < p} y_\nu$ includes all the terms x_ν with $\nu \leq n_r$. Then for any $q > p$ we have

$$\left|\left| \sum_{\nu = p}^{q} y_\nu \right|\right| \leq \sum_{\rho = r}^{\infty} \left|\left| \sum^{(\rho)} x_\nu \right|\right| \leq \sum_{\rho = r}^{\infty} \left(3 \sum_{\nu = n_{\rho+1}}^{n_{\rho+1}} c_\nu \right)^{1/2} < \epsilon.$$

Since B is complete it follows that $\sum y_\nu$ is convergent. As this is true for every rearrangement of $\sum x_\nu$, the series $\sum x_\nu$ is unconditionally convergent and Theorem 2 is proved.

We note that the completeness of B was used only to deduce the convergence of $\sum y_\nu$ from its Cauchy convergence. Hence we have (with obvious meaning of unconditional Cauchy convergence)

THEOREM 3. *Let N be an infinitely dimensional normed linear space over the reals and $\sum c_\nu$ be any convergent series of positive numbers. Then there exists an unconditionally Cauchy convergent series $\sum x_\nu$ of elements of N satisfying $||x_\nu||^2 = c_\nu (\nu = 1, 2, \ldots)$. In particular there exist such series with $\sum ||x_\nu|| = \infty$.*

Since a complex Banach space contains a real one, it is clear that Theorems 1 and 2 hold for complex Banach spaces. A similar remark applies to Theorem 3.

4. In this section we prove some geometrical results. The first result shows that Lemma 1 can be considerably improved in the special case where $r = n$ and $\lambda_1 = \pm 1, \ldots, \lambda_n = \pm 1$.

THEOREM 4. *Let C be a convex body with the origin O as center. Then there are points $P_1, \ldots, P_n$ on the boundary of C such that all the 2^n points $\pm P_1 \pm \ldots \pm P_n$ are in the body $2n^{3/4}C$.*

Proof: For $n > 1$ let q, r, s be the non-negative integers defined by

$$r(r-1) \leq 2n < r(r+1), \qquad n = qr + s, \qquad s < r. \quad (8)$$

Let $A_1, \ldots, A_r$ be the points thus denoted in Lemma 1 and for $t = 1, 2, \ldots, n$ put $P_t = A_{\nu(t)}$, where $\nu(t) \equiv t \pmod{r}$ and $1 \leq \nu(t) \leq r$.

Then all 2^n points $\pm P_1 \pm \ldots \pm P_n$ are of the form $\lambda_1 A_1 + \ldots + \lambda_r A_r$ where the integers λ_ν satisfy the inequalities

$$|\lambda_\nu| \leq q + 1 \text{ for } 1 \leq \nu \leq s, \qquad |\lambda_\nu| \leq q \text{ for } s < \nu \leq r.$$

Hence, by Lemma 1 all the 2^n points considered are in the body μC where

$$\mu^2 = \left[2 + \frac{r(r-1)}{n}\right] \sum_{\nu=1}^{r} \lambda_\nu^2 \leq 4\,[s(q+1)^2 + (r-s)q^2].$$

Taking account of (8) it is easily checked that $s(q+1)^2 + (r-s)q^2 < n^{1/2}$ for $n > 1$. The theorem being obvious for $n = 1$, is thus completely proved.

Remark: It is of some interest to find the exact dependence of μ on n. Our method, though capable of improving the constant 2 in this theorem, cannot improve the power in the estimate $\mu < 2n^{1/4}$. When C is a sphere then an enlargement by the factor $n^{1/2}$ is sufficient. Perhaps this is generally true, but we cannot prove it for $n \geq 3$.

We give a proof for $n = 2$ in the hope that it may be generalized to other values of n. Let B be the two-dimensional Banach space whose unit sphere is C. Given any point P_1 in this space with $||P_1|| = 1$ there exists, by continuity, a point P_2 satisfying $||P_2|| = 1$ and $||P_1 + P_2|| = ||P_1 - P_2||$. Let α denote this common norm, then also $||\pm P_1 \pm P_2|| = \alpha$. Now put $Q_1 = (P_1 + P_2)/\alpha$, $Q_2 = (P_1 - P_2)/\alpha$, then $||Q_1|| = ||Q_2|| = 1$ and $||\pm Q_1 \pm Q_2|| = 2/\alpha$. Since $\min.(\alpha, 2/\alpha) \leq 2^{1/2}$ the proof is completed.

The following results are simple consequences of the construction used in proving Lemma 1. We include them since they seem to be of some geometrical interest.

THEOREM 5A. *Let C be a convex body with the origin as center. Then there is an ellipsoid $\mathcal{E}$ contained in C and a parallelopiped $\mathcal{P}$ containing C with volumes $V(\mathcal{E})$ and $V(\mathcal{P})$ satisfying*

$$\frac{V(\mathcal{P})}{V(\mathcal{E})} \leq \frac{2^n}{J_n}\left(\frac{n^n}{n!}\right)^{1/2}, \tag{9}$$

where J_n is the volume of the unit n-dimensional sphere.

Proof: Take $\mathcal{E}$ to be an ellipsoid with O as center having the largest possible volume. As in the proof of Lemma 1, we may suppose without loss of generality that $\mathcal{E}$ is the unit sphere S and denote by $A_1, \ldots, A_n$ points of contact of C and S satisfying (3). As C contains S the only tac-plane to C at A_r is the plane $a_{r1}u_1 + \ldots + a_{rr}u_r = 1$. Thus C is contained in the parallelopiped $\mathcal{P}$ defined by $|a_{r1}u_1 + \ldots + a_{rr}u_r| \leq 1, r = 1, 2, \ldots, n$. By (3) the volume of $\mathcal{P}$ satisfies

$$V(\mathcal{P}) = 2^n\,|a_{11}a_{22}\ldots a_{nn}|^{-1} \leq 2^n\left(\frac{n^n}{n!}\right)^{1/2} = 2^n\left(\frac{n^n}{n!}\right)^{1/2}\frac{V(\mathcal{E})}{J_n}.$$

THEOREM 5B. *Under the conditions of Theorem 5A there is an ellipsoid* $\mathcal{E}$ *containing* C *and an "octahedron"* $\mathcal{O}$ *contained in* C, *with*

$$\frac{V(\mathcal{E})}{V(\mathcal{O})} \leq (1/2)^n J_n \, (n!n^n)^{1/2}. \tag{10}$$

Proof: The result follows immediately by application of Lemma 3 to the body K which is the polar reciprocal of C.

THEOREM 6. *Let* C *and* K *be convex bodies with the origin as center, which are polar reciprocal. Then their volumes satisfy*

$$\frac{2^n J_n}{(n!n^n)^{1/2}} \leq V(C) \cdot V(K) \leq 2^n J_n \left(\frac{n^n}{n!}\right)^{1/2}. \tag{11}$$

Proof: By Lemma 3 we may suppose without loss of generality that C contains the unit sphere S and is contained in a parallelopiped $\mathcal{P}$ with volume $V(\mathcal{P})$ satisfying

$$V(\mathcal{P}) \leq 2^n \left(\frac{n_n}{n!}\right)^{1/2}. \tag{12}$$

Then K is contained in S and contains an "octahedron" $\mathcal{O}$ with

$$V(\mathcal{O}) \geq \frac{2}{(n!n^n)^{1/2}} \tag{13}$$

The inequalities (11) now follow trivially from (12), (13) and the inclusion relations $S \subset K \subset \mathcal{P}$ and $\mathcal{O} \subset K \subset S$.

The bounds on the right of (9) and (10) can be written in the form $(\gamma_n n)^{n/2}$ where γ_n tends to a positive limit as n tends to infinity. It is easy to see that it is impossible to obtain such bounds with γ_n tending to zero as n tends to infinity. The bounds in (11) are considerably closer than those obtained by K. Mahler[7] but they are probably very far from the best possible.

[1] *Bull. Am. Math. Soc.*, **46**, 959–962 (1940).

[2] *Théorie des Opérations Linéaires*, Warsaw, 1932, p. 240.

[3] E.g., Orlicz, W., *Stud. Math.*, **4**, 51–47 (1933); Macphail, M. S., *Bull. Am. Math. Soc.*, **53**, 121–123 (1947).

[4] *Duke Math. J.*, **13**, 351–365 (1946).

[5] *Ibid.*, **15**, 971–985 (1948).

[6] I.e., the closure of a bounded open set.

[7] *Časopis Pěst. Mat. Fys.*, **68**, 93–102 (1939).