

SEEMOUS 2025 NKUA Team Report

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1 Introduction and Results

The 19th South Eastern European Mathematical Olympiad for University Students took place in Korçë, Albania, from the 4th till the 9th of March, 2025. NKUA sent 6 students to the competition (5 from the Department of Mathematics and 1 from the Department of Physics).

There were 4 problems in the contest, each carrying 10 points, with the whole exam totalling 40 points. The time allowed for work was 5 hours. The cut-offs for the Bronze, Silver and Gold medals, respectively, were 4, 16 and 30 points, respectively. NKUA performed strongly in SEEMOUS 2025, obtaining 2 Gold Medals, 1 Silver Medal and 3 Bronze Medals, placing 4th in the ranking of the 28 Universities that participated in the contest. Our results were the following:

Contestant	#1	#2	#3	#4	Total	Award
Orestis Lignos	10	5	10	10	35	Gold Medal
Anastasios Pastos	0	10	10	10	30	Gold Medal
Dimitrios Fotopoulos	2	10	0	10	22	Silver Medal
Lampros Tegos	10	0	1	3	14	Bronze Medal
Emmanouil Petrakis	10	1	2	0	13	Bronze Medal
Panagiotis-Nikolaos Glyptis	0	9	0	0	9	Bronze Medal

In the following, we present some of the solutions we gave in the contest problems.

2 Contest Problems

Problem 1: Let A be an $n \times n$ matrix with strictly positive elements and two vectors $u, v \in \mathbb{R}^n$, also with strictly positive elements, such that

$$Au = v \text{ and } Av = u.$$

Prove that $u = v$.

Problem 2: Calculate

$$\lim_{n \rightarrow \infty} n \int_0^\infty e^{-x} \sqrt[n]{e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} - \cdots - \frac{x^n}{n!}} dx.$$

Problem 3: Let $A \in \mathcal{M}_n(\mathbb{C})$ such that $A^* A^2 = A A^*$. Prove that $A^2 = A$. (Here we denote by A^* the conjugate transpose of A .)

Problem 4: Let $(a_n)_{n \geq 1}$ be a monotone decreasing sequence of real numbers that converges to 0. Prove that $\sum_{n=1}^{\infty} \frac{a_n}{n}$ is convergent if and only if the sequence $(a_n \ln n)_{n \geq 1}$ is bounded and $\sum_{n=1}^{\infty} (a_n - a_{n+1}) \ln n$ is convergent.

3 Solutions

3.1 Problem 1

Solution 1 (due to Emmanouil Petrakis): A^2 has positive elements (as a simple calculation shows), so let $A^2 = (a_{ij})_{1 \leq i, j \leq n}$ with $a_{ij} > 0$ for all i, j .

Note that $A^2 u = A(Au) = Av = u$ and $A^2 v = v$. Hence $A^2(u - tv) = u - tv$, (1) for all $t \in \mathbb{R}$. We choose $t = \min \left\{ \frac{u_i}{v_i}, i \in \{1, 2, \dots, n\} \right\}$, denoting by u_i, v_i the respective elements of u, v .

Hence, $u_i - tv_i \geq 0$ for all $i \in \{1, 2, \dots, n\}$, and moreover we can find a $j \in \{1, 2, \dots, n\}$ such that $u_j - tv_j = 0$. At relation (1), by looking the j -th row, we obtain:

$$\sum_{k=1}^n a_{jk}(u_k - tv_k) = u_j - tv_j = 0.$$

However, in this relation the left hand side is a sum of non-negative numbers, hence

$$a_{jk}(u_k - tv_k) = 0 \Rightarrow u_k = tv_k,$$

for all $k \in \{1, 2, \dots, n\}$.

We conclude that $u = tv$. Now,

$$Au = v \Rightarrow A(tv) = v \Rightarrow t(Av) = v \Rightarrow tu = v \Rightarrow t^2 v = v,$$

and so we conclude that $t^2 = 1$, that is $t \in \{-1, 1\}$. However, if $t = -1$ we obtain a clear contradiction since u, v have positive elements, hence $t = 1$, as desired.

Solution 2 (due to Orestis Lignos. In fact, the two solutions are more or less isomorphic to each other): Let us denote $A = (a_{ij})_{1 \leq i, j \leq n}$, $u = (u_1, u_2, \dots, u_n)$ and $v = (v_1, v_2, \dots, v_n)$. Note that all variables are positive from the problem statement. We know that

$$a_{i1}u_1 + a_{i2}u_2 + \dots + a_{in}u_n = v_i$$

and

$$a_{i1}v_1 + a_{i2}v_2 + \dots + a_{in}v_n = u_i$$

for all $1 \leq i \leq n$. Let us denote $k = \min_{1 \leq i \leq n} \left(\frac{u_i}{v_i} \right)$, and WLOG let $k = \frac{u_1}{v_1}$. Note that

$$\sum_{1 \leq j \leq n} a_{j1}(u_j u_j - v_j v_j) = u_1 v_1 - u_1 v_1 = 0, \quad (1)$$

for all $1 \leq i \leq n$. Moreover,

$$\sum_{1 \leq j \leq n} a_{ij}(u_j v_1 - u_1 v_j) = v_i v_1 - u_i u_1$$

and so

$$\sum_{2 \leq j \leq n} a_{ij}(u_j v_1 - u_1 v_j) = v_i v_1 - u_i u_1, \quad (2)$$

for all $1 \leq i \leq n$. Now, notice that $u_j v_1 - u_1 v_j \geq 0$ for all $1 \leq j \leq n$, and so by relation (2) we obtain that $v_i v_1 \geq u_i u_1$ for all $1 \leq i \leq n$. Therefore, putting $i = 1$ in relation (1) we obtain that the left hand side is ≤ 0 , and so equality must hold, i.e. $v_i v_1 = u_i u_1$ for all $1 \leq i \leq n$.

Putting $i = 1$ this readily implies that $k = 1$, that is $u_1 = v_1$, and subsequently $u_i = v_i$ for all $1 \leq i \leq n$, as desired.

3.2 Problem 2

Solution (due to Panagiotis-Nikolaos Glyptis): From Taylor's theorem (we use the Lagrange remainder form), there exists a $c \in (0, x)$ such that:

$$e^x - \sum_{k=1}^n \frac{1}{k!} = \frac{e^c}{(n+1)!} x^{n+1}$$

Now let

$$f_n(x) = \frac{n}{((n+1)!)^{\frac{1}{n}}} e^{\frac{c}{n}-x} x^{1+\frac{1}{n}}$$

and since we easily infer (for example, using Stirling's formula or the Cesaro-Stolz theorem) that

$$\lim_{n \rightarrow \infty} \frac{n}{((n+1)!)^{\frac{1}{n}}} = e,$$

we obtain that

$$f_n(x) \rightarrow e^{1-x} x$$

pointwise, and moreover

$$e^{\frac{c}{n}-x} < e^{\frac{x}{2}-x} \text{ for all } n \geq 2$$

and

$$64e^{\frac{x}{4}} > x^{1+\frac{1}{n}} \text{ for all } n \geq 2,$$

and since the sequence $\frac{n}{((n+1)!)^{\frac{1}{n}}}$ converges, it is bounded and so there exists a $M >$

0 such that $\frac{n}{((n+1)!)^{\frac{1}{n}}} < M$ for all $n \geq 1$. All in all, $f_n(x)$ is bounded by a function with a finite integral in the integration interval, and so by the dominated convergence theorem we finally may write

$$\lim_{n \rightarrow \infty} \int_0^\infty f_n(x) dx = \int_0^\infty e^{1-x} x dx = e,$$

and so the given limit equals e .

3.3 Problem 3

Solution 1 (due to Anastasios Pastos): We know that $A^*A^2 = AA^*$, (1). Our first claim is the following:

Claim 1: $\ker A = \ker A^*$, (2).

Proof: Let $u \in \ker A$. From relation (1) we obtain that

$$AA^*u = A^*A^2u = 0 \Rightarrow \langle AA^*u, u \rangle = 0 \Rightarrow \langle A^*u, A^*u \rangle = 0 \Rightarrow A^*u = 0 \Rightarrow u \in \ker A^*$$

Hence $\ker A \subseteq \ker A^*$. However,

$$\text{rank}(A^*) = \text{rank}(\overline{A}^T) = \text{rank } \overline{A} = \text{rank } A,$$

hence we obtain $\dim \text{Im } A^* = \dim \text{Im } A \Rightarrow \dim \ker A^* = \dim \ker A$, by virtue of the rank-nullity theorem. To sum up, we have that $\ker A \subseteq \ker A^*$ and their dimensions are equal, hence the claim follows ■

Moving on, we may write

$$(A^*)^2A = (A^*A^2)^* = (AA^*)^* = AA^* = A^*A^2,$$

that is $A^*(A^*A - A^2) = O_n$, and so by the above Claim (relation (2)) we obtain $A(A^*A - A^2) = O_n$, that is $AA^*A = A^3$, (3).

Multiplying relation (1) with A from the right, we obtain $A^*A^3 = AA^*A \Rightarrow A^*A^3 = A^3$, (4). Now, we move on to our next Claim:

Claim 2: $\ker A \perp \text{Im } A$.

Proof: It is well-known that $\ker A^* \perp \text{Im } A$ (indeed, if $w \in \ker A^*$ and $Av \in \text{Im } A$, then $\langle w, Av \rangle = \langle A^*w, v \rangle = \langle 0, v \rangle = 0$), hence using relation (2) we conclude that $\ker A \perp \text{Im } A$ ■

Using Claim 2, we infer that $\ker A + \text{Im } A$ is a direct sum of two subspaces. However,

$$\dim(\ker A \oplus \text{Im } A) = \dim \ker A + \dim \text{Im } A = n,$$

which implies that $\ker A \oplus \text{Im } A = \mathbb{C}^n$.

Our next step is to choose orthonormal bases $\hat{u} = (u_1, u_2, \dots, u_k)$ and $\hat{v} = (v_1, v_2, \dots, v_m)$, corresponding to the subspaces $\text{Im } A$ and $\ker A$, respectively. Note that if $A = O_n$ then $A^2 = A$ trivially holds. Else, $\text{Im } A \neq \{0\}$. From the above results we know that $\hat{w} = (u_1, u_2, \dots, u_k, v_1, v_2, \dots, v_m)$ is an orthonormal base of \mathbb{C}^n . We perform a change of basis from the ordinary base of \mathbb{C}^n to base \hat{w} .

Thus, we write

$$A = U^{-1} \left(\begin{array}{c|c} B & O_{k \times (n-k)} \\ \hline O_{(n-k) \times k} & O_{(n-k) \times (n-k)} \end{array} \right) U,$$

with B being a $k \times k$ invertible matrix (this follows as $\text{rank } B = \dim \text{Im } A = k$). Moreover, we know that U is a unitary matrix, since its columns are elements of \hat{w} , which themselves constitute an orthonormal basis of \mathbb{C}^n .

Hence, we obtain $A = U^* \left(\begin{array}{c|c} B & O \\ \hline O & O \end{array} \right) U$, and $A^* = U^* \left(\begin{array}{c|c} B^* & O \\ \hline O & O \end{array} \right) U$. Returning back, relation (4) easily turns into $B^* B^3 = B^3 \Rightarrow B^* B^3 B^{-3} = B^3 B^{-3}$, that is $B^* = I_k$, which means $B = I_k$. Thus, to sum up, we obtain $A = U^* \left(\begin{array}{c|c} I_k & O \\ \hline O & O \end{array} \right) U$, and now it is trivial to conclude that $A^2 = A$, as desired.

Solution 2 (due to Orestis Lignos): Let $B = AA^*$ and $C = A^*A$. Then, $A^*A^2 = AA^*$ is rewritten as $B = CA$, and taking $*$ in the previous relation we obtain $(A^*)^2 A = AA^*$, that is $B = A^*C$.

Claim 1: $(CB)^2 = B^3$.

Proof: Note that

$$\begin{aligned} (CB)^2 &= CB \cdot CB = A^*A \cdot AA^* \cdot A^*A \cdot AA^* = (A^*A^2) \cdot ((A^*)^2 A^2) \cdot A^* = \\ &= (AA^*)((A^*)^2 A \cdot A) \cdot A^* = (AA^*)(A \cdot A^*A) \cdot A^* = B^3 \end{aligned}$$

■

Claim 2: Matrices B and C commute.

Proof: We begin with a preliminary result: we prove that AA^*A is Hermitian. Indeed,

$$AA^*A = ((A^*)^2 A) \cdot A = (A^*)^2 A^2 = A^* \cdot (A^*A^2) = A^*AA^*,$$

hence $AA^*A = A^*AA^*$. Now, note that

$$\begin{aligned} BC &= A(A^*)^2 A = (AA^* \cdot A^*)A = (((A^*)^2 A) \cdot A^*)A = A^* \cdot (A^*AA^*) \cdot A = \\ &= A^* \cdot (AA^*A) \cdot A = (A^*A) \cdot (A^*A^2) = (A^*A) \cdot (AA^*) = CB, \end{aligned}$$

as desired (note that we used the first result in the end of the first line) ■

Since B and C are Hermitian and commute, they are simultaneously orthogonally diagonalizable. So, we may write

$$B = P\Delta_B P^* \text{ and } C = P\Delta_C P^*,$$

with P being a unitary matrix. Since $B = AA^*$ and $C = A^*A$, a well-known lemma (XY and YX have the same characteristic polynomial) implies that B and C , i.e. Δ_B and Δ_C , have the same characteristic polynomial, that is the same eigenvalues.

Assume that $\Delta_B = \text{diag}\{b_1, b_2, \dots, b_n\}$ and $\Delta_C = \text{diag}\{c_1, c_2, \dots, c_n\}$. Then, $\{b_i\}_{1 \leq i \leq n} = \{c_i\}_{1 \leq i \leq n}$, and $b_i, c_i \in \mathbb{R}$ (B and C are Hermitian matrices, so they have real eigenvalues). Using Claim 1,

$$(CB)^2 = B^3 \Rightarrow (\Delta_C \Delta_B)^2 = \Delta_B^3 \Rightarrow (c_i b_i)^2 = b_i^3, \text{ for all } i \in \{1, 2, \dots, n\} \Rightarrow$$

$$\Rightarrow b_i = 0 \text{ or } b_i = c_i^2, \text{ for all } i \in \{1, 2, \dots, n\}, (*)$$

We contend:

Claim 3: $b_i \in \{0, 1\}$ for all $i \in \{1, 2, \dots, n\}$.

Proof: Assume otherwise. Assume that $b_1 \neq 0$ and $b_1 \neq 1$. Then, using $(*)$, we obtain $c_1 = b_1^{1/2}$. Since $\{b_i\}_{1 \leq i \leq n} = \{c_i\}_{1 \leq i \leq n}$, there is an index j such that $c_1 = b_j$, hence $b_j = b_1^{1/2} \neq b_1$. Therefore, using $(*)$ again, we obtain that $c_j = b_1^{1/4} \notin \{b_1, b_1^{1/2}\}$.

Continuing, we produce infinitely many mutually distinct eigenvalues $b_1^{1/2^i}$ ($i \geq 0$), a contradiction ■

Now, returning back to $(*)$, we infer that if $b_i \neq 0$, then $1 = b_i = c_i^2$, hence $c_i = 1$.

This implies that the multiplicity of the eigenvalue 1 in Δ_C is at least as large as the multiplicity of the eigenvalue 1 in Δ_B . Since these two multiplicities must be equal (XY and YX have the same algebraic multiplicity in non-zero eigenvalues), we must have equality everywhere: these two matrices must have the zeros and the ones in the exact same places.

To sum up $\Delta_B = \Delta_C$, hence $AA^* = A^*A$, and so A, A^* commute. Therefore, A is orthogonally diagonalizable itself, that is $A = QDQ^*$ for some unitary matrix Q . Thus, $A^*A^2 = AA^*$ implies (assume that $\Delta = \text{diag}\{\lambda_i\}_{1 \leq i \leq n}$)

$$\overline{\lambda_i} \lambda_i^2 = \lambda \overline{\lambda_i},$$

and now this last relation easily implies $\lambda_i \in \{0, 1\}$, hence

$$A^2 = QD^2Q^* = QDQ^* = A,$$

as desired.

3.4 Problem 4

Solution (due to Orestis Lignos): In the following solution, we merely write $n \geq 1$ in all series, meaning that n ranges from 1 to $+\infty$. Moreover, we write $\log(\cdot)$ instead of $\ln(\cdot)$. We split the solution into two parts.

Part 1: If the series $\sum_{n \geq 1} \frac{a_n}{n}$ is convergent, then $(a_n \log n)$ is bounded and $\sum_{n \geq 1} (a_n - a_{n+1}) \log n$ is convergent. Note that

$$(a_1 - a_2) \log 1 + \dots + (a_n - a_{n+1}) \log n = a_1 \log 1 + a_2 \log \frac{2}{1} + \dots + a_n \log \frac{n}{n-1} - a_{n+1} \log n,$$

and since the sequence $((a_n - a_{n+1}) \log n)$ is positive, we need to prove that

$$s_n := a_1 \log 1 + a_2 \log \frac{2}{1} + \dots + a_n \log \frac{n}{n-1} - a_{n+1} \log n$$

is bounded. This follows using the inequality $\log x \leq x - 1$. Indeed,

$$s_n \leq a_1 \cdot 0 + a_2 \cdot \frac{1}{1} + \dots + a_n \cdot \frac{1}{n-1} - a_{n+1} \log n < \frac{a_1}{1} + \frac{a_2}{2} + \dots + \frac{a_n}{n},$$

and this last sum is obviously bounded, as the series $\sum_{n \geq 1} \frac{a_n}{n}$ is convergent.

To prove that $(a_n \log n)$ is bounded, note that

$$\log n < \frac{1}{1} + \frac{1}{2} + \dots + \frac{1}{n}$$

for all $n \geq 1$. Hence, it suffices to show that $\frac{a_n}{1} + \frac{a_n}{2} + \dots + \frac{a_n}{n}$ is bounded. However, this is trivially true, as (a_n) is decreasing, and $\frac{a_1}{1} + \frac{a_2}{2} + \dots + \frac{a_n}{n}$ is bounded, as we explained above.

Part 2: If $(a_n \log n)$ is bounded and $\sum_{n \geq 1} (a_n - a_{n+1}) \log n$ is convergent, then the series $\sum_{n \geq 1} \frac{a_n}{n}$ is bounded. The proof of this direction closely follows the proof in Part

1. Let $t = \frac{n}{n-1}$. Then, using the inequality $\log t \geq \frac{t-1}{t}$, we obtain

$$\log n - \log(n-1) = \log \frac{n}{n-1} = \log t \geq \frac{t-1}{t} = \frac{1}{n}.$$

Therefore,

$$\sum_{1 \leq n \leq k} \frac{a_n}{n} \leq a_1 + \sum_{2 \leq n \leq k} a_n (\log n - \log(n-1)) = a_1 + \sum_{1 \leq n \leq k-1} (a_n - a_{n+1}) \log n + a_k \log k.$$

To conclude, we observe that $a_k \log k$ and $\sum_{1 \leq n \leq k-1} (a_n - a_{n+1}) \log n$ are bounded when $k \rightarrow +\infty$, hence the partial sums of the series $\sum_{n \geq 1} \frac{a_n}{n}$ are bounded, and since the series has positive terms we conclude it must be convergent, as desired.