Uniqueness of translation invariant measures

joint work by The Class

Let G be an abelian group equiped with a σ-algebra $\mathcal{S} \subseteq \mathcal{P}(G)$ such that[1] *for all $g \in G$ the map $\lambda_g : G \to G : h \to g + h$ is measurable. Suppose μ and ν are two σ-finite measures on $\mathcal{S}$ which are **left invariant**, that is $\mu(\lambda_g(E)) = \mu(E)$ for all $g \in G$ and $E \in \mathcal{S}$, and similarly for ν. Then there exists $c > 0$ such that $\nu = c\mu$.*

Proof By σ-finiteness, there exists $B \in \mathcal{S}$ such that $\nu(B)$ and $\mu(B)$ are both nonzero and finite. Let $c = \frac{\nu(B)}{\mu(B)}$. Replacing μ by the measure $\mu'(E) = c\mu(E)$, we may assume that $\mu(B) = \nu(B)$ and we will show that $\mu = \nu$.

Claim Given any pair E, F of sets in $\mathcal{S}$, we claim that

$$\int \chi_E(x-y)\chi_F(x)d\mu(x) = \int \chi_E(x)\chi_F(y+x)d\mu(x) \qquad \text{for all } \ y \in G$$

Proof of the Claim Observe first that if $f : G \to [0, +\infty]$ is measurable, then

$$\int_G f(x+g)d\mu(x) = \int_G f(x)d\mu(x) \qquad \text{for all } \ g \in G$$

and similarly for ν. Indeed, if $f = \chi_E$, note that $f(x+g) = \chi_{E_g}(x)$, where $E_g = \lambda_g^{-1}(E)$, because $f(x+g) = f(\lambda_g(x)) = 1$ iff $\lambda_g(x) \in E$ iff $x \in \lambda_g^{-1}(E)$. But $\mu(E_g) = \mu(E)$ and thus

$$\int f(x+g)d\mu(x) = \int \chi_{E_g}(x)d\mu(x) = \mu(E_g) = \mu(E) = \int f d\mu.$$

By linearity of the integral the equality is valid whenever f is a nonnegative measurable simple function. For general f let s_n be an increasing sequence of nonnegative measurable simple functions such that $s_n \to f$ pointwise, observe that $s_n \circ \lambda_g \to f \circ \lambda_g$ and apply the monotone convergence theorem.

If $y \in G$, applying this to the function f_y given by $f_y(x) = \chi_E(x)\chi_F(x+y)$ gives

$$\int \chi_E(x-y)\chi_F(x)d\mu(x) = \int f_y(x-y)d\mu(x) = \int f_y(x)d\mu(x) = \int \chi_E(x)\chi_F(y+x)d\mu(x)$$

which proves the Claim.

It follows that

$$\int \left(\int \chi_E(x-y)\chi_F(x)d\mu(x) \right) d\nu(y) = \int \left(\int \chi_E(x)\chi_F(y+x)d\mu(x) \right) d\nu(y).$$

[1]haar, 5 feb 09

Since the integrands are both non-negative and $\mathcal{S}\otimes\mathcal{S}$ measurable functions on $\times G$, using Tonelli's Theorem, we get

$$\int\left(\int \chi_E(x-y)\chi_F(x)d\nu(y)\right)d\mu(x) = \int\left(\int \chi_E(x)\chi_F(y+x)d\nu(y)\right)d\mu(x).$$

Now we calculate

$$\begin{aligned}\int\left(\int \chi_E(x)\chi_F(y+x)d\nu(y)\right)d\mu(x) &= \int\left(\chi_E(x)\int \chi_{\lambda_{-x}(F)}(y)d\nu(y)\right)d\mu(x)\\ &= \int \chi_E(x)\nu(\lambda_{-x}(F))d\mu(x) = \int \chi_E(x)\nu(F)d\mu(x)\\ &= \mu(E)\nu(F)\end{aligned}$$

(since $\nu(\lambda_{-x}(F)) = \nu(F)$) and

$$\begin{aligned}\int\left(\int \chi_E(x-y)\chi_F(x)d\nu(y)\right)d\mu(x) &= \int\left(\chi_F(x)\int \chi_{-E}(y-x)d\nu(y)\right)d\mu(x)\\ &= \int\left(\chi_F(x)\int \chi_{\lambda_x(-E)}(y)d\nu(y)\right)d\mu(x)\\ &= \int \chi_F(x)\nu(\lambda_x(-E))d\mu(x) = \int \chi_F(x)\nu(-E)d\mu(x)\\ &= \nu(-E)\mu(F)\end{aligned}$$

and so

$$\mu(E)\nu(F) = \nu(-E)\mu(F) \qquad \text{for all } \ E, F \in \mathcal{S}. \tag{1}$$

Set $F = B$ in (1) to obtain

$$\begin{aligned}\mu(E)\nu(B) = \nu(-E)\mu(B) &= \nu(-E)\nu(B)\\ \text{and so } \ \mu(E) = \nu(-E) &\qquad \text{for all } \ E \in \mathcal{S}\end{aligned}$$

since $0 < \nu(B) < +\infty$. Applying the last equality to B we obtain $\mu(B) = \nu(-B)$ and now (1) for $E = B$ gives

$$\mu(B)\nu(F) = \nu(-B)\mu(F)$$

and so finally

$$\nu(F) = \mu(F) \qquad \text{for all } \ F \in \mathcal{S}. \qquad \square$$

Remarks [A.K.] *Note that we have actually shown that $\mu(E) = \mu(-E)$ for all $E \in \mathcal{S}$: thus any translation invariant measure is automatically reflection invariant. This is not true in general for non-abelian groups.*

For the case $G = \mathbb{R}^n, \mathcal{S} = \mathcal{B}_G$, the above shows that any translation invariant Borel measure on $\mathbb{R}^n$ is a multiple of Lebesgue measure.

It is known that any (not necessarily abelian) locally compact (Hausdorff) group G admits a (left-) translation invariant regular Borel measure μ, called ***Haar measure****. The proof is non-trivial.*

Any other left translation invariant regular Borel measure on G is a positive multiple of Haar measure. The proof in the non-abelian case also uses Tonelli's theorem.