International Competition in Mathematics for Universtiy Students in Plovdiv, Bulgaria 1994

PROBLEMS AND SOLUTIONS

First day — July 29, 1994

Problem 1. (13 points)

a) Let A be a $n \times n$, $n \geq 2$, symmetric, invertible matrix with real positive elements. Show that $z_n \leq n^2 - 2n$, where z_n is the number of zero elements in A^{-1}.

b) How many zero elements are there in the inverse of the $n \times n$ matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 2 & 2 & \dots & 2 \\ 1 & 2 & 1 & 1 & \dots & 1 \\ 1 & 2 & 1 & 2 & \dots & 2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 2 & 1 & 2 & \dots & \dots \end{pmatrix} ?$$

Solution. Denote by a_{ij} and b_{ij} the elements of A and A^{-1}, respectively. Then for $k \neq m$ we have $\sum_{i=0}^{n} a_{ki} b_{im} = 0$ and from the positivity of a_{ij} we conclude that at least one of $\{b_{im} : i = 1, 2, \dots, n\}$ is positive and at least one is negative. Hence we have at least two non-zero elements in every column of A^{-1}. This proves part a). For part b) all b_{ij} are zero except $b_{1,1} = 2$, $b_{n,n} = (-1)^n$, $b_{i,i+1} = b_{i+1,i} = (-1)^i$ for $i = 1, 2, \dots, n-1$.

Problem 2. (13 points)

Let $f \in C^1(a,b)$, $\lim\limits_{x \to a+} f(x) = +\infty$, $\lim\limits_{x \to b-} f(x) = -\infty$ and $f'(x) + f^2(x) \geq -1$ for $x \in (a,b)$. Prove that $b - a \geq \pi$ and give an example where $b - a = \pi$.

Solution. From the inequality we get

$$\frac{d}{dx}(\operatorname{arctg} f(x) + x) = \frac{f'(x)}{1 + f^2(x)} + 1 \geq 0$$

for $x \in (a,b)$. Thus $\operatorname{arctg} f(x) + x$ is non-decreasing in the interval and using the limits we get $\frac{\pi}{2} + a \leq -\frac{\pi}{2} + b$. Hence $b - a \geq \pi$. One has equality for $f(x) = \operatorname{cotg} x$, $a = 0$, $b = \pi$.

Problem 3. (13 points)

Given a set S of $2n-1$, $n \in \mathbb{N}$, different irrational numbers. Prove that there are n different elements $x_1, x_2, \dots, x_n \in S$ such that for all non-negative rational numbers $a_1, a_2, \dots, a_n$ with $a_1 + a_2 + \dots + a_n > 0$ we have that $a_1x_1 + a_2x_2 + \dots + a_nx_n$ is an irrational number.

Solution. Let $\mathbb{I}$ be the set of irrational numbers, $\mathbb{Q}$ – the set of rational numbers, $\mathbb{Q}^+ = \mathbb{Q} \cap [0, \infty)$. We work by induction. For $n = 1$ the statement is trivial. Let it be true for $n - 1$. We start to prove it for n. From the induction argument there are $n - 1$ different elements $x_1, x_2, \dots, x_{n-1} \in S$ such that

$$(1) \qquad \begin{array}{l} a_1x_1 + a_2x_2 + \dots + a_{n-1}x_{n-1} \in \mathbb{I} \\ \text{for all } \ a_1, a_2, \dots, a_n \in \mathbb{Q}^+ \text{ with } a_1 + a_2 + \dots + a_{n-1} > 0. \end{array}$$

Denote the other elements of S by $x_n, x_{n+1}, \dots, x_{2n-1}$. Assume the statement is not true for n. Then for $k = 0, 1, \dots, n-1$ there are $r_k \in \mathbb{Q}$ such that

$$(2) \qquad \sum_{i=1}^{n-1} b_{ik}x_i + c_kx_{n+k} = r_k \ \text{ for some } b_{ik}, c_k \in \mathbb{Q}^+, \ \sum_{i=1}^{n-1} b_{ik} + c_k > 0.$$

Also

$$(3) \qquad \sum_{k=0}^{n-1} d_kx_{n+k} = R \ \text{ for some } d_k \in \mathbb{Q}^+, \ \sum_{k=0}^{n-1} d_k > 0, \quad R \in \mathbb{Q}.$$

If in (2) $c_k = 0$ then (2) contradicts (1). Thus $c_k \neq 0$ and without loss of generality one may take $c_k = 1$. In (2) also $\sum\limits_{i=1}^{n-1} b_{ik} > 0$ in view of $x_{n+k} \in \mathbb{I}$. Replacing (2) in (3) we get

$$\sum_{k=0}^{n-1} d_k \left(-\sum_{i=1}^{n-1} b_{ik}x_i + r_k \right) = R \quad \text{or} \quad \sum_{i=1}^{n-1} \left(\sum_{k=0}^{n-1} d_kb_{ik} \right) x_i \in \mathbb{Q},$$

which contradicts (1) because of the conditions on b's and d's.

Problem 4. (18 points)

Let $\alpha \in \mathbb{R} \setminus \{0\}$ and suppose that F and G are linear maps (operators) from $\mathbb{R}^n$ into $\mathbb{R}^n$ satisfying $F \circ G - G \circ F = \alpha F$.

a) Show that for all $k \in \mathbb{N}$ one has $F^k \circ G - G \circ F^k = \alpha k F^k$.

b) Show that there exists $k \geq 1$ such that $F^k = 0$.

Solution. For a) using the assumptions we have

$$\begin{aligned} F^k \circ G - G \circ F^k &= \sum_{i=1}^{k} \left(F^{k-i+1} \circ G \circ F^{i-1} - F^{k-i} \circ G \circ F^i \right) = \\ &= \sum_{i=1}^{k} F^{k-i} \circ (F \circ G - G \circ F) \circ F^{i-1} = \\ &= \sum_{i=1}^{k} F^{k-i} \circ \alpha F \circ F^{i-1} = \alpha k F^k. \end{aligned}$$

b) Consider the linear operator $L(F) = F \circ G - G \circ F$ acting over all $n \times n$ matrices F. It may have at most n^2 different eigenvalues. Assuming that $F^k \neq 0$ for every k we get that L has infinitely many different eigenvalues αk in view of a) – a contradiction.

Problem 5. (18 points)

a) Let $f \in C[0,b]$, $g \in C(\mathbb{R})$ and let g be periodic with period b. Prove that $\int_0^b f(x)g(nx)dx$ has a limit as $n \to \infty$ and

$$\lim_{n\to\infty} \int_0^b f(x)g(nx)dx = \frac{1}{b} \int_0^b f(x)dx \cdot \int_0^b g(x)dx.$$

b) Find

$$\lim_{n\to\infty} \int_0^\pi \frac{\sin x}{1 + 3\cos^2 nx} dx.$$

Solution. Set $\|g\|_1 = \int_0^b |g(x)|dx$ and

$$\omega(f,t) = \sup\left\{ |f(x) - f(y)| \ : \ x, y \in [0,b], \ |x-y| \le t \right\}.$$

In view of the uniform continuity of f we have $\omega(f,t) \to 0$ as $t \to 0$. Using the periodicity of g we get

$$\begin{aligned} &\int_0^b f(x)g(nx)dx = \sum_{k=1}^{n} \int_{b(k-1)/n}^{bk/n} f(x)g(nx)dx \\ &= \sum_{k=1}^{n} f(bk/n) \int_{b(k-1)/n}^{bk/n} g(nx)dx + \sum_{k=1}^{n} \int_{b(k-1)/n}^{bk/n} \{f(x) - f(bk/n)\} g(nx)dx \\ &= \frac{1}{n} \sum_{k=1}^{n} f(bk/n) \int_0^b g(x)dx + O(\omega(f, b/n)\|g\|_1) \end{aligned}$$

$$= \frac{1}{b}\sum_{k=1}^{n}\int_{b(k-1)/n}^{bk/n} f(x)dx \int_0^b g(x)dx$$
$$+ \frac{1}{b}\sum_{k=1}^{n}\left(\frac{b}{n}f(bk/n) - \int_{b(k-1)/n}^{bk/n} f(x)dx\right)\int_0^b g(x)dx + O(\omega(f,b/n)\|g\|_1)$$
$$= \frac{1}{b}\int_0^b f(x)dx \int_0^b g(x)dx + O(\omega(f,b/n)\|g\|_1).$$

This proves a). For b) we set $b=\pi$, $f(x)=\sin x$, $g(x)=(1+3\cos^2 x)^{-1}$. From a) and

$$\int_0^\pi \sin x dx = 2, \quad \int_0^\pi (1+3\cos^2 x)^{-1}dx = \frac{\pi}{2}$$

we get

$$\lim_{n\to\infty}\int_0^\pi \frac{\sin x}{1+3\cos^2 nx}dx = 1.$$

Problem 6. (25 points)

Let $f \in C^2[0,N]$ and $|f'(x)| < 1$, $f''(x) > 0$ for every $x \in [0,N]$. Let $0 \le m_0 < m_1 < \cdots < m_k \le N$ be integers such that $n_i = f(m_i)$ are also integers for $i = 0,1,\ldots,k$. Denote $b_i = n_i - n_{i-1}$ and $a_i = m_i - m_{i-1}$ for $i = 1,2,\ldots,k$.

a) Prove that

$$-1 < \frac{b_1}{a_1} < \frac{b_2}{a_2} < \cdots < \frac{b_k}{a_k} < 1.$$

b) Prove that for every choice of $A > 1$ there are no more than N/A indices j such that $a_j > A$.

c) Prove that $k \le 3N^{2/3}$ (i.e. there are no more than $3N^{2/3}$ integer points on the curve $y = f(x)$, $x \in [0,N]$).

Solution. a) For $i = 1,2,\ldots,k$ we have

$$b_i = f(m_i) - f(m_{i-1}) = (m_i - m_{i-1})f'(x_i)$$

for some $x_i \in (m_{i-1}, m_i)$. Hence $\frac{b_i}{a_i} = f'(x_i)$ and so $-1 < \frac{b_i}{a_i} < 1$. From the convexity of f we have that f' is increasing and $\frac{b_i}{a_i} = f'(x_i) < f'(x_{i+1}) = \frac{b_{i+1}}{a_{i+1}}$ because of $x_i < m_i < x_{i+1}$.

b) Set $S_A = \{j \in \{0,1,\dots,k\} \ : \ a_j > A\}$. Then

$$N \geq m_k - m_0 = \sum_{i=1}^{k} a_i \geq \sum_{j \in S_A} a_j > A|S_A|$$

and hence $|S_A| < N/A$.

c) All different fractions in $(-1,1)$ with denominators less or equal A are no more $2A^2$. Using b) we get $k < N/A + 2A^2$. Put $A = N^{1/3}$ in the above estimate and get $k < 3N^{2/3}$.

Second day — July 30, 1994

Problem 1. (14 points)

Let $f \in C^1[a,b]$, $f(a) = 0$ and suppose that $\lambda \in \mathbb{R}$, $\lambda > 0$, is such that

$$|f'(x)| \leq \lambda |f(x)|$$

for all $x \in [a,b]$. Is it true that $f(x) = 0$ for all $x \in [a,b]$?

Solution. Assume that there is $y \in (a,b]$ such that $f(y) \neq 0$. Without loss of generality we have $f(y) > 0$. In view of the continuity of f there exists $c \in [a,y)$ such that $f(c) = 0$ and $f(x) > 0$ for $x \in (c,y]$. For $x \in (c,y]$ we have $|f'(x)| \leq \lambda f(x)$. This implies that the function $g(x) = \ln f(x) - \lambda x$ is not increasing in $(c,y]$ because of $g'(x) = \dfrac{f'(x)}{f(x)} - \lambda \leq 0$. Thus $\ln f(x) - \lambda x \geq \ln f(y) - \lambda y$ and $f(x) \geq e^{\lambda x - \lambda y} f(y)$ for $x \in (c,y]$. Thus

$$0 = f(c) = f(c+0) \geq e^{\lambda c - \lambda y} f(y) > 0$$

— a contradiction. Hence one has $f(x) = 0$ for all $x \in [a,b]$.

Problem 2. (14 points)

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x,y) = (x^2 - y^2)e^{-x^2-y^2}$.

a) Prove that f attains its minimum and its maximum.

b) Determine all points (x,y) such that $\dfrac{\partial f}{\partial x}(x,y) = \dfrac{\partial f}{\partial y}(x,y) = 0$ and determine for which of them f has global or local minimum or maximum.

Solution. We have $f(1,0) = e^{-1}$, $f(0,1) = -e^{-1}$ and $te^{-t} \leq 2e^{-2}$ for $t \geq 2$. Therefore $|f(x,y)| \leq (x^2+y^2)e^{-x^2-y^2} \leq 2e^{-2} < e^{-1}$ for $(x,y) \notin M = \{(u,v) \ : \ u^2+v^2 \leq 2\}$ and f cannot attain its minimum and its maximum outside M. Part a) follows from the compactness of M and the continuity of f. Let (x, y) be a point from part b). From $\frac{\partial f}{\partial x}(x, y) = 2x(1 - x^2 + y^2)e^{-x^2-y^2}$ we get

$$x(1 - x^2 + y^2) = 0. \tag{1}$$

Similarly

$$y(1 + x^2 - y^2) = 0. \tag{2}$$

All solutions (x, y) of the system (1), (2) are $(0, 0)$, $(0, 1)$, $(0, -1)$, $(1, 0)$ and $(-1, 0)$. One has $f(1, 0) = f(-1, 0) = e^{-1}$ and f has global maximum at the points $(1, 0)$ and $(-1, 0)$. One has $f(0, 1) = f(0, -1) = -e^{-1}$ and f has global minimum at the points $(0, 1)$ and $(0, -1)$. The point $(0, 0)$ is not an extrema point because of $f(x, 0) = x^2e^{-x^2} > 0$ if $x \neq 0$ and $f(y, 0) = -y^2e^{-y^2} < 0$ if $y \neq 0$.

Problem 3. (14 points)

Let f be a real-valued function with $n + 1$ derivatives at each point of $\mathbb{R}$. Show that for each pair of real numbers a, b, $a < b$, such that

$$\ln\left(\frac{f(b) + f'(b) + \cdots + f^{(n)}(b)}{f(a) + f'(a) + \cdots + f^{(n)}(a)}\right) = b - a$$

there is a number c in the open interval (a, b) for which

$$f^{(n+1)}(c) = f(c).$$

Note that ln denotes the natural logarithm.

Solution. Set $g(x) = \left(f(x) + f'(x) + \cdots + f^{(n)}(x)\right) e^{-x}$. From the assumption one get $g(a) = g(b)$. Then there exists $c \in (a, b)$ such that $g'(c) = 0$. Replacing in the last equality $g'(x) = \left(f^{(n+1)}(x) - f(x)\right) e^{-x}$ we finish the proof.

Problem 4. (18 points)

Let A be a $n \times n$ diagonal matrix with characteristic polynomial

$$(x - c_1)^{d_1}(x - c_2)^{d_2} \ldots (x - c_k)^{d_k},$$

where $c_1, c_2, \ldots, c_k$ are distinct (which means that c_1 appears d_1 times on the diagonal, c_2 appears d_2 times on the diagonal, etc. and $d_1+d_2+\cdots+d_k = n$).

Let V be the space of all $n \times n$ matrices B such that $AB = BA$. Prove that the dimension of V is

$$d_1^2 + d_2^2 + \cdots + d_k^2.$$

Solution. Set $A = (a_{ij})_{i,j=1}^n$, $B = (b_{ij})_{i,j=1}^n$, $AB = (x_{ij})_{i,j=1}^n$ and $BA = (y_{ij})_{i,j=1}^n$. Then $x_{ij} = a_{ii}b_{ij}$ and $y_{ij} = a_{jj}b_{ij}$. Thus $AB = BA$ is equivalent to $(a_{ii} - a_{jj})b_{ij} = 0$ for $i, j = 1, 2, \dots, n$. Therefore $b_{ij} = 0$ if $a_{ii} \neq a_{jj}$ and b_{ij} may be arbitrary if $a_{ii} = a_{jj}$. The number of indices (i, j) for which $a_{ii} = a_{jj} = c_m$ for some $m = 1, 2, \dots, k$ is d_m^2. This gives the desired result.

Problem 5. (18 points)

Let $x_1, x_2, \dots, x_k$ be vectors of m-dimensional Euclidian space, such that $x_1 + x_2 + \cdots + x_k = 0$. Show that there exists a permutation π of the integers $\{1, 2, \dots, k\}$ such that

$$\left\| \sum_{i=1}^{n} x_{\pi(i)} \right\| \leq \left(\sum_{i=1}^{k} \|x_i\|^2 \right)^{1/2} \quad \text{for each } n = 1, 2, \dots, k.$$

Note that $\|\cdot\|$ denotes the Euclidian norm.

Solution. We define π inductively. Set $\pi(1) = 1$. Assume π is defined for $i = 1, 2, \dots, n$ and also

$$\left\| \sum_{i=1}^{n} x_{\pi(i)} \right\|^2 \leq \sum_{i=1}^{n} \|x_{\pi(i)}\|^2. \tag{1}$$

Note (1) is true for $n = 1$. We choose $\pi(n+1)$ in a way that (1) is fulfilled with $n+1$ instead of n. Set $y = \sum\limits_{i=1}^{n} x_{\pi(i)}$ and $A = \{1, 2, \dots, k\} \setminus \{\pi(i) : i = 1, 2, \dots, n\}$. Assume that $(y, x_r) > 0$ for all $r \in A$. Then $\left(y, \sum\limits_{r \in A} x_r \right) > 0$ and in view of $y + \sum\limits_{r \in A} x_r = 0$ one gets $-(y, y) > 0$, which is impossible. Therefore there is $r \in A$ such that

$$(y, x_r) \leq 0. \tag{2}$$

Put $\pi(n+1) = r$. Then using (2) and (1) we have

$$\left\| \sum_{i=1}^{n+1} x_{\pi(i)} \right\|^2 = \|y + x_r\|^2 = \|y\|^2 + 2(y, x_r) + \|x_r\|^2 \leq \|y\|^2 + \|x_r\|^2 \leq$$

$$\leq \sum_{i=1}^{n} \|x_{\pi(i)}\|^2 + \|x_r\|^2 = \sum_{i=1}^{n+1} \|x_{\pi(i)}\|^2,$$

which verifies (1) for $n+1$. Thus we define π for every $n = 1, 2, \ldots, k$. Finally from (1) we get

$$\left\| \sum_{i=1}^{n} x_{\pi(i)} \right\|^2 \leq \sum_{i=1}^{n} \|x_{\pi(i)}\|^2 \leq \sum_{i=1}^{k} \|x_i\|^2.$$

Problem 6. (22 points)

Find $\lim\limits_{N\to\infty} \dfrac{\ln^2 N}{N} \sum\limits_{k=2}^{N-2} \dfrac{1}{\ln k \cdot \ln(N-k)}$. Note that ln denotes the natural logarithm.

Solution. Obviously

$$(1) \qquad A_N = \frac{\ln^2 N}{N} \sum_{k=2}^{N-2} \frac{1}{\ln k \cdot \ln(N-k)} \geq \frac{\ln^2 N}{N} \cdot \frac{N-3}{\ln^2 N} = 1 - \frac{3}{N}.$$

Take M, $2 \leq M < N/2$. Then using that $\dfrac{1}{\ln k \cdot \ln(N-k)}$ is decreasing in $[2, N/2]$ and the symmetry with respect to $N/2$ one get

$$A_N = \frac{\ln^2 N}{N} \left\{ \sum_{k=2}^{M} + \sum_{k=M+1}^{N-M-1} + \sum_{k=N-M}^{N-2} \right\} \frac{1}{\ln k \cdot \ln(N-k)} \leq$$

$$\leq \frac{\ln^2 N}{N} \left\{ 2\frac{M-1}{\ln 2 \cdot \ln(N-2)} + \frac{N-2M-1}{\ln M \cdot \ln(N-M)} \right\} \leq$$

$$\leq \frac{2}{\ln 2} \cdot \frac{M \ln N}{N} + \left(1 - \frac{2M}{N}\right) \frac{\ln N}{\ln M} + O\left(\frac{1}{\ln N}\right).$$

Choose $M = \left[\dfrac{N}{\ln^2 N}\right] + 1$ to get

$$(2) \;\; A_N \leq \left(1 - \frac{2}{N \ln^2 N}\right) \frac{\ln N}{\ln N - 2\ln\ln N} + O\left(\frac{1}{\ln N}\right) \leq 1 + O\left(\frac{\ln\ln N}{\ln N}\right).$$

Estimates (1) and (2) give

$$\lim_{N\to\infty} \frac{\ln^2 N}{N} \sum_{k=2}^{N-2} \frac{1}{\ln k \cdot \ln(N-k)} = 1.$$

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PROBLEMS AND SOLUTIONS

First day

Problem 1. (10 points)

Let X be a nonsingular matrix with columns $X_1, X_2, \dots, X_n$. Let Y be a matrix with columns $X_2, X_3, \dots, X_n, 0$. Show that the matrices $A = YX^{-1}$ and $B = X^{-1}Y$ have rank $n-1$ and have only 0's for eigenvalues.

Solution. Let $J = (a_{ij})$ be the $n \times n$ matrix where $a_{ij} = 1$ if $i = j+1$ and $a_{ij} = 0$ otherwise. The rank of J is $n-1$ and its only eigenvalues are $0'$s. Moreover $Y = XJ$ and $A = YX^{-1} = XJX^{-1}$, $B = X^{-1}Y = J$. It follows that both A and B have rank $n-1$ with only $0'$s for eigenvalues.

Problem 2. (15 points)

Let f be a continuous function on $[0,1]$ such that for every $x \in [0,1]$ we have $\int_x^1 f(t)dt \geq \frac{1-x^2}{2}$. Show that $\int_0^1 f^2(t)dt \geq \frac{1}{3}$.

Solution. From the inequality

$$0 \leq \int_0^1 (f(x)-x)^2\,dx = \int_0^1 f^2(x)dx - 2\int_0^1 xf(x)dx + \int_0^1 x^2dx$$

we get

$$\int_0^1 f^2(x)dx \geq 2\int_0^1 xf(x)dx - \int_0^1 x^2dx = 2\int_0^1 xf(x)dx - \frac{1}{3}.$$

From the hypotheses we have $\int_0^1 \int_x^1 f(t)dtdx \geq \int_0^1 \frac{1-x^2}{2}dx$ or $\int_0^1 tf(t)dt \geq \frac{1}{3}$. This completes the proof.

Problem 3. (15 points)

Let f be twice continuously differentiable on $(0,+\infty)$ such that $\lim_{x\to 0+} f'(x) = -\infty$ and $\lim_{x\to 0+} f''(x) = +\infty$. Show that

$$\lim_{x\to 0+} \frac{f(x)}{f'(x)} = 0.$$

Solution. Since f' tends to $-\infty$ and f'' tends to $+\infty$ as x tends to $0+$, there exists an interval $(0,r)$ such that $f'(x) < 0$ and $f''(x) > 0$ for all $x \in (0,r)$. Hence f is decreasing and f' is increasing on $(0,r)$. By the mean value theorem for every $0 < x < x_0 < r$ we obtain

$$f(x) - f(x_0) = f'(\xi)(x - x_0) > 0,$$

for some $\xi \in (x, x_0)$. Taking into account that f' is increasing, $f'(x) < f'(\xi) < 0$, we get

$$x - x_0 < \frac{f'(\xi)}{f'(x)}(x - x_0) = \frac{f(x) - f(x_0)}{f'(x)} < 0.$$

Taking limits as x tends to $0+$ we obtain

$$-x_0 \leq \liminf_{x \to 0+} \frac{f(x)}{f'(x)} \leq \limsup_{x \to 0+} \frac{f(x)}{f'(x)} \leq 0.$$

Since this happens for all $x_0 \in (0,r)$ we deduce that $\lim\limits_{x \to 0+} \dfrac{f(x)}{f'(x)}$ exists and $\lim\limits_{x \to 0+} \dfrac{f(x)}{f'(x)} = 0$.

Problem 4. (15 points)
Let $F : (1,\infty) \to \mathbb{R}$ be the function defined by

$$F(x) := \int_x^{x^2} \frac{dt}{\ln t}.$$

Show that F is one-to-one (i.e. injective) and find the range (i.e. set of values) of F.

Solution. From the definition we have

$$F'(x) = \frac{x-1}{\ln x}, \quad x > 1.$$

Therefore $F'(x) > 0$ for $x \in (1,\infty)$. Thus F is strictly increasing and hence one-to-one. Since

$$F(x) \geq (x^2 - x) \min\left\{\frac{1}{\ln t} \ : \ x \leq t \leq x^2\right\} = \frac{x^2 - x}{\ln x^2} \to \infty$$

as $x \to \infty$, it follows that the range of F is $(F(1+), \infty)$. In order to determine $F(1+)$ we substitute $t = e^v$ in the definition of F and we get

$$F(x) = \int_{\ln x}^{2\ln x} \frac{e^v}{v} dv.$$

Hence

$$F(x) < e^{2\ln x} \int_{\ln x}^{2\ln x} \frac{1}{v} dv = x^2 \ln 2$$

and similarly $F(x) > x \ln 2$. Thus $F(1+) = \ln 2$.

Problem 5. (20 points)

Let A and B be real $n \times n$ matrices. Assume that there exist $n + 1$ different real numbers $t_1, t_2, \ldots, t_{n+1}$ such that the matrices

$$C_i = A + t_i B, \quad i = 1, 2, \ldots, n + 1,$$

are nilpotent (i.e. $C_i^n = 0$).

Show that both A and B are nilpotent.

Solution. We have that

$$(A + tB)^n = A^n + tP_1 + t^2 P_2 + \cdots + t^{n-1} P_{n-1} + t^n B^n$$

for some matrices $P_1, P_2, \ldots, P_{n-1}$ not depending on t.

Assume that $a, p_1, p_2, \ldots, p_{n-1}, b$ are the (i, j)-th entries of the corresponding matrices $A^n, P_1, P_2, \ldots, P_{n-1}, B^n$. Then the polynomial

$$bt^n + p_{n-1}t^{n-1} + \cdots + p_2 t^2 + p_1 t + a$$

has at least $n + 1$ roots $t_1, t_2, \ldots, t_{n+1}$. Hence all its coefficients vanish. Therefore $A^n = 0$, $B^n = 0$, $P_i = 0$; and A and B are nilpotent.

Problem 6. (25 points)

Let $p > 1$. Show that there exists a constant $K_p > 0$ such that for every $x, y \in \mathbb{R}$ satisfying $|x|^p + |y|^p = 2$, we have

$$(x - y)^2 \leq K_p \left(4 - (x + y)^2\right).$$

Solution. Let $0 < \delta < 1$. First we show that there exists $K_{p,\delta} > 0$ such that

$$f(x,y) = \frac{(x-y)^2}{4-(x+y)^2} \le K_{p,\delta}$$

for every $(x,y) \in D_\delta = \{(x,y) \ : \ |x-y| \ge \delta, |x|^p + |y|^p = 2\}$.

Since D_δ is compact it is enough to show that f is continuous on D_δ. For this we show that the denominator of f is different from zero. Assume the contrary. Then $|x+y| = 2$, and $\left|\frac{x+y}{2}\right|^p = 1$. Since $p > 1$, the function $g(t) = |t|^p$ is strictly convex, in other words $\left|\frac{x+y}{2}\right|^p < \frac{|x|^p + |y|^p}{2}$ whenever $x \ne y$. So for some $(x,y) \in D_\delta$ we have $\left|\frac{x+y}{2}\right|^p < \frac{|x|^p+|y|^p}{2} = 1 = \left|\frac{x+y}{2}\right|^p$. We get a contradiction.

If x and y have different signs then $(x,y) \in D_\delta$ for all $0 < \delta < 1$ because then $|x-y| \ge \max\{|x|,|y|\} \ge 1 > \delta$. So we may further assume without loss of generality that $x > 0$, $y > 0$ and $x^p + y^p = 2$. Set $x = 1 + t$. Then

$$\begin{aligned}
y &= (2-x^p)^{1/p} = (2-(1+t)^p)^{1/p} = \left(2 - (1 + pt + \frac{p(p-1)}{2}t^2 + o(t^2))\right)^{1/p} \\
&= \left(1 - pt - \frac{p(p-1)}{2}t^2 + o(t^2)\right)^{1/p} \\
&= 1 + \frac{1}{p}\left(-pt - \frac{p(p-1)}{2}t^2 + o(t^2)\right) + \frac{1}{2p}\left(\frac{1}{p} - 1\right)(-pt + o(t))^2 + o(t^2) \\
&= 1 - t - \frac{p-1}{2}t^2 + o(t^2) - \frac{p-1}{2}t^2 + o(t^2) \\
&= 1 - t - (p-1)t^2 + o(t^2).
\end{aligned}$$

We have

$$(x-y)^2 = (2t + o(t))^2 = 4t^2 + o(t^2)$$

and

$$4-(x+y)^2 = 4-(2-(p-1)t^2+o(t^2))^2 = 4-4+4(p-1)t^2+o(t^2) = 4(p-1)t^2+o(t^2).$$

So there exists $\delta_p > 0$ such that if $|t| < \delta_p$ we have $(x-y)^2 < 5t^2$, $4-(x+y)^2 > 3(p-1)t^2$. Then

$$(*) \qquad (x-y)^2 < 5t^2 = \frac{5}{3(p-1)} \cdot 3(p-1)t^2 < \frac{5}{3(p-1)}(4-(x+y)^2)$$

if $|x-1|<\delta_p$. From the symmetry we have that $(*)$ also holds when $|y-1|<\delta_p$.

To finish the proof it is enough to show that $|x-y|\geq 2\delta_p$ whenever $|x-1|\geq\delta_p$, $|y-1|\geq\delta_p$ and $x^p+y^p=2$. Indeed, since $x^p+y^p=2$ we have that $\max\{x,y\}\geq 1$. So let $x-1\geq\delta_p$. Since $\left(\frac{x+y}{2}\right)^p\leq\frac{x^p+y^p}{2}=1$ we get $x+y\leq 2$. Then $x-y\geq 2(x-1)\geq 2\delta_p$.

Second day

Problem 1. (10 points)

Let A be 3×3 real matrix such that the vectors Au and u are orthogonal for each column vector $u\in\mathbb{R}^3$. Prove that:

a) $A^\top=-A$, where $A^\top$ denotes the transpose of the matrix A;

b) there exists a vector $v\in\mathbb{R}^3$ such that $Au=v\times u$ for every $u\in\mathbb{R}^3$, where $v\times u$ denotes the vector product in $\mathbb{R}^3$.

Solution. a) Set $A=(a_{ij})$, $u=(u_1,u_2,u_3)^\top$. If we use the orthogonality condition

$$(Au,u)=0 \tag{1}$$

with $u_i=\delta_{ik}$ we get $a_{kk}=0$. If we use (1) with $u_i=\delta_{ik}+\delta_{im}$ we get

$$a_{kk}+a_{km}+a_{mk}+a_{mm}=0$$

and hence $a_{km}=-a_{mk}$.

b) Set $v_1=-a_{23}$, $v_2=a_{13}$, $v_3=-a_{12}$. Then

$$Au=(v_2u_3-v_3u_2,v_3u_1-v_1u_3,v_1u_2-v_2u_1)^\top=v\times u.$$

Problem 2. (15 points)

Let $\{b_n\}_{n=0}^\infty$ be a sequence of positive real numbers such that $b_0=1$, $b_n=2+\sqrt{b_{n-1}}-2\sqrt{1+\sqrt{b_{n-1}}}$. Calculate

$$\sum_{n=1}^\infty b_n 2^n.$$

Solution. Put $a_n = 1 + \sqrt{b_n}$ for $n \geq 0$. Then $a_n > 1$, $a_0 = 2$ and

$$a_n = 1 + \sqrt{1 + a_{n-1} - 2\sqrt{a_{n-1}}} = \sqrt{a_{n-1}},$$

so $a_n = 2^{2^{-n}}$. Then

$$\begin{aligned}
\sum_{n=1}^{N} b_n 2^n &= \sum_{n=1}^{N} (a_n - 1)^2 2^n = \sum_{n=1}^{N} [a_n^2 2^n - a_n 2^{n+1} + 2^n] \\
&= \sum_{n=1}^{N} [(a_{n-1} - 1)2^n - (a_n - 1)2^{n+1}] \\
&= (a_0 - 1)2^1 - (a_N - 1)2^{N+1} = 2 - 2\frac{2^{2^{-N}} - 1}{2^{-N}}.
\end{aligned}$$

Put $x = 2^{-N}$. Then $x \to 0$ as $N \to \infty$ and so

$$\sum_{n=1}^{\infty} b_n 2^N = \lim_{N\to\infty} \left(2 - 2\frac{2^{2^{-N}} - 1}{2^{-N}}\right) = \lim_{x\to 0} \left(2 - 2\frac{2^x - 1}{x}\right) = 2 - 2\ln 2.$$

Problem 3. (15 points)

Let all roots of an n-th degree polynomial $P(z)$ with complex coefficients lie on the unit circle in the complex plane. Prove that all roots of the polynomial

$$2zP'(z) - nP(z)$$

lie on the same circle.

Solution. It is enough to consider only polynomials with leading coefficient 1. Let $P(z) = (z - \alpha_1)(z - \alpha_2)\dots(z - \alpha_n)$ with $|\alpha_j| = 1$, where the complex numbers $\alpha_1, \alpha_2, \dots, \alpha_n$ may coincide.

We have

$$\begin{aligned}
\widetilde{P}(z) \equiv{}& 2zP'(z) - nP(z) = (z + \alpha_1)(z - \alpha_2)\dots(z - \alpha_n) + \\
&+(z - \alpha_1)(z + \alpha_2)\dots(z - \alpha_n) + \dots + (z - \alpha_1)(z - \alpha_2)\dots(z + \alpha_n).
\end{aligned}$$

Hence, $\dfrac{\widetilde{P}(z)}{P(z)} = \displaystyle\sum_{k=1}^{n} \frac{z + \alpha_k}{z - \alpha_k}$. Since $\mathrm{Re}\dfrac{z + \alpha}{z - \alpha} = \dfrac{|z|^2 - |\alpha|^2}{|z - \alpha|^2}$ for all complex z, α, $z \neq \alpha$, we deduce that in our case $\mathrm{Re}\dfrac{\widetilde{P}(z)}{P(z)} = \displaystyle\sum_{k=1}^{n} \frac{|z|^2 - 1}{|z - \alpha_k|^2}$. From $|z| \neq 1$ it follows that $\mathrm{Re}\dfrac{\widetilde{P}(z)}{P(z)} \neq 0$. Hence $\widetilde{P}(z) = 0$ implies $|z| = 1$.

Problem 4. (15 points)

a) Prove that for every $\varepsilon > 0$ there is a positive integer n and real numbers $\lambda_1, \dots, \lambda_n$ such that

$$\max_{x\in[-1,1]} \left| x - \sum_{k=1}^{n} \lambda_k x^{2k+1} \right| < \varepsilon.$$

b) Prove that for every odd continuous function f on $[-1,1]$ and for every $\varepsilon > 0$ there is a positive integer n and real numbers $\mu_1, \dots, \mu_n$ such that

$$\max_{x\in[-1,1]} \left| f(x) - \sum_{k=1}^{n} \mu_k x^{2k+1} \right| < \varepsilon.$$

Recall that f is odd means that $f(x) = -f(-x)$ for all $x \in [-1,1]$.

Solution. a) Let n be such that $(1-\varepsilon^2)^n \le \varepsilon$. Then $|x(1-x^2)^n| < \varepsilon$ for every $x \in [-1,1]$. Thus one can set $\lambda_k = (-1)^{k+1}\binom{n}{k}$ because then

$$x - \sum_{k=1}^{n} \lambda_k x^{2k+1} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} x^{2k+1} = x(1-x^2)^n.$$

b) From the Weierstrass theorem there is a polynomial, say $p \in \Pi_m$, such that

$$\max_{x\in[-1,1]} |f(x) - p(x)| < \frac{\varepsilon}{2}.$$

Set $q(x) = \frac{1}{2}\{p(x) - p(-x)\}$. Then

$$f(x) - q(x) = \frac{1}{2}\{f(x) - p(x)\} - \frac{1}{2}\{f(-x) - p(-x)\}$$

and

$$(1)\quad \max_{|x|\le 1} |f(x) - q(x)| \le \frac{1}{2}\max_{|x|\le 1}|f(x) - p(x)| + \frac{1}{2}\max_{|x|\le 1}|f(-x) - p(-x)| < \frac{\varepsilon}{2}.$$

But q is an odd polynomial in Π_m and it can be written as

$$q(x) = \sum_{k=0}^{m} b_k x^{2k+1} = b_0 x + \sum_{k=1}^{m} b_k x^{2k+1}.$$

If $b_0 = 0$ then (1) proves b). If $b_0 \neq 0$ then one applies a) with $\dfrac{\varepsilon}{2|b_0|}$ instead of ε to get

$$\max_{|x|\le 1}\left|b_0 x - \sum_{k=1}^{n} b_0\lambda_k x^{2k+1}\right| < \frac{\varepsilon}{2} \tag{2}$$

for appropriate n and $\lambda_1, \lambda_2, \ldots, \lambda_n$. Now b) follows from (1) and (2) with $\max\{n, m\}$ instead of n.

Problem 5. (10+15 points)
a) Prove that every function of the form

$$f(x) = \frac{a_0}{2} + \cos x + \sum_{n=2}^{N} a_n \cos(nx)$$

with $|a_0| < 1$, has positive as well as negative values in the period $[0, 2\pi)$.
b) Prove that the function

$$F(x) = \sum_{n=1}^{100} \cos(n^{\frac{3}{2}} x)$$

has at least 40 zeros in the interval $(0, 1000)$.

Solution. a) Let us consider the integral

$$\int_0^{2\pi} f(x)(1 \pm \cos x)dx = \pi(a_0 \pm 1).$$

The assumption that $f(x) \geq 0$ implies $a_0 \geq 1$. Similarly, if $f(x) \leq 0$ then $a_0 \leq -1$. In both cases we have a contradiction with the hypothesis of the problem.

b) We shall prove that for each integer N and for each real number $h \geq 24$ and each real number y the function

$$F_N(x) = \sum_{n=1}^{N} \cos(xn^{\frac{3}{2}})$$

changes sign in the interval $(y, y+h)$. The assertion will follow immediately from here.

Consider the integrals

$$I_1 = \int_y^{y+h} F_N(x)dx, \qquad I_2 = \int_y^{y+h} F_N(x)\cos x\, dx.$$

If $F_N(x)$ does not change sign in $(y, y+h)$ then we have

$$|I_2| \leq \int_y^{y+h} |F_N(x)|dx = \left| \int_y^{y+h} F_N(x)dx \right| = |I_1|.$$

Hence, it is enough to prove that

$$|I_2| > |I_1|.$$

Obviously, for each $\alpha \neq 0$ we have

$$\left| \int_y^{y+h} \cos(\alpha x)dx \right| \leq \frac{2}{|\alpha|}.$$

Hence

$$(1) \qquad |I_1| = \left| \sum_{n=1}^{N} \int_y^{y+h} \cos(xn^{\frac{3}{2}})dx \right| \leq 2 \sum_{n=1}^{N} \frac{1}{n^{\frac{3}{2}}} < 2\left(1 + \int_1^{\infty} \frac{dt}{t^{\frac{3}{2}}}\right) = 6.$$

On the other hand we have

$$\begin{aligned} I_2 &= \sum_{n=1}^{N} \int_y^{y+h} \cos x \cos(xn^{\frac{3}{2}})dx \\ &= \frac{1}{2} \int_y^{y+h} (1 + \cos(2x))dx + \\ &\quad + \frac{1}{2} \sum_{n=2}^{N} \int_y^{y+h} \left(\cos\left(x(n^{\frac{3}{2}} - 1)\right) + \cos\left(x(n^{\frac{3}{2}} + 1)\right)\right) dx \\ &= \frac{1}{2}h + \Delta, \end{aligned}$$

where

$$|\Delta| \leq \frac{1}{2}\left(1 + 2\sum_{n=2}^{N}\left(\frac{1}{n^{\frac{3}{2}} - 1} + \frac{1}{n^{\frac{3}{2}} + 1}\right)\right) \leq \frac{1}{2} + 2\sum_{n=2}^{N} \frac{1}{n^{\frac{3}{2}} - 1}.$$

We use that $n^{\frac{3}{2}} - 1 \geq \frac{2}{3} n^{\frac{3}{2}}$ for $n \geq 3$ and we get

$$|\Delta| \leq \frac{1}{2} + \frac{2}{2^{\frac{3}{2}} - 1} + 3 \sum_{n=3}^{N} \frac{1}{n^{\frac{3}{2}}} < \frac{1}{2} + \frac{2}{2\sqrt{2} - 1} + 3 \int_2^{\infty} \frac{dt}{t^{\frac{3}{2}}} < 6.$$

Hence

$$|I_2| > \frac{1}{2} h - 6. \tag{2}$$

We use that $h \geq 24$ and inequalities (1), (2) and we obtain $|I_2| > |I_1|$. The proof is completed.

Problem 6. (20 points)

Suppose that $\{f_n\}_{n=1}^{\infty}$ is a sequence of continuous functions on the interval $[0,1]$ such that

$$\int_0^1 f_m(x) f_n(x) dx = \begin{cases} 1 & \text{if} \quad n = m \\ 0 & \text{if} \quad n \neq m \end{cases}$$

and

$$\sup\{|f_n(x)| \ : \ x \in [0,1] \text{ and } n = 1, 2, \ldots\} < +\infty.$$

Show that there exists no subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $\lim\limits_{k \to \infty} f_{n_k}(x)$ exists for all $x \in [0,1]$.

Solution. It is clear that one can add some functions, say $\{g_m\}$, which satisfy the hypothesis of the problem and the closure of the finite linear combinations of $\{f_n\} \cup \{g_m\}$ is $L_2[0,1]$. Therefore without loss of generality we assume that $\{f_n\}$ generates $L_2[0,1]$.

Let us suppose that there is a subsequence $\{n_k\}$ and a function f such that

$$f_{n_k}(x) \underset{k \to \infty}{\longrightarrow} f(x) \quad \text{for every} \quad x \in [0,1].$$

Fix $m \in \mathbb{N}$. From Lebesgue's theorem we have

$$0 = \int_0^1 f_m(x) f_{n_k}(x) dx \underset{k \to \infty}{\longrightarrow} \int_0^1 f_m(x) f(x) dx.$$

Hence $\int_0^1 f_m(x) f(x) dx = 0$ for every $m \in \mathbb{N}$, which implies $f(x) = 0$ almost everywhere. Using once more Lebesgue's theorem we get

$$1 = \int_0^1 f_{n_k}^2(x) dx \underset{k \to \infty}{\longrightarrow} \int_0^1 f^2(x) dx = 0.$$

The contradiction proves the statement.

International Competition in Mathematics for Universtiy Students in Plovdiv, Bulgaria 1996

PROBLEMS AND SOLUTIONS

First day — August 2, 1996

Problem 1. (10 points)

Let for $j = 0, \dots, n$, $a_j = a_0 + jd$, where a_0, d are fixed real numbers. Put

$$A = \begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_n \\ a_1 & a_0 & a_1 & \dots & a_{n-1} \\ a_2 & a_1 & a_0 & \dots & a_{n-2} \\ \hdotsfor{5} \\ a_n & a_{n-1} & a_{n-2} & \dots & a_0 \end{pmatrix}.$$

Calculate $\det(A)$, where $\det(A)$ denotes the determinant of A.

Solution. Adding the first column of A to the last column we get that

$$\det(A) = (a_0 + a_n)\ \det \begin{pmatrix} a_0 & a_1 & a_2 & \dots & 1 \\ a_1 & a_0 & a_1 & \dots & 1 \\ a_2 & a_1 & a_0 & \dots & 1 \\ \hdotsfor{5} \\ a_n & a_{n-1} & a_{n-2} & \dots & 1 \end{pmatrix}.$$

Subtracting the n-th row of the above matrix from the $(n+1)$-st one, $(n-1)$-st from n-th, $\dots$, first from second we obtain that

$$\det(A) = (a_0 + a_n)\ \det \begin{pmatrix} a_0 & a_1 & a_2 & \dots & 1 \\ d & -d & -d & \dots & 0 \\ d & d & -d & \dots & 0 \\ \hdotsfor{5} \\ d & d & d & \dots & 0 \end{pmatrix}.$$

Hence,

$$\det(A) = (-1)^n (a_0 + a_n)\ \det \begin{pmatrix} d & -d & -d & \dots & -d \\ d & d & -d & \dots & -d \\ d & d & d & \dots & -d \\ \hdotsfor{5} \\ d & d & d & \dots & d \end{pmatrix}.$$

Adding the last row of the above matrix to the other rows we have

$$\det(A) = (-1)^n(a_0+a_n)\det\begin{pmatrix} 2d & 0 & 0 & \dots & 0 \\ 2d & 2d & 0 & \dots & 0 \\ 2d & 2d & 2d & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ d & d & d & \dots & d \end{pmatrix} = (-1)^n(a_0+a_n)2^{n-1}d^n.$$

Problem 2. (10 points)
Evaluate the definite integral

$$\int_{-\pi}^{\pi} \frac{\sin nx}{(1+2^x)\sin x}dx,$$

where n is a natural number.

Solution. We have

$$\begin{aligned} I_n &= \int_{-\pi}^{\pi} \frac{\sin nx}{(1+2^x)\sin x}dx \\ &= \int_{0}^{\pi} \frac{\sin nx}{(1+2^x)\sin x}dx + \int_{-\pi}^{0} \frac{\sin nx}{(1+2^x)\sin x}dx. \end{aligned}$$

In the second integral we make the change of variable $x = -x$ and obtain

$$\begin{aligned} I_n &= \int_{0}^{\pi} \frac{\sin nx}{(1+2^x)\sin x}dx + \int_{0}^{\pi} \frac{\sin nx}{(1+2^{-x})\sin x}dx \\ &= \int_{0}^{\pi} \frac{(1+2^x)\sin nx}{(1+2^x)\sin x}dx \\ &= \int_{0}^{\pi} \frac{\sin nx}{\sin x}dx. \end{aligned}$$

For $n \geq 2$ we have

$$\begin{aligned} I_n - I_{n-2} &= \int_{0}^{\pi} \frac{\sin nx - \sin{(n-2)x}}{\sin x}dx \\ &= 2\int_{0}^{\pi} \cos{(n-1)x}dx = 0. \end{aligned}$$

The answer

$$I_n = \begin{cases} 0 & \text{if } n \text{ is even,} \\ \pi & \text{if } n \text{ is odd} \end{cases}$$

follows from the above formula and $I_0 = 0$, $I_1 = \pi$.

Problem 3. (15 points)

The linear operator A on the vector space V is called an involution if $A^2 = E$ where E is the identity operator on V. Let $\dim V = n < \infty$.

(i) Prove that for every involution A on V there exists a basis of V consisting of eigenvectors of A.

(ii) Find the maximal number of distinct pairwise commuting involutions on V.

Solution.

(i) Let $B = \frac{1}{2}(A + E)$. Then

$$B^2 = \frac{1}{4}(A^2 + 2AE + E) = \frac{1}{4}(2AE + 2E) = \frac{1}{2}(A + E) = B.$$

Hence B is a projection. Thus there exists a basis of eigenvectors for B, and the matrix of B in this basis is of the form $\mathrm{diag}(1, \dots, 1, 0, \dots, 0)$.

Since $A = 2B - E$ the eigenvalues of A are ± 1 only.

(ii) Let $\{A_i : i \in I\}$ be a set of commuting diagonalizable operators on V, and let A_1 be one of these operators. Choose an eigenvalue λ of A_1 and denote $V_\lambda = \{v \in V : A_1 v = \lambda v\}$. Then V_λ is a subspace of V, and since $A_1 A_i = A_i A_1$ for each $i \in I$ we obtain that V_λ is invariant under each A_i. If $V_\lambda = V$ then A_1 is either E or $-E$, and we can start with another operator A_i. If $V_\lambda \neq V$ we proceed by induction on $\dim V$ in order to find a common eigenvector for all A_i. Therefore $\{A_i : i \in I\}$ are simultaneously diagonalizable.

If they are involutions then $|I| \leq 2^n$ since the diagonal entries may equal 1 or -1 only.

Problem 4. (15 points)

Let $a_1 = 1$, $a_n = \frac{1}{n}\sum_{k=1}^{n-1} a_k a_{n-k}$ for $n \geq 2$. Show that

(i) $\limsup_{n\to\infty} |a_n|^{1/n} < 2^{-1/2}$;

(ii) $\limsup_{n\to\infty} |a_n|^{1/n} \geq 2/3$.

Solution.

(i) We show by induction that

$$(*) \qquad a_n \leq q^n \quad \text{for} \quad n \geq 3,$$

where $q = 0.7$ and use that $0.7 < 2^{-1/2}$. One has $a_1 = 1$, $a_2 = \frac{1}{2}$, $a_3 = \frac{1}{3}$, $a_4 = \frac{11}{48}$. Therefore $(*)$ is true for $n = 3$ and $n = 4$. Assume $(*)$ is true for $n \le N - 1$ for some $N \ge 5$. Then

$$a_N = \frac{2}{N}a_{N-1} + \frac{1}{N}a_{N-2} + \frac{1}{N}\sum_{k=3}^{N-3} a_k a_{N-k} \le \frac{2}{N}q^{N-1} + \frac{1}{N}q^{N-2} + \frac{N-5}{N}q^N \le q^N$$

because $\frac{2}{q} + \frac{1}{q^2} \le 5$.

(ii) We show by induction that

$$a_n \ge q^n \quad \text{for} \quad n \ge 2,$$

where $q = \frac{2}{3}$. One has $a_2 = \frac{1}{2} > \left(\frac{2}{3}\right)^2 = q^2$. Going by induction we have for $N \ge 3$

$$a_N = \frac{2}{N}a_{N-1} + \frac{1}{N}\sum_{k=2}^{N-2} a_k a_{N-k} \ge \frac{2}{N}q^{N-1} + \frac{N-3}{N}q^N = q^N$$

because $\frac{2}{q} = 3$.

Problem 5. (25 points)

(i) Let a, b be real numbers such that $b \le 0$ and $1 + ax + bx^2 \ge 0$ for every x in $[0, 1]$. Prove that

$$\lim_{n\to+\infty} n \int_0^1 (1 + ax + bx^2)^n dx = \begin{cases} -\dfrac{1}{a} & \text{if } a < 0, \\ +\infty & \text{if } a \ge 0. \end{cases}$$

(ii) Let $f : [0, 1] \to [0, \infty)$ be a function with a continuous second derivative and let $f''(x) \le 0$ for every x in $[0, 1]$. Suppose that $L = \lim_{n\to\infty} n \int_0^1 (f(x))^n \, dx$ exists and $0 < L < +\infty$. Prove that f' has a constant sign and $\min_{x\in[0,1]} |f'(x)| = L^{-1}$.

Solution. (i) With a linear change of the variable (i) is equivalent to:

(i′) Let a, b, A be real numbers such that $b \le 0$, $A > 0$ and $1+ax+bx^2 > 0$ for every x in $[0, A]$. Denote $I_n = n\int_0^A (1 + ax + bx^2)^n dx$. Prove that $\lim_{n\to+\infty} I_n = -\frac{1}{a}$ when $a < 0$ and $\lim_{n\to+\infty} I_n = +\infty$ when $a \ge 0$.

Let $a < 0$. Set $f(x) = e^{ax} - (1 + ax + bx^2)$. Using that $f(0) = f'(0) = 0$ and $f''(x) = a^2 e^{ax} - 2b$ we get for $x > 0$ that

$$0 < e^{ax} - (1 + ax + bx^2) < cx^2$$

where $c = \dfrac{a^2}{2} - b$. Using the mean value theorem we get

$$0 < e^{anx} - (1 + ax + bx^2)^n < cx^2 n e^{a(n-1)x}.$$

Therefore

$$0 < n \int_0^A e^{anx} dx - n \int_0^A (1 + ax + bx^2)^n dx < cn^2 \int_0^A x^2 e^{a(n-1)x} dx.$$

Using that

$$n \int_0^A e^{anx} dx = \frac{e^{anA} - 1}{a} \underset{n\to\infty}{\longrightarrow} -\frac{1}{a}$$

and

$$\int_0^A x^2 e^{a(n-1)x} dx < \frac{1}{|a|^3 (n-1)^3} \int_0^\infty t^2 e^{-t} dt$$

we get (i$'$) in the case $a < 0$.

Let $a \geq 0$. Then for $n > \max\{A^{-2}, -b\} - 1$ we have

$$\begin{aligned} n \int_0^A (1 + ax + bx^2)^n dx &> n \int_0^{\frac{1}{\sqrt{n+1}}} (1 + bx^2)^n dx \\ &> n \cdot \frac{1}{\sqrt{n+1}} \cdot \left(1 + \frac{b}{n+1}\right)^n \\ &> \frac{n}{\sqrt{n+1}} e^b \underset{n\to\infty}{\longrightarrow} \infty. \end{aligned}$$

(i) is proved.

(ii) Denote $I_n = n \displaystyle\int_0^1 (f(x))^n dx$ and $M = \max\limits_{x\in[0,1]} f(x)$.

For $M < 1$ we have $I_n \leq nM^n \underset{n\to\infty}{\longrightarrow} 0$, a contradiction.

If $M > 1$ since f is continuous there exists an interval $I \subset [0,1]$ with $|I| > 0$ such that $f(x) > 1$ for every $x \in I$. Then $I_n \geq n|I| \underset{n\to\infty}{\longrightarrow} +\infty$, a contradiction. Hence $M = 1$. Now we prove that f' has a constant sign. Assume the opposite. Then $f'(x_0) = 0$ for some $x \in (0,1)$. Then

$f(x_0) = M = 1$ because $f'' \le 0$. For x_0+h in $[0,1]$, $f(x_0+h) = 1+\dfrac{h^2}{2}f''(\xi)$, $\xi \in (x_0, x_0+h)$. Let $m = \min\limits_{x\in[0,1]} f''(x)$. So, $f(x_0+h) \ge 1 + \frac{h^2}{2}m$.

Let $\delta > 0$ be such that $1 + \dfrac{\delta^2}{2}m > 0$ and $x_0 + \delta < 1$. Then

$$I_n \ge n\int_{x_0}^{x_0+\delta} (f(x))^n dx \ge n\int_0^{\delta}\left(1+\frac{m}{2}h^2\right)^n dh \underset{n\to\infty}{\longrightarrow} \infty$$

in view of (i$'$) – a contradiction. Hence f is monotone and $M = f(0)$ or $M = f(1)$.

Let $M = f(0) = 1$. For h in $[0,1]$

$$1 + hf'(0) \ge f(h) \ge 1 + hf'(0) + \frac{m}{2}h^2,$$

where $f'(0) \ne 0$, because otherwise we get a contradiction as above. Since $f(0) = M$ the function f is decreasing and hence $f'(0) < 0$. Let $0 < A < 1$ be such that $1 + Af'(0) + \dfrac{m}{2}A^2 > 0$. Then

$$n\int_0^A (1+hf'(0))^n dh \ge n\int_0^A (f(x))^n dx \ge n\int_0^A \left(1+hf'(0)+\frac{m}{2}h^2\right)^n dh.$$

From (i$'$) the first and the third integral tend to $-\dfrac{1}{f'(0)}$ as $n\to\infty$, hence so does the second.

Also $n\displaystyle\int_A^1 (f(x))^n dx \le n(f(A))^n \underset{n\to\infty}{\longrightarrow} 0$ $(f(A) < 1)$. We get $L = -\dfrac{1}{f'(0)}$ in this case.

If $M = f(1)$ we get in a similar way $L = \dfrac{1}{f'(1)}$.

Problem 6. (25 points)

Upper content of a subset E of the plane $\mathbb{R}^2$ is defined as

$$\mathcal{C}(E) = \inf\left\{\sum_{i=1}^n \operatorname{diam}(E_i)\right\}$$

where inf is taken over all finite families of sets $E_1,\dots,E_n$, $n\in\mathbb{N}$, in $\mathbb{R}^2$ such that $E \subset \bigcup\limits_{i=1}^n E_i$.

Lower content of E is defined as

$$\mathcal{K}(E) = \sup\{\text{lenght}(L) \;:\; L \text{ is a closed line segment onto which } E \text{ can be contracted}\}.$$

Show that

(a) $\mathcal{C}(L) = \text{lenght}(L)$ if L is a closed line segment;

(b) $\mathcal{C}(E) \geq \mathcal{K}(E)$;

(c) the equality in (b) needs not hold even if E is compact.

Hint. If $E = T \cup T'$ where T is the triangle with vertices $(-2,2)$, $(2,2)$ and $(0,4)$, and T' is its reflexion about the x-axis, then $\mathcal{C}(E) = 8 > \mathcal{K}(E)$.

Remarks: All *distances* used in this problem are Euclidian. *Diameter* of a set E is $\text{diam}(E) = \sup\{\text{dist}(x,y) \;:\; x,y \in E\}$. *Contraction* of a set E to a set F is a mapping $f : E \mapsto F$ such that $\text{dist}(f(x), f(y)) \leq \text{dist}(x,y)$ for all $x, y \in E$. A set E can be contracted *onto* a set F if there is a contraction f of E to F which is onto, i.e., such that $f(E) = F$. *Triangle* is defined as the union of the three segments joining its vertices, i.e., it does not contain the interior.

Solution.

(a) The choice $E_1 = L$ gives $\mathcal{C}(L) \leq \text{lenght}(L)$. If $E \subset \cup_{i=1}^n E_i$ then $\sum_{i=1}^n \text{diam}(E_i) \geq \text{lenght}(L)$: By induction, n=1 obvious, and assuming that E_{n+1} contains the end point a of L, define the segment $L_\varepsilon = \{x \in L \;:\; \text{dist}(x,a) \geq \text{diam}(E_{n+1}) + \varepsilon\}$ and use induction assumption to get $\sum_{i=1}^{n+1} \text{diam}(E_i) \geq \text{lenght}(L_\varepsilon) + \text{diam}(E_{n+1}) \geq \text{lenght}(L) - \varepsilon$; but $\varepsilon > 0$ is arbitrary.

(b) If f is a contraction of E onto L and $E \subset \cup_{n=1}^n E_i$, then $L \subset \cup_{i=1}^n f(E_i)$ and $\text{lenght}(L) \leq \sum_{i=1}^n \text{diam}(f(E_i)) \leq \sum_{i=1}^n \text{diam}(E_i)$.

(c1) Let $E = T \cup T'$ where T is the triangle with vertices $(-2,2)$, $(2,2)$ and $(0,4)$, and T' is its reflexion about the x-axis. Suppose $E \subset \bigcup_{i=1}^n E_i$. If no set among E_i meets both T and T', then E_i may be partitioned into covers of segments $[(-2,2),(2,2)]$ and $[(-2,-2),(2,-2)]$, both of length 4, so $\sum_{i=1}^n \text{diam}(E_i) \geq 8$. If at least one set among E_i, say E_k, meets both T and T', choose $a \in E_k \cap T$ and $b \in E_k \cap T'$ and note that the sets $E_i' = E_i$ for $i \neq k$, $E_k' = E_k \cup [a,b]$ cover $T \cup T' \cup [a,b]$, which is a set of upper content at least 8, since its orthogonal projection onto y-axis is a segment of length 8. Since $\mathrm{diam}(E_j) = \mathrm{diam}(E'_j)$, we get $\sum_{i=1}^{n} \mathrm{diam}(E_i) \geq 8$.

(c2) Let f be a contraction of E onto $L = [a', b']$. Choose $a = (a_1, a_2)$, $b = (b_1, b_2) \in E$ such that $f(a) = a'$ and $f(b) = b'$. Since lenght$(L) = \mathrm{dist}(a', b') \leq \mathrm{dist}(a, b)$ and since the triangles have diameter only 4, we may assume that $a \in T$ and $b \in T'$. Observe that if $a_2 \leq 3$ then a lies on one of the segments joining some of the points $(-2, 2)$, $(2, 2)$, $(-1, 3)$, $(1, 3)$; since all these points have distances from vertices, and so from points, of T_2 at most $\sqrt{50}$, we get that lenght$(L) \leq \mathrm{dist}(a, b) \leq \sqrt{50}$. Similarly if $b_2 \geq -3$. Finally, if $a_2 > 3$ and $b_2 < -3$, we note that every vertex, and so every point of T is in the distance at most $\sqrt{10}$ for a and every vertex, and so every point, of T' is in the distance at most $\sqrt{10}$ of b. Since f is a contraction, the image of T lies in a segment containing a' of length at most $\sqrt{10}$ and the image of T' lies in a segment containing b' of length at most $\sqrt{10}$. Since the union of these two images is L, we get lenght$(L) \leq 2\sqrt{10} \leq \sqrt{50}$. Thus $\mathcal{K}(E) \leq \sqrt{50} < 8$.

Second day — August 3, 1996

Problem 1. (10 points)

Prove that if $f : [0, 1] \to [0, 1]$ is a continuous function, then the sequence of iterates $x_{n+1} = f(x_n)$ converges if and only if

$$\lim_{n\to\infty} (x_{n+1} - x_n) = 0.$$

Solution. The "only if" part is obvious. Now suppose that $\lim_{n\to\infty} (x_{n+1} - x_n) = 0$ and the sequence $\{x_n\}$ does not converge. Then there are two cluster points $K < L$. There must be points from the interval (K, L) in the sequence. There is an $x \in (K, L)$ such that $f(x) \neq x$. Put $\varepsilon = \dfrac{|f(x) - x|}{2} > 0$. Then from the continuity of the function f we get that for some $\delta > 0$ for all $y \in (x - \delta, x + \delta)$ it is $|f(y) - y| > \varepsilon$. On the other hand for n large enough it is $|x_{n+1} - x_n| < 2\delta$ and $|f(x_n) - x_n| = |x_{n+1} - x_n| < \varepsilon$. So the sequence cannot come into the interval $(x - \delta, x + \delta)$, but also cannot jump over this interval. Then all cluster points have to be at most $x - \delta$ (a contradiction with L being a cluster point), or at least $x + \delta$ (a contradiction with K being a cluster point).

Problem 2. (10 points)

Let θ be a positive real number and let $\cosh t = \dfrac{e^t + e^{-t}}{2}$ denote the hyperbolic cosine. Show that if $k \in \mathbb{N}$ and both $\cosh k\theta$ and $\cosh (k+1)\theta$ are rational, then so is $\cosh \theta$.

Solution. First we show that

(1) If $\cosh t$ is rational and $m \in \mathbb{N}$, then $\cosh mt$ is rational.

Since $\cosh 0.t = \cosh 0 = 1 \in \mathbb{Q}$ and $\cosh 1.t = \cosh t \in \mathbb{Q}$, (1) follows inductively from

$$\cosh (m+1)t = 2\cosh t.\cosh mt - \cosh (m-1)t.$$

The statement of the problem is obvious for $k = 1$, so we consider $k \geq 2$. For any m we have

(2)
$$\begin{aligned}\cosh \theta &= \cosh ((m+1)\theta - m\theta) = \\ &= \cosh (m+1)\theta.\cosh m\theta - \sinh (m+1)\theta.\sinh m\theta \\ &= \cosh (m+1)\theta.\cosh m\theta - \sqrt{\cosh^2 (m+1)\theta - 1}.\sqrt{\cosh^2 m\theta - 1}\end{aligned}$$

Set $\cosh k\theta = a$, $\cosh (k+1)\theta = b$, $a, b \in \mathbb{Q}$. Then (2) with $m = k$ gives

$$\cosh \theta = ab - \sqrt{a^2-1}\sqrt{b^2-1}$$

and then

(3)
$$\begin{aligned}(a^2-1)(b^2-1) &= (ab - \cosh \theta)^2 \\ &= a^2b^2 - 2ab\cosh \theta + \cosh^2 \theta.\end{aligned}$$

Set $\cosh (k^2-1)\theta = A$, $\cosh k^2\theta = B$. From (1) with $m = k-1$ and $t = (k+1)\theta$ we have $A \in \mathbb{Q}$. From (1) with $m = k$ and $t = k\theta$ we have $B \in \mathbb{Q}$. Moreover $k^2 - 1 > k$ implies $A > a$ and $B > b$. Thus $AB > ab$. From (2) with $m = k^2 - 1$ we have

(4)
$$\begin{aligned}(A^2-1)(B^2-1) &= (AB - \cosh \theta)^2 \\ &= A^2B^2 - 2AB\cosh \theta + \cosh^2 \theta.\end{aligned}$$

So after we cancel the $\cosh^2 \theta$ from (3) and (4) we have a non-trivial linear equation in $\cosh \theta$ with rational coefficients.

Problem 3. (15 points)
Let G be the subgroup of $GL_2(\mathbb{R})$, generated by A and B, where

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Let H consist of those matrices $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ in G for which $a_{11}{=}a_{22}{=}1$.
(a) Show that H is an abelian subgroup of G.
(b) Show that H is not finitely generated.

Remarks. $GL_2(\mathbb{R})$ denotes, as usual, the group (under matrix multiplication) of all 2×2 invertible matrices with real entries (elements). *Abelian* means commutative. A group is *finitely generated* if there are a finite number of elements of the group such that every other element of the group can be obtained from these elements using the group operation.

Solution.
(a) All of the matrices in G are of the form

$$\begin{bmatrix} * & * \\ 0 & * \end{bmatrix}.$$

So all of the matrices in H are of the form

$$M(x) = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix},$$

so they commute. Since $M(x)^{-1} = M(-x)$, H is a subgroup of G.

(b) A generator of H can only be of the form $M(x)$, where x is a binary rational, i.e., $x = \dfrac{p}{2^n}$ with integer p and non-negative integer n. In H it holds

$$M(x)M(y) = M(x+y)$$
$$M(x)M(y)^{-1} = M(x-y).$$

The matrices of the form $M\left(\dfrac{1}{2^n}\right)$ are in H for all $n \in \mathbb{N}$. With only finite number of generators all of them cannot be achieved.

Problem 4. (20 points)

Let B be a bounded closed convex symmetric (with respect to the origin) set in $\mathbb{R}^2$ with boundary the curve Γ. Let B have the property that the ellipse of maximal area contained in B is the disc D of radius 1 centered at the origin with boundary the circle C. Prove that $A \cap \Gamma \neq \emptyset$ for any arc A of C of length $l(A) \geq \dfrac{\pi}{2}$.

Solution. Assume the contrary – there is an arc $A \subset C$ with length $l(A) = \dfrac{\pi}{2}$ such that $A \subset B \backslash \Gamma$. Without loss of generality we may assume that the ends of A are $M = (1/\sqrt{2}, 1/\sqrt{2})$, $N = (1/\sqrt{2}, -1/\sqrt{2})$. A is compact and Γ is closed. From $A \cap \Gamma = \emptyset$ we get $\delta > 0$ such that $\operatorname{dist}(x, y) > \delta$ for every $x \in A$, $y \in \Gamma$.

Given $\varepsilon > 0$ with E_ε we denote the ellipse with boundary: $\dfrac{x^2}{(1+\varepsilon)^2} + \dfrac{y^2}{b^2} = 1$, such that $M, N \in E_\varepsilon$. Since $M \in E_\varepsilon$ we get

$$b^2 = \frac{(1+\varepsilon)^2}{2(1+\varepsilon)^2 - 1}.$$

Then we have

$$\operatorname{area} E_\varepsilon = \pi \frac{(1+\varepsilon)^2}{\sqrt{2(1+\varepsilon)^2 - 1}} > \pi = \operatorname{area} D.$$

In view of the hypotheses, $E_\varepsilon \setminus B \neq \emptyset$ for every $\varepsilon > 0$. Let $S = \{(x, y) \in \mathbb{R}^2 \ : \ |x| > |y|\}$. ¿From $E_\varepsilon \setminus S \subset D \subset B$ it follows that $E_\varepsilon \setminus B \subset S$. Taking $\varepsilon < \delta$ we get that

$$\emptyset \neq E_\varepsilon \setminus B \subset E_\varepsilon \cap S \subset D_{1+\varepsilon} \cap S \subset B$$

– a contradiction (we use the notation $D_t = \{(x, y) \in \mathbb{R}^2 \ : \ x^2 + y^2 \leq t^2\}$).

Remark. The ellipse with maximal area is well known as John's ellipse. Any coincidence with the President of the Jury is accidental.

Problem 5. (20 points)

(i) Prove that

$$\lim_{x \to +\infty} \sum_{n=1}^{\infty} \frac{nx}{(n^2 + x)^2} = \frac{1}{2}.$$

(ii) Prove that there is a positive constant c such that for every $x \in [1, \infty)$ we have

$$\left| \sum_{n=1}^{\infty} \frac{nx}{(n^2 + x)^2} - \frac{1}{2} \right| \leq \frac{c}{x}.$$

Solution.

(i) Set $f(t) = \dfrac{t}{(1+t^2)^2}$, $h = \dfrac{1}{\sqrt{x}}$. Then

$$\sum_{n=1}^{\infty} \frac{nx}{(n^2+x)^2} = h\sum_{n=1}^{\infty} f(nh) \underset{h\to 0}{\longrightarrow} \int_0^{\infty} f(t)dt = \frac{1}{2}.$$

The convergence holds since $h\sum_{n=1}^{\infty} f(nh)$ is a Riemann sum of the integral $\int_0^{\infty} f(t)dt$. There are no problems with the infinite domain because f is integrable and $f \downarrow 0$ for $x \to \infty$ (thus $h\sum_{n=N}^{\infty} f(nh) \geq \int_{nN}^{\infty} f(t)dt \geq h\sum_{n=N+1}^{\infty} f(nh)$).

(ii) We have

$$\begin{aligned}(1)\qquad \left|\sum_{n=1}^{\infty} \frac{nx}{(n^2+x)^2} - \frac{1}{2}\right| &= \left|\sum_{n=1}^{\infty}\left(hf(nh) - \int_{nh-\frac{h}{2}}^{nh+\frac{h}{2}} f(t)dt\right) - \int_0^{\frac{h}{2}} f(t)dt\right| \\ &\leq \sum_{n=1}^{\infty}\left|hf(nh) - \int_{nh-\frac{h}{2}}^{nh+\frac{h}{2}} f(t)dt\right| + \int_0^{\frac{h}{2}} f(t)dt\end{aligned}$$

Using twice integration by parts one has

$$(2)\qquad 2bg(a) - \int_{a-b}^{a+b} g(t)dt = -\frac{1}{2}\int_0^b (b-t)^2(g''(a+t) + g''(a-t))dt$$

for every $g \in C^2[a-b, a+b]$. Using $f(0) = 0$, $f \in C^2[0, h/2]$ one gets

$$(3)\qquad \int_0^{h/2} f(t)dt = O(h^2).$$

From (1), (2) and (3) we get

$$\begin{aligned}\left|\sum_{n=1}^{\infty} \frac{nx}{(n^2+x)^2} - \frac{1}{2}\right| &\leq \sum_{n=1}^{\infty} h^2 \int_{nh-\frac{h}{2}}^{nh+\frac{h}{2}} |f''(t)|dt + O(h^2) = \\ &= h^2 \int_{\frac{h}{2}}^{\infty} |f''(t)|dt + O(h^2) = O(h^2) = O(x^{-1}).\end{aligned}$$

Problem 6. (Carleman's inequality) (25 points)

(i) Prove that for every sequence $\{a_n\}_{n=1}^{\infty}$, such that $a_n > 0$, $n = 1, 2, \ldots$ and $\sum_{n=1}^{\infty} a_n < \infty$, we have

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} a_n,$$

where e is the natural log base.

(ii) Prove that for every $\varepsilon > 0$ there exists a sequence $\{a_n\}_{n=1}^{\infty}$, such that $a_n > 0$, $n = 1, 2, \ldots$, $\sum_{n=1}^{\infty} a_n < \infty$ and

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} > (e - \varepsilon) \sum_{n=1}^{\infty} a_n.$$

Solution.

(i) Put for $n \in \mathbb{N}$

$$c_n = (n+1)^n / n^{n-1}. \tag{1}$$

Observe that $c_1 c_2 \cdots c_n = (n+1)^n$. Hence, for $n \in \mathbb{N}$,

$$\begin{aligned}(a_1 a_2 \cdots a_n)^{1/n} &= (a_1 c_1 a_2 c_2 \cdots a_n c_n)^{1/n} / (n+1) \\ &\le (a_1 c_1 + \cdots + a_n c_n) / n(n+1).\end{aligned}$$

Consequently,

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} \le \sum_{n=1}^{\infty} a_n c_n \left(\sum_{m=n}^{\infty} (m(m+1))^{-1} \right). \tag{2}$$

Since

$$\sum_{m=n}^{\infty} (m(m+1))^{-1} = \sum_{m=n}^{\infty} \left(\frac{1}{m} - \frac{1}{m+1} \right) = 1/n$$

we have

$$\begin{aligned}\sum_{n=1}^{\infty} a_n c_n \left(\sum_{m=n}^{\infty} (m(m+1))^{-1} \right) &= \sum_{n=1}^{\infty} a_n c_n / n \\ &= \sum_{n=1}^{\infty} a_n ((n+1)/n)^n < e \sum_{n=1}^{\infty} a_n\end{aligned}$$

(by (1)). Combining the last inequality with (2) we get the result.

(ii) Set $a_n = n^{n-1}(n+1)^{-n}$ for $n = 1, 2, \dots, N$ and $a_n = 2^{-n}$ for $n > N$, where N will be chosen later. Then

$$(a_1 \cdots a_n)^{1/n} = \frac{1}{n+1} \tag{3}$$

for $n \le N$. Let $K = K(\varepsilon)$ be such that

$$\left(\frac{n+1}{n}\right)^n > e - \frac{\varepsilon}{2} \quad \text{for} \quad n > K. \tag{4}$$

Choose N from the condition

$$\sum_{n=1}^{K} a_n + \sum_{n=1}^{\infty} 2^{-n} \le \frac{\varepsilon}{(2e-\varepsilon)(e-\varepsilon)} \sum_{n=K+1}^{N} \frac{1}{n}, \tag{5}$$

which is always possible because the harmonic series diverges. Using (3), (4) and (5) we have

$$\begin{aligned}\sum_{n=1}^{\infty} a_n &= \sum_{n=1}^{K} a_n + \sum_{n=N+1}^{\infty} 2^{-n} + \sum_{n=K+1}^{N} \frac{1}{n}\left(\frac{n}{n+1}\right)^n < \\ &< \frac{\varepsilon}{(2e-\varepsilon)(e-\varepsilon)} \sum_{n=K+1}^{N} \frac{1}{n} + \left(e - \frac{\varepsilon}{2}\right)^{-1} \sum_{n=K+1}^{N} \frac{1}{n} = \\ &= \frac{1}{e-\varepsilon} \sum_{n=K+1}^{N} \frac{1}{n} \le \frac{1}{e-\varepsilon} \sum_{n=1}^{\infty} (a_1 \cdots a_n)^{1/n}.\end{aligned}$$

FOURTH INTERNATIONAL COMPETITION FOR UNIVERSITY STUDENTS IN MATHEMATICS
July 30 – August 4, 1997, Plovdiv, BULGARIA

First day — August 1, 1997

Problems and Solutions

Problem 1.

Let $\{\varepsilon_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers, such that $\lim\limits_{n\to\infty}\varepsilon_n = 0$. Find

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\ln\left(\frac{k}{n}+\varepsilon_n\right),$$

where ln denotes the natural logarithm.

Solution.

It is well known that

$$-1=\int_0^1 \ln x dx = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\ln\left(\frac{k}{n}\right)$$

(Riemman's sums). Then

$$\frac{1}{n}\sum_{k=1}^{n}\ln\left(\frac{k}{n}+\varepsilon_n\right) \geq \frac{1}{n}\sum_{k=1}^{n}\ln\left(\frac{k}{n}\right)\underset{n\to\infty}{\longrightarrow} -1.$$

Given $\varepsilon>0$ there exist n_0 such that $0<\varepsilon_n\leq\varepsilon$ for all $n\geq n_0$. Then

$$\frac{1}{n}\sum_{k=1}^{n}\ln\left(\frac{k}{n}+\varepsilon_n\right) \leq \frac{1}{n}\sum_{k=1}^{n}\ln\left(\frac{k}{n}+\varepsilon\right).$$

Since

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\ln\left(\frac{k}{n}+\varepsilon\right) &= \int_0^1 \ln(x+\varepsilon)dx \\ &= \int_{\varepsilon}^{1+\varepsilon}\ln x dx\end{aligned}$$

we obtain the result when ε goes to 0 and so

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\ln\left(\frac{k}{n}+\varepsilon_n\right)=-1.$$

Problem 2.

Suppose $\sum\limits_{n=1}^{\infty} a_n$ converges. Do the following sums have to converge as well?

a) $a_1+a_2+a_4+a_3+a_8+a_7+a_6+a_5+a_{16}+a_{15}+\cdots+a_9+a_{32}+\cdots$

b) $a_1+a_2+a_3+a_4+a_5+a_7+a_6+a_8+a_9+a_{11}+a_{13}+a_{15}+a_{10}+a_{12}+a_{14}+a_{16}+a_{17}+a_{19}+\cdots$

Justify your answers.

Solution.

a) Yes. Let $S=\sum\limits_{n=1}^{\infty} a_n$, $S_n=\sum\limits_{k=1}^{n} a_k$. Fix $\varepsilon>0$ and a number n_0 such that $|S_n-S|<\varepsilon$ for $n>n_0$. The partial sums of the permuted series have the form $L_{2^{n-1}+k}=S_{2^{n-1}}+S_{2^n}-S_{2^n-k}$, $0\le k<2^{n-1}$ and for $2^{n-1}>n_0$ we have $|L_{2^{n-1}+k}-S|<3\varepsilon$, i.e. the permuted series converges.

b) No. Take $a_n=\dfrac{(-1)^{n+1}}{\sqrt{n}}$.Then $L_{3.2^{n-2}}=S_{2^{n-1}}+\sum\limits_{k=2^{n-2}}^{2^{n-1}-1}\dfrac{1}{\sqrt{2k+1}}$ and $L_{3.2^{n-2}}-S_{2^{n-1}}\ge 2^{n-2}\dfrac{1}{\sqrt{2^n}}\underset{n\to\infty}{\longrightarrow}\infty$, so $L_{3.2^{n-2}}\underset{n\to\infty}{\longrightarrow}\infty$.

Problem 3.

Let A and B be real $n\times n$ matrices such that $A^2+B^2=AB$. Prove that if $BA-AB$ is an invertible matrix then n is divisible by 3.

Solution.

Set $S=A+\omega B$, where $\omega=-\dfrac{1}{2}+i\dfrac{\sqrt{3}}{2}$. We have

$$\begin{aligned} S\overline{S} &= (A+\omega B)(A+\overline{\omega}B)=A^2+\omega BA+\overline{\omega}AB+B^2\\ &= AB+\omega BA+\overline{\omega}AB=\omega(BA-AB), \end{aligned}$$

because $\overline{\omega}+1=-\omega$. Since $\det(S\overline{S})=\det S.\det\overline{S}$ is a real number and $\det\omega(BA-AB)=\omega^n\det(BA-AB)$ and $\det(BA-AB)\neq 0$, then ω^n is a real number. This is possible only when n is divisible by 3.

Problem 4.

Let α be a real number, $1 < \alpha < 2$.

a) Show that α has a unique representation as an infinite product

$$\alpha = \left(1 + \frac{1}{n_1}\right)\left(1 + \frac{1}{n_2}\right)\dots$$

where each n_i is a positive integer satisfying

$$n_i^2 \leq n_{i+1}.$$

b) Show that α is rational if and only if its infinite product has the following property:

For some m and all $k \geq m$,

$$n_{k+1} = n_k^2.$$

Solution.

a) We construct inductively the sequence $\{n_i\}$ and the ratios

$$\theta_k = \frac{\alpha}{\prod_1^k (1 + \frac{1}{n_i})}$$

so that

$$\theta_k > 1 \quad \text{for all} \quad k.$$

Choose n_k to be the least n for which

$$1 + \frac{1}{n} < \theta_{k-1}$$

($\theta_0 = \alpha$) so that for each k,

$$1 + \frac{1}{n_k} < \theta_{k-1} \leq 1 + \frac{1}{n_k - 1}. \tag{1}$$

Since

$$\theta_{k-1} \leq 1 + \frac{1}{n_k - 1}$$

we have

$$1 + \frac{1}{n_{k+1}} < \theta_k = \frac{\theta_{k-1}}{1 + \frac{1}{n_k}} \leq \frac{1 + \frac{1}{n_k - 1}}{1 + \frac{1}{n_k}} = 1 + \frac{1}{n_k^2 - 1}.$$

Hence, for each k, $n_{k+1} \geq n_k^2$.

Since $n_1 \geq 2$, $n_k \to \infty$ so that $\theta_k \to 1$. Hence

$$\alpha = \prod_1^\infty \left(1 + \frac{1}{n_k}\right).$$

The uniqueness of the infinite product will follow from the fact that on every step n_k has to be determine by (1).

Indeed, if for some k we have

$$1 + \frac{1}{n_k} \geq \theta_{k-1}$$

then $\theta_k \leq 1$, $\theta_{k+1} < 1$ and hence $\{\theta_k\}$ does not converge to 1.

Now observe that for $M > 1$,

$$(2) \quad \left(1 + \frac{1}{M}\right)\left(1 + \frac{1}{M^2}\right)\left(1 + \frac{1}{M^4}\right)\cdots = 1 + \frac{1}{M} + \frac{1}{M^2} + \frac{1}{M^3} + \cdots = 1 + \frac{1}{M-1}.$$

Assume that for some k we have

$$1 + \frac{1}{n_k - 1} < \theta_{k-1}.$$

Then we get

$$\begin{aligned} \frac{\alpha}{(1+\frac{1}{n_1})(1+\frac{1}{n_2})\dots} &= \frac{\theta_{k-1}}{(1+\frac{1}{n_k})(1+\frac{1}{n_{k+1}})\dots} \\ &\geq \frac{\theta_{k-1}}{(1+\frac{1}{n_k})(1+\frac{1}{n_k^2})\dots} = \frac{\theta_{k-1}}{1+\frac{1}{n_k-1}} > 1 \end{aligned}$$

– a contradiction.

b) From (2) α is rational if its product ends in the stated way.

Conversely, suppose α is the rational number $\frac{p}{q}$. Our aim is to show that for some m,

$$\theta_{m-1} = \frac{n_m}{n_m - 1}.$$

Suppose this is not the case, so that for every m,

$$(3) \qquad \theta_{m-1} < \frac{n_m}{n_m - 1}.$$

For each k we write

$$\theta_k = \frac{p_k}{q_k}$$

as a fraction (not necessarily in lowest terms) where

$$p_0 = p, \quad q_0 = q$$

and in general

$$p_k = p_{k-1} n_k, \quad q_k = q_{k-1}(n_k + 1).$$

The numbers $p_k - q_k$ are positive integers: to obtain a contradiction it suffices to show that this sequence is strictly decreasing. Now,

$$\begin{aligned} p_k - q_k - (p_{k-1} - q_{k-1}) &= n_k p_{k-1} - (n_k + 1) q_{k-1} - p_{k-1} + q_{k-1} \\ &= (n_k - 1) p_{k-1} - n_k q_{k-1} \end{aligned}$$

and this is negative because

$$\frac{p_{k-1}}{q_{k-1}} = \theta_{k-1} < \frac{n_k}{n_k - 1}$$

by inequality (3).

Problem 5. For a natural n consider the hyperplane

$$R_0^n = \left\{ x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=1}^{n} x_i = 0 \right\}$$

and the lattice $Z_0^n = \{y \in R_0^n : \text{ all } y_i \text{ are integers}\}$. Define the (quasi–)norm in $\mathbb{R}^n$ by $\|x\|_p = \left(\sum\limits_{i=1}^{n} |x_i|^p \right)^{1/p}$ if $0 < p < \infty$, and $\|x\|_\infty = \max\limits_i |x_i|$.

a) Let $x \in R_0^n$ be such that

$$\max_i x_i - \min_i x_i \leq 1.$$

For every $p \in [1, \infty]$ and for every $y \in Z_0^n$ prove that

$$\|x\|_p \leq \|x + y\|_p.$$

b) For every $p \in (0, 1)$, show that there is an n and an $x \in R_0^n$ with $\max\limits_i x_i - \min\limits_i x_i \leq 1$ and an $y \in Z_0^n$ such that

$$\|x\|_p > \|x + y\|_p.$$

Solution.

a) For $x = 0$ the statement is trivial. Let $x \neq 0$. Then $\max_i x_i > 0$ and $\min_i x_i < 0$. Hence $\|x\|_\infty < 1$. From the hypothesis on x it follows that:

i) If $x_j \leq 0$ then $\max_i x_i \leq x_j + 1$.

ii) If $x_j \geq 0$ then $\min_i x_i \geq x_j - 1$.

Consider $y \in Z_0^n$, $y \neq 0$. We split the indices $\{1, 2, \dots, n\}$ into five sets:

$$I(0) = \{i : y_i = 0\},$$

$$I(+,+) = \{i : y_i > 0, x_i \geq 0\}, \quad I(+,-) = \{i : y_i > 0, x_i < 0\},$$

$$I(-,+) = \{i : y_i < 0, x_i > 0\}, \quad I(-,-) = \{i : y_i < 0, x_i \leq 0\}.$$

As least one of the last four index sets is not empty. If $I(+,+) \neq \emptyset$ or $I(-,-) \neq \emptyset$ then $\|x+y\|_\infty \geq 1 > \|x\|_\infty$. If $I(+,+) = I(-,-) = \emptyset$ then $\sum y_i = 0$ implies $I(+,-) \neq \emptyset$ and $I(-,+) \neq \emptyset$. Therefore i) and ii) give $\|x+y\|_\infty \geq \|x\|_\infty$ which completes the case $p = \infty$.

Now let $1 \leq p < \infty$. Then using i) for every $j \in I(+,-)$ we get $|x_j + y_j| = y_j - 1 + x_j + 1 \geq |y_j| - 1 + \max_i x_i$. Hence

$$|x_j + y_j|^p \geq |y_j| - 1 + |x_k|^p \quad \text{for every } k \in I(-,+) \quad \text{and} \quad j \in I(+,-).$$

Similarly

$$|x_j + y_j|^p \geq |y_j| - 1 + |x_k|^p \quad \text{for every } k \in I(+,-) \quad \text{and} \quad j \in I(-,+);$$

$$|x_j + y_j|^p \geq |y_j| + |x_j|^p \quad \text{for every } j \in I(+,+) \cup I(-,-).$$

Assume that $\sum_{j \in I(+,-)} 1 \geq \sum_{j \in I(-,+)} 1$. Then

$$\begin{aligned}
&\|x+y\|_p^p - \|x\|_p^p \\
&= \sum_{j \in I(+,+) \cup I(-,-)} (|x_j + y_j|^p - |x_j|^p) + \left(\sum_{j \in I(+,-)} |x_j + y_j|^p - \sum_{k \in I(-,+)} |x_k|^p \right) \\
&\quad + \left(\sum_{j \in I(-,+)} |x_j + y_j|^p - \sum_{k \in I(+,-)} |x_k|^p \right) \\
&\geq \sum_{j \in I(+,+) \cup I(-,-)} |y_j| + \sum_{j \in I(+,-)} (|y_j| - 1)
\end{aligned}$$

$$+\left(\sum_{j\in I(-,+)}(|y_j|-1)-\sum_{j\in I(+,-)}1+\sum_{j\in I(-,+)}1\right)$$

$$=\sum_{i=1}^{n}|y_i|-2\sum_{j\in I(+,-)}1=2\sum_{j\in I(+,-)}(y_j-1)+2\sum_{j\in I(+,+)}y_j\geq 0.$$

The case $\sum_{j\in I(+,-)} 1 \leq \sum_{j\in I(-,+)} 1$ is similar. This proves the statement.

b) Fix $p \in (0,1)$ and a rational $t \in (\frac{1}{2}, 1)$. Choose a pair of positive integers m and l such that $mt = l(1-t)$ and set $n = m + l$. Let

$$x_i = t, \quad i = 1,2,\ldots,m; \quad x_i = t-1, \quad i = m+1, m+2, \ldots, n;$$

$$y_i = -1, \quad i = 1,2,\ldots,m; \quad y_{m+1} = m; \quad y_i = 0, \ i = m+2,\ldots,n.$$

Then $x \in R_0^n$, $\max_i x_i - \min_i x_i = 1$, $y \in Z_0^n$ and

$$\|x\|_p^p - \|x+y\|_p^p = m(t^p - (1-t)^p) + (1-t)^p - (m-1+t)^p,$$

which is possitive for m big enough.

Problem 6. Suppose that F is a family of finite subsets of $\mathbb{N}$ and for any two sets $A, B \in F$ we have $A \cap B \neq \emptyset$.

a) Is it true that there is a finite subset Y of $\mathbb{N}$ such that for any $A, B \in F$ we have $A \cap B \cap Y \neq \emptyset$?

b) Is the statement a) true if we suppose in addition that all of the members of F have the same size?

Justify your answers.

Solution.

a) No. Consider $F = \{A_1, B_1, \ldots, A_n, B_n, \ldots\}$, where $A_n = \{1, 3, 5, \ldots, 2n-1, 2n\}$, $B_n = \{2, 4, 6, \ldots, 2n, 2n+1\}$.

b) Yes. We will prove inductively a stronger statement:

Suppose F, G are two families of finite subsets of $\mathbb{N}$ such that:

1) For every $A \in F$ and $B \in G$ we have $A \cap B \neq \emptyset$;

2) All the elements of F have the same size r, and elements of G – size s. (we shall write $\#(F) = r$, $\#(G) = s$).

Then there is a finite set Y such that $A \cup B \cup Y \neq \emptyset$ for every $A \in F$ and $B \in G$.

The problem b) follows if we take $F = G$.

Proof of the statement: The statement is obvious for $r = s = 1$. Fix the numbers r, s and suppose the statement is proved for all pairs F', G' with $\#(F') < r$, $\#(G') < s$. Fix $A_0 \in F$, $B_0 \in G$. For any subset $C \subset A_0 \cup B_0$, denote

$$F(C) = \{A \in F : A \cap (A_0 \cup B_0) = C\}.$$

Then $F = \bigcup_{\emptyset \neq C \subset A_0 \cup B_0} F(C)$. It is enough to prove that for any pair of non-empty sets $C, D \subset A_0 \cup B_0$ the families $F(C)$ and $G(D)$ satisfy the statement.

Indeed, if we denote by $Y_{C,D}$ the corresponding finite set, then the finite set $\bigcup_{C,D \subset A_0 \cup B_0} Y_{C,D}$ will satisfy the statement for F and G. The proof for $F(C)$ and $G(D)$.

If $C \cap D \neq \emptyset$, it is trivial.

If $C \cap D = \emptyset$, then any two sets $A \in F(C)$, $B \in G(D)$ must meet outside $A_0 \cup B_0$. Then if we denote $\tilde{F}(C) = \{A \setminus C : A \in F(C)\}$, $\tilde{G}(D) = \{B \setminus D : B \in G(D)\}$, then $\tilde{F}(C)$ and $\tilde{G}(D)$ satisfy the conditions 1) and 2) above, with $\#(\tilde{F}(C)) = \#(F) - \#C < r$, $\#(\tilde{G}(D)) = \#(G) - \#D < s$, and the inductive assumption works.

FOURTH INTERNATIONAL COMPETITION FOR UNIVERSITY STUDENTS IN MATHEMATICS
July 30 – August 4, 1997, Plovdiv, BULGARIA

Second day — August 2, 1997

Problems and Solutions

Problem 1.

Let f be a $C^3(\mathbb{R})$ non-negative function, $f(0)=f'(0)=0$, $0 < f''(0)$. Let

$$g(x) = \left(\frac{\sqrt{f(x)}}{f'(x)}\right)'$$

for $x \neq 0$ and $g(0) = 0$. Show that g is bounded in some neighbourhood of 0. Does the theorem hold for $f \in C^2(\mathbb{R})$?

Solution.

Let $c = \frac{1}{2}f''(0)$. We have

$$g = \frac{(f')^2 - 2ff''}{2(f')^2\sqrt{f}},$$

where

$$f(x) = cx^2 + O(x^3), \quad f'(x) = 2cx + O(x^2), \quad f''(x) = 2c + O(x).$$

Therefore $(f'(x))^2 = 4c^2x^2 + O(x^3)$,

$$2f(x)f''(x) = 4c^2x^2 + O(x^3)$$

and

$$2(f'(x))^2\sqrt{f(x)} = 2(4c^2x^2 + O(x^3))|x|\sqrt{c + O(x)}.$$

g is bounded because

$$\frac{2(f'(x))^2\sqrt{f(x)}}{|x|^3} \underset{x\to 0}{\longrightarrow} 8c^{5/2} \neq 0$$

and $f'(x)^2 - 2f(x)f''(x) = O(x^3)$.

The theorem does not hold for some C^2-functions.

Let $f(x) = (x + |x|^{3/2})^2 = x^2 + 2x^2\sqrt{|x|} + |x|^3$, so f is C^2. For $x > 0$,

$$g(x) = \frac{1}{2}\left(\frac{1}{1 + \frac{3}{2}\sqrt{x}}\right)' = -\frac{1}{2} \cdot \frac{1}{(1 + \frac{3}{2}\sqrt{x})^2} \cdot \frac{3}{4} \cdot \frac{1}{\sqrt{x}} \underset{x \to 0}{\longrightarrow} -\infty.$$

Problem 2.

Let M be an invertible matrix of dimension $2n \times 2n$, represented in block form as

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad \text{and} \quad M^{-1} = \begin{bmatrix} E & F \\ G & H \end{bmatrix}.$$

Show that $\det M . \det H = \det A$.

Solution.

Let I denote the identity $n \times n$ matrix. Then

$$\det M . \det H = \det \begin{bmatrix} A & B \\ C & D \end{bmatrix} \cdot \det \begin{bmatrix} I & F \\ 0 & H \end{bmatrix} = \det \begin{bmatrix} A & 0 \\ C & I \end{bmatrix} = \det A.$$

Problem 3.

Show that $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n-1}\sin(\log n)}{n^\alpha}$ converges if and only if $\alpha > 0$.

Solution.

Set $f(t) = \dfrac{\sin(\log t)}{t^\alpha}$. We have

$$f'(t) = \frac{-\alpha}{t^{\alpha+1}}\sin(\log t) + \frac{\cos(\log t)}{t^{\alpha+1}}.$$

So $|f'(t)| \le \dfrac{1+\alpha}{t^{\alpha+1}}$ for $\alpha > 0$. Then from Mean value theorem for some $\theta \in (0,1)$ we get $|f(n+1) - f(n)| = |f'(n+\theta)| \le \dfrac{1+\alpha}{n^{\alpha+1}}$. Since $\sum \dfrac{1+\alpha}{n^{\alpha+1}} < +\infty$ for $\alpha > 0$ and $f(n) \underset{n\to\infty}{\longrightarrow} 0$ we get that $\sum\limits_{n=1}^{\infty} (-1)^{n-1} f(n) = \sum\limits_{n=1}^{\infty} (f(2n-1) - f(2n))$ converges.

Now we have to prove that $\dfrac{\sin(\log n)}{n^\alpha}$ does not converge to 0 for $\alpha \le 0$. It suffices to consider $\alpha = 0$.

We show that $a_n = \sin(\log n)$ does not tend to zero. Assume the contrary. There exist $k_n \in \mathbb{N}$ and $\lambda_n \in \left[-\dfrac{1}{2}, \dfrac{1}{2}\right]$ for $n > e^2$ such that $\dfrac{\log n}{\pi} = k_n + \lambda_n$. Then $|a_n| = \sin \pi |\lambda_n|$. Since $a_n \to 0$ we get $\lambda_n \to 0$.

We have $k_{n+1} - k_n =$

$$= \frac{\log(n+1) - \log n}{\pi} - (\lambda_{n+1} - \lambda_n) = \frac{1}{\pi} \log\left(1 + \frac{1}{n}\right) - (\lambda_{n+1} - \lambda_n).$$

Then $|k_{n+1} - k_n| < 1$ for all n big enough. Hence there exists n_0 so that $k_n = k_{n_0}$ for $n > n_0$. So $\dfrac{\log n}{\pi} = k_{n_0} + \lambda_n$ for $n > n_0$. Since $\lambda_n \to 0$ we get contradiction with $\log n \to \infty$.

Problem 4.

a) Let the mapping $f : M_n \to \mathbb{R}$ from the space $M_n = \mathbb{R}^{n^2}$ of $n \times n$ matrices with real entries to reals be linear, i.e.:

$$f(A+B) = f(A) + f(B), \quad f(cA) = cf(A) \tag{1}$$

for any $A, B \in M_n$, $c \in \mathbb{R}$. Prove that there exists a unique matrix $C \in M_n$ such that $f(A) = \mathrm{tr}(AC)$ for any $A \in M_n$. (If $A = \{a_{ij}\}_{i,j=1}^n$ then $\mathrm{tr}(A) = \sum_{i=1}^n a_{ii}$).

b) Suppose in addition to (1) that

$$f(A.B) = f(B.A) \tag{2}$$

for any $A, B \in M_n$. Prove that there exists $\lambda \in \mathbb{R}$ such that $f(A) = \lambda.\mathrm{tr}(A)$.

Solution.

a) If we denote by E_{ij} the standard basis of M_n consisting of elementary matrix (with entry 1 at the place (i,j) and zero elsewhere), then the entries c_{ij} of C can be defined by $c_{ij} = f(E_{ji})$. b) Denote by L the n^2-1-dimensional linear subspace of M_n consisting of all matrices with zero trace. The elements E_{ij} with $i \neq j$ and the elements $E_{ii} - E_{nn}$, $i = 1, \ldots, n-1$ form a linear basis for L. Since

$$\begin{aligned} E_{ij} &= E_{ij}.E_{jj} - E_{jj}.E_{ij}, \ i \neq j \\ E_{ii} - E_{nn} &= E_{in}.E_{ni} - E_{ni}.E_{in}, i = 1, \ldots, n-1, \end{aligned}$$

then the property (2) shows that f is vanishing identically on L. Now, for any $A \in M_n$ we have $A - \frac{1}{n}\mathrm{tr}(A).E \in L$, where E is the identity matrix, and therefore $f(A) = \frac{1}{n} f(E).\mathrm{tr}(A)$.

Problem 5.

Let X be an arbitrary set, let f be an one-to-one function mapping X onto itself. Prove that there exist mappings $g_1, g_2 : X \to X$ such that $f = g_1 \circ g_2$ and $g_1 \circ g_1 = id = g_2 \circ g_2$, where id denotes the identity mapping on X.

Solution.

Let $f^n = \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times}}$, $f^0 = id$, $f^{-n} = (f^{-1})^n$ for every natural number n. Let $T(x) = \{f^n(x) : n \in \mathbb{Z}\}$ for every $x \in X$. The sets $T(x)$ for different x's either coinside or do not intersect. Each of them is mapped by f onto itself. It is enough to prove the theorem for every such set. Let $A = T(x)$. If A is finite, then we can think that A is the set of all vertices of a regular n polygon and that f is rotation by $\dfrac{2\pi}{n}$. Such rotation can be obtained as a composition of 2 symmetries mapping the n polygon onto itself (if n is even then there are axes of symmetry making $\dfrac{\pi}{n}$ angle; if $n = 2k+1$ then there are axes making $k\dfrac{2\pi}{n}$ angle). If A is infinite then we can think that $A = \mathbb{Z}$ and $f(m) = m+1$ for every $m \in \mathbb{Z}$. In this case we define g_1 as a symmetry relative to $\dfrac{1}{2}$, g_2 as a symmetry relative to 0.

Problem 6.

Let $f : [0,1] \to \mathbb{R}$ be a continuous function. Say that f "crosses the axis" at x if $f(x) = 0$ but in any neighbourhood of x there are y, z with $f(y) < 0$ and $f(z) > 0$.

a) Give an example of a continuous function that "crosses the axis" infiniteley often.

b) Can a continuous function "cross the axis" uncountably often? Justify your answer.

Solution.

a) $f(x) = x \sin \dfrac{1}{x}$.

b) Yes. The Cantor set is given by

$$C = \{x \in [0,1) : x = \sum_{j=1}^{\infty} b_j 3^{-j},\ b_j \in \{0,2\}\}.$$

There is an one-to-one mapping $f : [0,1) \to C$. Indeed, for $x = \sum\limits_{j=1}^{\infty} a_j 2^{-j}$, $a_j \in \{0,1\}$ we set $f(x) = \sum\limits_{j=1}^{\infty} (2a_j) 3^{-j}$. Hence C is uncountable.

For $k = 1, 2, \ldots$ and $i = 0, 1, 2, \ldots, 2^{k-1} - 1$ we set

$$a_{k,i} = 3^{-k}\left(6\sum_{j=0}^{k-2} a_j 3^j + 1\right), \quad b_{k,i} = 3^{-k}\left(6\sum_{j=0}^{k-2} a_j 3^j + 2\right),$$

where $i = \sum\limits_{j=0}^{k-2} a_j 2^j$, $a_j \in \{0, 1\}$. Then

$$[0,1) \setminus C = \bigcup_{k=1}^{\infty} \bigcup_{i=0}^{2^{k-1}-1} (a_{k,i}, b_{k,i}),$$

i.e. the Cantor set consists of all points which have a trinary representation with 0 and 2 as digits and the points of its compliment have some 1's in their trinary representation. Thus, $\bigcup\limits_{i=0}^{2^{k-1}-1} (a_{k,i}, b_{k,i})$ are all points (exept $a_{k,i}$) which have 1 on k-th place and 0 or 2 on the j-th ($j < k$) places.

Noticing that the points with at least one digit equals to 1 are everywhere dence in [0,1] we set

$$f(x) = \sum_{k=1}^{\infty} (-1)^k g_k(x).$$

where g_k is a piece-wise linear continuous functions with values at the knots $g_k\left(\dfrac{a_{k,i} + b_{k,i}}{2}\right) = 2^{-k}$, $g_k(0) = g_k(1) = g_k(a_{k,i}) = g_k(b_{k,i}) = 0$, $i = 0, 1, \ldots, 2^{k-1} - 1$.

Then f is continuous and f "crosses the axis" at every point of the Cantor set.

5^{th} INTERNATIONAL MATHEMATICS COMPETITION FOR UNIVERSITY STUDENTS

July 29 - August 3, 1998, Blagoevgrad, Bulgaria

First day

PROBLEMS AND SOLUTIONS

Problem 1. *(20 points)* Let V be a 10-dimensional real vector space and U_1 and U_2 two linear subspaces such that $U_1 \subseteq U_2$, $\dim_{\mathbf{R}} U_1 = 3$ and $\dim_{\mathbf{R}} U_2 = 6$. Let $\mathcal{E}$ be the set of all linear maps $T : V \longrightarrow V$ which have U_1 and U_2 as invariant subspaces (i.e., $T(U_1) \subseteq U_1$ and $T(U_2) \subseteq U_2$). Calculate the dimension of $\mathcal{E}$ as a real vector space.

Solution First choose a basis $\{v_1, v_2, v_3\}$ of U_1. It is possible to extend this basis with vectors v_4,v_5 and v_6 to get a basis of U_2. In the same way we can extend a basis of U_2 with vectors $v_7, \ldots, v_{10}$ to get as basis of V.

Let $T \in \mathcal{E}$ be an endomorphism which has U_1 and U_2 as invariant subspaces. Then its matrix, relative to the basis $\{v_1, \ldots, v_{10}\}$ is of the form

$$\begin{bmatrix} * & * & * & * & * & * & * & * & * & * \\ & * & * & * & * & * & * & * & * & * \\ & * & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * \end{bmatrix}.$$

So $\dim_{\mathbf{R}} \mathcal{E} = 9 + 18 + 40 = 67$.

Problem 2. Prove that the following proposition holds for $n = 3$ *(5 points)* and $n = 5$ *(7 points)*, and does not hold for $n = 4$ *(8 points)*.

"For any permutation π_1 of $\{1, 2, \ldots, n\}$ different from the identity there is a permutation π_2 such that any permutation π can be obtained from π_1 and π_2 using only compositions (for example, $\pi = \pi_1 \circ \pi_1 \circ \pi_2 \circ \pi_1$)."

Solution

Let S_n be the group of permutations of $\{1, 2, \ldots, n\}$.

1) When $n = 3$ the proposition is obvious: if $x = (12)$ we choose $y = (123)$; if $x = (123)$ we choose $y = (12)$.

2) $n = 4$. Let $x = (12)(34)$. Assume that there exists $y \in S_n$, such that $S_4 = \langle x, y \rangle$. Denote by K the invariant subgroup

$$K = \{id,\ (12)(34), (13)(24), (14)(23)\}.$$

By the fact that x and y generate the whole group S_4, it follows that the factor group S_4/K contains only powers of $\bar{y} = yK$, i.e., S_4/K is cyclic. It is easy to see that this factor-group is not comutative (something more this group is not isomorphic to S_3).

3) $n = 5$

a) If $x = (12)$, then for y we can take $y = (12345)$.

b) If $x = (123)$, we set $y = (124)(35)$. Then $y^3 x y^3 = (125)$ and $y^4 = (124)$. Therefore $(123), (124), (125) \in \langle x, y \rangle$- the subgroup generated by x and y. From the fact that $(123), (124), (125)$ generate the alternating subgroup A_5, it follows that $A_5 \subset \langle x, y \rangle$. Moreover y is an odd permutation, hence $\langle x, y \rangle = S_5$.

c) If $x = (123)(45)$, then as in b) we see that for y we can take the element (124).

d) If $x = (1234)$, we set $y = (12345)$. Then $(yx)^3 = (24) \in \langle x, y \rangle$, $x^2(24) = (13) \in \langle x, y \rangle$ and $y^2 = (13524) \in \langle x, y \rangle$. By the fact $(13) \in \langle x, y \rangle$ and $(13524) \in \langle x, y \rangle$, it follows that $\langle x, y \rangle = S_5$.

e) If $x = (12)(34)$, then for y we can take $y = (1354)$. Then $y^2x = (125)$, $y^3x = (124)(53)$ and by c) $S_5 = \langle x, y\rangle$.

f) If $x = (12345)$, then it is clear that for y we can take the element $y = (12)$.

Problem 3. Let $f(x) = 2x(1-x)$, $x \in \mathbb{R}$. Define

$$f_n = \overbrace{f \circ \ldots \circ f}^{n}.$$

a) *(10 points)* Find $\lim_{n\to\infty} \int_0^1 f_n(x)dx$.

b) *(10 points)* Compute $\int_0^1 f_n(x)dx$ for $n = 1, 2, \ldots$.

Solution. a) Fix $x = x_0 \in (0,1)$. If we denote $x_n = f_n(x_0)$, $n = 1, 2, \ldots$ it is easy to see that $x_1 \in (0, 1/2]$, $x_1 \le f(x_1) \le 1/2$ and $x_n \le f(x_n) \le 1/2$ (by induction). Then $(x_n)_n$ is a bounded non-decreasing sequence and, since $x_{n+1} = 2x_n(1-x_n)$, the limit $l = \lim_{n\to\infty} x_n$ satisfies $l = 2l(1-l)$, which implies $l = 1/2$. Now the monotone convergence theorem implies that

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = 1/2.$$

b) We prove by induction that

$$f_n(x) = \frac{1}{2} - 2^{2^n-1}\left(x - \frac{1}{2}\right)^{2^n} \tag{1}$$

holds for $n = 1, 2, \ldots$. For $n = 1$ this is true, since $f(x) = 2x(1-x) = \frac{1}{2} - 2(x - \frac{1}{2})^2$. If (1) holds for some $n = k$, then we have

$$\begin{aligned} f_{k+1}(x) &= f_k\left(f(x)\right) = \tfrac{1}{2} - 2^{2^k-1}\left(\left(\tfrac{1}{2} - 2(x-\tfrac{1}{2})^2\right) - \tfrac{1}{2}\right)^{2^k} \\ &= \tfrac{1}{2} - 2^{2^k-1}\left(-2(x-\tfrac{1}{2})^2\right)^{2^k} \\ &= \tfrac{1}{2} - 2^{2^{k+1}-1}(x-\tfrac{1}{2})^{2^{k+1}} \end{aligned}$$

which is (2) for $n = k+1$.

Using (1) we can compute the integral,

$$\int_0^1 f_n(x)dx = \left[\frac{1}{2}x - \frac{2^{2^n-1}}{2^n+1}\left(x - \frac{1}{2}\right)^{2^n+1}\right]_{x=0}^{1} = \frac{1}{2} - \frac{1}{2(2^n+1)}.$$

Problem 4. *(20 points)* The function $f : \mathbb{R} \to \mathbb{R}$ is twice differentiable and satisfies $f(0) = 2$, $f'(0) = -2$ and $f(1) = 1$. Prove that there exists a real number $\xi \in (0,1)$ for which

$$f(\xi) \cdot f'(\xi) + f''(\xi) = 0.$$

Solution. Define the function

$$g(x) = \frac{1}{2}f^2(x) + f'(x).$$

Because $g(0) = 0$ and

$$f(x) \cdot f'(x) + f''(x) = g'(x),$$

it is enough to prove that there exists a real number $0 < \eta \le 1$ for which $g(\eta) = 0$.

a) If f is never zero, let

$$h(x) = \frac{x}{2} - \frac{1}{f(x)}.$$

Because $h(0) = h(1) = -\frac{1}{2}$, there exists a real number $0 < \eta < 1$ for which $h'(\eta) = 0$. But $g = f^2 \cdot h'$, and we are done.

b) If f has at least one zero, let z_1 be the first one and z_2 be the last one. (The set of the zeros is closed.) By the conditions, $0 < z_1 \le z_2 < 1$.

The function f is positive on the intervals $[0, z_1)$ and $(z_2, 1]$; this implies that $f'(z_1) \le 0$ and $f'(z_2) \ge 0$.

Then $g(z_1) = f'(z_1) \le 0$ and $g(z_2) = f'(z_2) \ge 0$, and there exists a real number $\eta \in [z_1, z_2]$ for which $g(\eta) = 0$.

Remark. For the function $f(x) = \frac{2}{x+1}$ the conditions hold and $f \cdot f' + f''$ is constantly 0.

Problem 5. Let P be an algebraic polynomial of degree n having only real zeros and real coefficients.

a) *(15 points)* Prove that for every real x the following inequality holds:

$$(n-1)(P'(x))^2 \ge nP(x)P''(x). \tag{2}$$

b) *(5 points)* Examine the cases of equality.

Solution. Observe that both sides of (2) are identically equal to zero if $n = 1$. Suppose that $n > 1$. Let $x_1, \ldots, x_n$ be the zeros of P. Clearly (2) is true when $x = x_i$, $i \in \{1, \ldots, n\}$, and equality is possible only if $P'(x_i) = 0$, i.e., if x_i is a multiple zero of P. Now suppose that x is not a zero of P. Using the identities

$$\frac{P'(x)}{P(x)} = \sum_{i=1}^{n} \frac{1}{x - x_i}, \quad \frac{P''(x)}{P(x)} = \sum_{1 \le i < j \le n} \frac{2}{(x - x_i)(x - x_j)},$$

we find

$$(n-1)\left(\frac{P'(x)}{P(x)}\right)^2 - n\frac{P''(x)}{P(x)} = \sum_{i=1}^{n} \frac{n-1}{(x - x_i)^2} - \sum_{1 \le i < j \le n} \frac{2}{(x - x_i)(x - x_j)}.$$

But this last expression is simply

$$\sum_{1 \le i < j \le n} \left(\frac{1}{x - x_i} - \frac{1}{x - x_j}\right)^2,$$

and therefore is positive. The inequality is proved. In order that (2) holds with equality sign for every real x it is necessary that $x_1 = x_2 = \ldots = x_n$. A direct verification shows that indeed, if $P(x) = c(x - x_1)^n$, then (2) becomes an identity.

Problem 6. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function with the property that for any x and y in the interval,

$$xf(y) + yf(x) \le 1.$$

a) *(15 points)* Show that

$$\int_0^1 f(x)dx \le \frac{\pi}{4}.$$

b) *(5 points)* Find a function, satisfying the condition, for which there is equality.

Solution Observe that the integral is equal to

$$\int_0^{\frac{\pi}{2}} f(\sin\theta)\cos\theta d\theta$$

and to

$$\int_0^{\frac{\pi}{2}} f(\cos\theta)\sin\theta d\theta$$

So, twice the integral is at most

$$\int_0^{\frac{\pi}{2}} 1 d\theta = \frac{\pi}{2}.$$

Now let $f(x) = \sqrt{1 - x^2}$. If $x = \sin\theta$ and $y = \sin\phi$ then

$$xf(y) + yf(x) = \sin\theta\cos\phi + \sin\phi\cos\theta = \sin(\theta + \phi) \le 1.$$

5^{th} INTERNATIONAL MATHEMATICS COMPETITION FOR UNIVERSITY STUDENTS

July 29 - August 3, 1998, Blagoevgrad, Bulgaria

Second day

PROBLEMS AND SOLUTION

Problem 1. *(20 points)* Let V be a real vector space, and let $f, f_1, f_2, \ldots, f_k$ be linear maps from V to $\mathbb{R}$. Suppose that $f(x) = 0$ whenever $f_1(x) = f_2(x) = \ldots = f_k(x) = 0$. Prove that f is a linear combination of f_1, f_2, ..., f_k.

Solution. We use induction on k. By passing to a subset, we may assume that $f_1, \ldots, f_k$ are linearly independent.

Since f_k is independent of $f_1, \ldots, f_{k-1}$, by induction there exists a vector $a_k \in V$ such that $f_1(a_k) = \ldots = f_{k-1}(a_k) = 0$ and $f_k(a_k) \neq 0$. After normalising, we may assume that $f_k(a_k) = 1$. The vectors $a_1, \ldots, a_{k-1}$ are defined similarly to get

$$f_i(a_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

For an arbitrary $x \in V$ and $1 \le i \le k$, $f_i(x - f_1(x)a_1 - f_2(x)a_2 - \cdots - f_k(x)a_k) = f_i(x) - \sum_{j=1}^{k} f_j(x) f_i(a_j) = f_i(x) - f_i(x) f_i(a_i) = 0$, thus $f(x - f_1(x)a_1 - \cdots - f_k(x)a_k) = 0$. By the linearity of f this implies $f(x) = f_1(x)f(a_1) + \cdots + f_k(x)f(a_k)$, which gives $f(x)$ as a linear combination of $f_1(x), \ldots, f_k(x)$.

Problem 2. *(20 points)* Let

$$\mathcal{P} = \{f : f(x) = \sum_{k=0}^{3} a_k x^k,\ a_k \in \mathbb{R},\ |f(\pm 1)| \le 1,\ |f(\pm \frac{1}{2})| \le 1\}.$$

Evaluate

$$\sup_{f \in \mathcal{P}} \max_{-1 \le x \le 1} |f''(x)|$$

and find all polynomials $f \in \mathcal{P}$ for which the above "sup" is attained.

Solution. Denote $x_0 = 1, x_1 = -\frac{1}{2}, x_2 = \frac{1}{2}, x_3 = 1$,

$$w(x) = \prod_{i=0}^{3} (x - x_i),$$

$$w_k(x) = \frac{w(x)}{x - x_k},\ k = 0, \ldots, 3,$$

$$l_k(x) = \frac{w_k(x)}{w_k(x_k)}.$$

Then for every $f \in \mathcal{P}$

$$f''(x) = \sum_{k=0}^{3} l_k''(x) f(x_k),$$

$$|f''(x)| \le \sum_{k=0}^{3} |l_k''(x)|.$$

Since f'' is a linear function $\max_{-1\le x\le 1}|f''(x)|$ is attained either at $x=-1$ or at $x=1$. Without loss of generality let the maximum point is $x=1$. Then

$$\sup_{f\in\mathcal{P}}\max_{-1\le x\le 1}|f''(x)|=\sum_{k=0}^{3}|l_k''(1)|.$$

In order to have equality for the extremal polynomial f_* there must hold

$$f_*(x_k)=\operatorname{sign} l_k''(1),\ k=0,1,2,3.$$

It is easy to see that $\{l_k''(1)\}_{k=0}^3$ alternate in sign, so $f_*(x_k)=(-1)^{k-1},\ k=0,\dots,3$. Hence $f_*(x)=T_3(x)=4x^3-3x$, the Chebyshev polynomial of the first kind, and $f_*''(1)=24$. The other extremal polynomial, corresponding to $x=-1$, is $-T_3$.

Problem 3. *(20 points)* Let $0<c<1$ and

$$f(x)=\begin{cases}\frac{x}{c} & \text{for } x\in[0,c],\\[2ex] \frac{1-x}{1-c} & \text{for } x\in[c,1].\end{cases}$$

We say that p is an n-periodic point if

$$\underbrace{f(f(\dots f}_{n}(p)))=p$$

and n is the smallest number with this property. Prove that for every $n\ge 1$ the set of n-periodic points is non-empty and finite.

Solution. Let $f_n(x)=\underbrace{f(f(\dots f}_{n}(x)))$. It is easy to see that $f_n(x)$ is a picewise monotone function and its graph contains 2^n linear segments; one endpoint is always on $\{(x,y):0\le x\le 1,\ y=0\}$, the other is on $\{(x,y):0\le x\le 1, y=1\}$. Thus the graph of the identity function intersects each segment once, so the number of points for which $f_n(x)=x$ is 2^n.

Since for each n-periodic points we have $f_n(x)=x$, the number of n-periodic points is finite.

A point x is n-periodic if $f_n(x)=x$ but $f_k(x)\ne x$ for $k=1,\dots,n-1$. But as we saw before $f_k(x)=x$ holds only at 2^k points, so there are at most $2^1+2^2+\dots+2^{n-1}=2^n-2$ points x for which $f_k(x)=x$ for at least one $k\in\{1,2,\dots,n-1\}$. Therefore at least two of the 2^n points for which $f_n(x)=x$ are n-periodic points.

Problem 4. *(20 points)* Let $A_n=\{1,2,\dots,n\}$, where $n\ge 3$. Let $\mathcal{F}$ be the family of all non-constant functions $f\colon A_n\to A_n$ satisfying the following conditions:

(1) $f(k)\le f(k+1)$ for $k=1,2,\dots,n-1$,

(2) $f(k)=f(f(k+1))$ for $k=1,2,\dots,n-1$.

Find the number of functions in $\mathcal{F}$.

Solution. It is clear that $id:A_n\longrightarrow A_n$, given by $id(x)=x$, does not verify condition (2). Since id is the only increasing injection on A_n, $\mathcal{F}$ does not contain injections. Let us take any $f\in\mathcal{F}$ and suppose that $\#\left(f^{-1}(k)\right)\ge 2$. Since f is increasing, there exists $i\in A_n$ such that $f(i)=f(i+1)=k$. In view of (2), $f(k)=f\left(f(i+1)\right)=f(i)=k$. If $\{i<k:f(i)<k\}=\emptyset$, then taking $j=\max\{i<k:f(i)<k\}$ we get $f(j)<f(j+1)=k=f\left(f(j+1)\right)$, a contradiction. Hence $f(i)=k$ for $i\le k$. If $\#\left(f^{-1}(\{l\})\right)\ge 2$ for some $l\ge k$, then the similar consideration shows that $f(i)=l=k$ for $i\le k$. Hence $\#\left(f^{-1}\{i\}\right)=0$ or 1 for every $i>k$. Therefore $f(i)\le i$ for $i>k$. If $f(l)=l$, then taking $j=\max\{i<l:f(i)<l\}$ we get $f(j)<f(j+1)=l=f\left(f(j+1)\right)$, a contradiction. Thus, $f(i)\le i-1$ for $i>k$. Let $m=\max\{i:f(i)=k\}$. Since f is non-constant $m\le n-1$. Since $k=f(m)=f\left(f(m+1)\right)$, $f(m+1)\in[k+1,m]$. If $f(l)>l-1$ for some $l>m+1$, then $l-1$ and $f(l)$ belong to $f^{-1}\left(f(l)\right)$ and this contradicts the facts above. Hence $f(i) = i-1$ for $i > m+1$. Thus we show that every function f in $\mathcal{F}$ is defined by natural numbers k, l, m, where $1 \le k < l = f(m+1) \le m \le n-1$.

$$f(i) = \begin{cases} k & \text{if } i \le m \\ l & \text{if } i = m \\ i-1 & \text{if } i > m+1. \end{cases}$$

Then

$$\#(\mathcal{F}) = \begin{pmatrix} n \\ 3 \end{pmatrix}.$$

Problem 5. *(20 points)* Suppose that $\mathcal{S}$ is a family of spheres (i.e., surfaces of balls of positive radius) in $\mathbb{R}^n$, $n \ge 2$, such that the intersection of any two contains at most one point. Prove that the set M of those points that belong to at least two different spheres from $\mathcal{S}$ is countable.

Solution. For every $x \in M$ choose spheres $S, T \in \mathcal{S}$ such that $S \ne T$ and $x \in S \cap T$; denote by U, V, W the three components of $\mathbf{R}^n \setminus (S \cup T)$, where the notation is such that $\partial U = S$, $\partial V = T$ and x is the only point of $\overline{U} \cap \overline{V}$, and choose points with rational coordinates $u \in U$, $v \in V$, and $w \in W$. We claim that x is uniquely determined by the triple $\langle u, v, w\rangle$; since the set of such triples is countable, this will finish the proof.

To prove the claim, suppose, that from some $x' \in M$ we arrived to the same $\langle u, v, w\rangle$ using spheres $S', T' \in \mathcal{S}$ and components U', V', W' of $\mathbf{R}^n \setminus (S' \cup T')$. Since $S \cap S'$ contains at most one point and since $U \cap U' \ne \emptyset$, we have that $U \subset U'$ or $U' \subset U$; similarly for V's and W's. Exchanging the role of x and x' and/or of U's and V's if necessary, there are only two cases to consider: (a) $U \supset U'$ and $V \supset V'$ and (b) $U \subset U'$, $V \supset V'$ and $W \subset W'$. In case (a) we recall that $\overline{U} \cap \overline{V}$ contains only x and that $x' \in \overline{U'} \cap \overline{V'}$, so $x = x'$. In case (b) we get from $W \subset W'$ that $U' \subset \overline{U \cup V}$; so since U' is open and connected, and $\overline{U} \cap \overline{V}$ is just one point, we infer that $U' = U$ and we are back in the already proved case (a).

Problem 6. *(20 points)* Let $f\colon (0,1) \to [0, \infty)$ be a function that is zero except at the distinct points $a_1, a_2, \ldots$. Let $b_n = f(a_n)$.

(a) Prove that if $\sum\limits_{n=1}^{\infty} b_n < \infty$, then f is differentiable at at least one point $x \in (0,1)$.

(b) Prove that for any sequence of non-negative real numbers $(b_n)_{n=1}^{\infty}$, with $\sum\limits_{n=1}^{\infty} b_n = \infty$, there exists a sequence $(a_n)_{n=1}^{\infty}$ such that the function f defined as above is nowhere differentiable.

Solution

a) We first construct a sequence c_n of positive numbers such that $c_n \to \infty$ and $\sum\limits_{n=1}^{\infty} c_n b_n < \frac{1}{2}$. Let $B = \sum\limits_{n=1}^{\infty} b_n$, and for each $k = 0, 1, \ldots$ denote by N_k the first positive integer for which

$$\sum_{n=N_k}^{\infty} b_n \le \frac{B}{4^k}.$$

Now set $c_n = \frac{2^k}{5B}$ for each n, $N_k \le n < N_{k+1}$. Then we have $c_n \to \infty$ and

$$\sum_{n=1}^{\infty} c_n b_n = \sum_{k=0}^{\infty} \sum_{N_k \le n < N_{k+1}} c_n b_n \le \sum_{k=0}^{\infty} \frac{2^k}{5B} \sum_{n=N_k}^{\infty} b_n \le \sum_{k=0}^{\infty} \frac{2^k}{5B} \cdot \frac{B}{4^k} = \frac{2}{5}.$$

Consider the intervals $I_n = (a_n - c_n b_n, a_n + c_n b_n)$. The sum of their lengths is $2 \sum c_n b_n < 1$, thus there exists a point $x_0 \in (0,1)$ which is not contained in any I_n. We show that f is differentiable at x_0, and $f'(x_0) = 0$. Since x_0 is outside of the intervals I_n, $x_0 \neq a_n$ for any n and $f(x_0) = 0$. For arbitrary $x \in (0,1) \setminus \{x_0\}$, if $x = a_n$ for some n, then

$$\left| \frac{f(x) - f(x_0)}{x - x_0} \right| = \frac{f(a_n) - 0}{|a_n - x_0|} \leq \frac{b_n}{c_n b_n} = \frac{1}{c_n},$$

otherwise $\frac{f(x)-f(x_0)}{x-x_0} = 0$. Since $c_n \to \infty$, this implies that for arbitrary $\varepsilon > 0$ there are only finitely many $x \in (0,1) \setminus \{x_0\}$ for which

$$\left| \frac{f(x) - f(x_0)}{x - x_0} \right| < \varepsilon$$

does not hold, and we are done.

Remark. The variation of f is finite, which implies that f is differentiable almost everywhere .

b) We remove the zero elements from sequence b_n. Since $f(x) = 0$ except for a countable subset of $(0,1)$, if f is differentiable at some point x_0, then $f(x_0)$ and $f'(x_0)$ must be 0.

It is easy to construct a sequence β_n satisfying $0 < \beta_n \leq b_n$, $b_n \to 0$ and $\sum_{n=1}^{\infty} \beta_n = \infty$.

Choose the numbers $a_1, a_2, \dots$ such that the intervals $I_n = (a_n - \beta_n,\, a_n + \beta_n)$ $(n = 1, 2, \dots)$ cover each point of (0, 1) infinitely many times (it is possible since the sum of lengths is $2\sum b_n = \infty$). Then for arbitrary $x_0 \in (0,\, 1)$, $f(x_0) = 0$ and $\varepsilon > 0$ there is an n for which $\beta_n < \varepsilon$ and $x_0 \in I_n$ which implies

$$\frac{|f(a_n) - f(x_0)|}{|a_n - x_0|} > \frac{b_n}{\beta_n} \geq 1.$$

6th INTERNATIONAL COMPETITION FOR UNIVERSITY STUDENTS IN MATHEMATICS

Keszthely, 1999.

Problems and solutions on the first day

1. a) Show that for any $m \in \mathbf{N}$ there exists a real $m \times m$ matrix A such that $A^3 = A + I$, where I is the $m \times m$ identity matrix. (6 points)
b) Show that $\det A > 0$ for every real $m \times m$ matrix satisfying $A^3 = A + I$. (14 points)

Solution. a) The diagonal matrix

$$A = \lambda I = \begin{pmatrix} \lambda & & 0 \\ & \ddots & \\ 0 & & \lambda \end{pmatrix}$$

is a solution for equation $A^3 = A + I$ if and only if $\lambda^3 = \lambda + 1$, because $A^3 - A - I = (\lambda^3 - \lambda - 1)I$. This equation, being cubic, has real solution.

b) It is easy to check that the polynomial $p(x) = x^3 - x - 1$ has a positive real root λ_1 (because $p(0) < 0$) and two conjugated complex roots λ_2 and λ_3 (one can check the discriminant of the polynomial, which is $\left(\frac{-1}{3}\right)^3 + \left(\frac{-1}{2}\right)^2 = \frac{23}{108} > 0$, or the local minimum and maximum of the polynomial).

If a matrix A satisfies equation $A^3 = A + I$, then its eigenvalues can be only λ_1, λ_2 and λ_3. The multiplicity of λ_2 and λ_3 must be the same, because A is a real matrix and its characteristic polynomial has only real coefficients. Denoting the multiplicity of λ_1 by α and the common multiplicity of λ_2 and λ_3 by β,

$$\det A = \lambda_1^\alpha \lambda_2^\beta \lambda_3^\beta = \lambda_1^\alpha \cdot (\lambda_2\lambda_3)^\beta.$$

Because λ_1 and $\lambda_2\lambda_3 = |\lambda_2|^2$ are positive, the product on the right side has only positive factors.

2. Does there exist a bijective map $\pi : \mathbf{N} \to \mathbf{N}$ such that

$$\sum_{n=1}^{\infty} \frac{\pi(n)}{n^2} < \infty?$$

(20 points)

Solution 1. No. For, let π be a permutation of $\mathbf{N}$ and let $N \in \mathbf{N}$. We shall argue that

$$\sum_{n=N+1}^{3N} \frac{\pi(n)}{n^2} > \frac{1}{9}.$$

In fact, of the $2N$ numbers $\pi(N+1), \ldots, \pi(3N)$ only N can be $\le N$ so that at least N of them are $> N$. Hence

$$\sum_{n=N+1}^{3N} \frac{\pi(n)}{n^2} \ge \frac{1}{(3N)^2} \sum_{n=N+1}^{3N} \pi(n) > \frac{1}{9N^2} \cdot N \cdot N = \frac{1}{9}.$$

Solution 2. Let π be a permutation of $\mathbf{N}$. For any $n \in \mathbf{N}$, the numbers $\pi(1), \ldots, \pi(n)$ are distinct positive integers, thus $\pi(1) + \ldots + \pi(n) \ge 1 + \ldots + n = \frac{n(n+1)}{2}$. By this inequality,

$$\sum_{n=1}^{\infty} \frac{\pi(n)}{n^2} = \sum_{n=1}^{\infty} \left(\pi(1) + \ldots + \pi(n)\right) \left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right) \ge$$

$$\ge \sum_{n=1}^{\infty} \frac{n(n+1)}{2} \cdot \frac{2n+1}{n^2(n+1)^2} = \sum_{n=1}^{\infty} \frac{2n+1}{2n(n+1)} \ge \sum_{n=1}^{\infty} \frac{1}{n+1} = \infty.$$

3. Suppose that a function $f:\mathbf{R}\to\mathbf{R}$ satisfies the inequality

$$\left|\sum_{k=1}^{n} 3^k\big(f(x+ky)-f(x-ky)\big)\right| \le 1 \tag{1}$$

for every positive integer n and for all $x, y\in\mathbf{R}$. Prove that f is a constant function. (20 points)

Solution. Writing (1) with $n-1$ instead of n,

$$\left|\sum_{k=1}^{n-1} 3^k\big(f(x+ky)-f(x-ky)\big)\right| \le 1. \tag{2}$$

From the difference of (1) and (2),

$$\big|3^n\big(f(x+ny)-f(x-ny)\big)\big| \le 2;$$

which means

$$\big|f(x+ny)-f(x-ny)\big| \le \frac{2}{3^n}. \tag{3}$$

For arbitrary $u, v\in\mathbf{R}$ and $n\in\mathbf{N}$ one can choose x and y such that $x-ny=u$ and $x+ny=v$, namely $x=\frac{u+v}{2}$ and $y=\frac{v-u}{2n}$. Thus, (3) yields

$$\big|f(u)-f(v)\big| \le \frac{2}{3^n}$$

for arbitrary positive integer n. Because $\frac{2}{3^n}$ can be arbitrary small, this implies $f(u)=f(v)$.

4. Find all strictly monotonic functions $f:(0,+\infty)\to(0,+\infty)$ such that $f\big(\frac{x^2}{f(x)}\big)\equiv x$. (20 points)

Solution. Let $g(x)=\dfrac{f(x)}{x}$. We have $g(\dfrac{x}{g(x)})=g(x)$. By induction it follows that $g(\dfrac{x}{g^n(x)})=g(x)$, i.e.

(1) $$f(\frac{x}{g^n(x)})=\frac{x}{g^{n-1}(x)},\ \ n\in\mathbf{N}.$$

On the other hand, let substitute x by $f(x)$ in $f(\dfrac{x^2}{f(x)})=x$. ¿From the injectivity of f we get $\dfrac{f^2(x)}{f(f(x))}=x$, i.e. $g(xg(x))=g(x)$. Again by induction we deduce that $g(xg^n(x))=g(x)$ which can be written in the form

(2) $$f(xg^n(x))=xg^{n-1}(x),\ \ n\in\mathbf{N}.$$

Set $f^{(m)}=\underbrace{f\circ f\circ\ldots\circ f}_{m\ times}$. It follows from (1) and (2) that

(3) $$f^{(m)}(xg^n(x))=xg^{n-m}(x),\ \ m,n\in\mathbf{N}.$$

Now, we shall prove that g is a constant. Assume $g(x_1)<g(x_2)$. Then we may find $n\in\mathbf{N}$ such that $x_1g^n(x_1)\le x_2g^n(x_2)$. On the other hand, if m is even then $f^{(m)}$ is strictly increasing and from (3) it follows that $x_1^m g^{n-m}(x_1)\le x_2^m g^{n-m}(x_2)$. But when n is fixed the opposite inequality holds $\forall m\gg 1$. This contradiction shows that g is a constant, i.e. $f(x)=Cx,\ C>0$.

Conversely, it is easy to check that the functions of this type verify the conditions of the problem.

5. Suppose that $2n$ points of an $n\times n$ grid are marked. Show that for some $k>1$ one can select $2k$ distinct marked points, say $a_1,\ldots,a_{2k}$, such that a_1 and a_2 are in the same row, a_2 and a_3 are in the same column, $\ldots$, a_{2k-1} and a_{2k} are in the same row, and a_{2k} and a_1 are in the same column. (20 points)

Solution 1. We prove the more general statement that if at least $n+k$ points are marked in an $n\times k$ grid, then the required sequence of marked points can be selected.

If a row or a column contains at most one marked point, delete it. This decreases $n+k$ by 1 and the number of the marked points by at most 1, so the condition remains true. Repeat this step until each row and column contains at least two marked points. Note that the condition implies that there are at least two marked points, so the whole set of marked points cannot be deleted.

We define a sequence $b_1, b_2, \ldots$ of marked points. Let b_1 be an arbitrary marked point. For any positive integer n, let b_{2n} be an other marked point in the row of b_{2n-1} and b_{2n+1} be an other marked point in the column of b_{2n}.

Let m be the first index for which b_m is the same as one of the earlier points, say $b_m = b_l$, $l < m$.

If $m-l$ is even, the line segments $b_l b_{l+1}$, $b_{l+1}b_{l+2}$, ..., $b_{m-1}b_l = b_{m-1}b_m$ are alternating horizontal and vertical. So one can choose $2k = m-l$, and $(a_1,\ldots,a_{2k}) = (b_l,\ldots,b_{m-1})$ or $(a_1,\ldots,a_{2k}) = (b_{l+1},\ldots,b_m)$ if l is odd or even, respectively.

If $m-l$ is odd, then the points $b_l = b_m$, b_{l+1} and b_{m-1} are in the same row/column. In this case chose $2k = m-l-1$. Again, the line segments $b_{l+1}b_{l+2}$, $b_{l+2}b_{l+3}$, ..., $b_{m-1}b_{l+1}$ are alternating horizontal and vertical and one can choose $(a_1,\ldots,a_{2k}) = (b_{l+1},\ldots,b_{m-1})$ or $(a_1,\ldots,a_{2k}) = (b_{l+2},\ldots,b_{m-1},b_{l+1})$ if l is even or odd, respectively.

Solution 2. Define the graph G in the following way: Let the vertices of G be the rows and the columns of the grid. Connect a row r and a column c with an edge if the intersection point of r and c is marked.

The graph G has $2n$ vertices and $2n$ edges. As is well known, if a graph of N vertices contains no circle, it can have at most $N-1$ edges. Thus G does contain a circle. A circle is an alternating sequence of rows and columns, and the intersection of each neighbouring row and column is a marked point. The required sequence consists of these intersection points.

6. a) For each $1 < p < \infty$ find a constant $c_p < \infty$ for which the following statement holds: If $f : [-1,1] \to \mathbf{R}$ is a continuously differentiable function satisfying $f(1) > f(-1)$ and $|f'(y)| \le 1$ for all $y \in [-1,1]$, then there is an $x \in [-1,1]$ such that $f'(x) > 0$ and $|f(y) - f(x)| \le c_p \big(f'(x)\big)^{1/p} |y-x|$ for all $y \in [-1,1]$. (10 points)
b) Does such a constant also exist for $p = 1$? (10 points)

Solution. (a) Let $g(x) = \max(0, f'(x))$. Then $0 < \int_{-1}^1 f'(x)dx = \int_{-1}^1 g(x)dx + \int_{-1}^1 (f'(x) - g(x))dx$, so we get $\int_{-1}^1 |f'(x)|dx = \int_{-1}^1 g(x)dx + \int_{-1}^1 (g(x) - f'(x))dx < 2\int_{-1}^1 g(x)dx$. Fix p and c (to be determined at the end). Given any $t > 0$, choose for every x such that $g(x) > t$ an interval $I_x = [x,y]$ such that $|f(y) - f(x)| > cg(x)^{1/p}|y-x| > ct^{1/p}|I_x|$ and choose disjoint I_{x_i} that cover at least one third of the measure of the set $\{g > t\}$. For $I = \bigcup_i I_i$ we thus have $ct^{1/p}|I| \le \int_I f'(x)dx \le \int_{-1}^1 |f'(x)|dx < 2\int_{-1}^1 g(x)dx$; so $|\{g > t\}| \le 3|I| < (6/c)t^{-1/p}\int_{-1}^1 g(x)dx$. Integrating the inequality, we get $\int_{-1}^1 g(x)dx = \int_0^1 |\{g > t\}|dt < (6/c)p/(p-1)\int_{-1}^1 g(x)dx$; this is a contradiction e.g. for $c_p = (6p)/(p-1)$.

(b) No. Given $c > 1$, denote $\alpha = 1/c$ and choose $0 < \varepsilon < 1$ such that $((1+\varepsilon)/(2\varepsilon))^{-\alpha} < 1/4$. Let $g : [-1,1] \to [-1,1]$ be continuous, even, $g(x) = -1$ for $|x| \le \varepsilon$ and $0 \le g(x) < \alpha((|x|+\varepsilon)/(2\varepsilon))^{-\alpha-1}$ for $\varepsilon < |x| \le 1$ is chosen such that $\int_\varepsilon^1 g(t)dt > -\varepsilon/2 + \int_\varepsilon^1 \alpha((|x|+\varepsilon)/(2\varepsilon))^{-\alpha-1}dt = -\varepsilon/2 + 2\varepsilon(1 - ((1+\varepsilon)/(2\varepsilon))^{-\alpha}) > \varepsilon$. Let $f = \int g(t)dt$. Then $f(1) - f(-1) \ge -2\varepsilon + 2\int_\varepsilon^1 g(t)dt > 0$. If $\varepsilon < x < 1$ and $y = -\varepsilon$, then $|f(x) - f(y)| \ge 2\varepsilon - \int_\varepsilon^x g(t)dt \ge 2\varepsilon - \int_\varepsilon^x \alpha((t+\varepsilon)/(2\varepsilon))^{-\alpha-1} = 2\varepsilon((x+\varepsilon)/(2\varepsilon))^{-\alpha} > g(x)|x-y|/\alpha = f'(x)|x-y|/\alpha$; symmetrically for $-1 < x < -\varepsilon$ and $y = \varepsilon$.

6th INTERNATIONAL COMPETITION FOR UNIVERSITY STUDENTS IN MATHEMATICS

Keszthely, 1999.

Problems and solutions on the second day

1. Suppose that in a not necessarily commutative ring R the square of any element is 0. Prove that $abc + abc = 0$ for any three elements a, b, c. (20 points)

Solution. From $0 = (a + b)^2 = a^2 + b^2 + ab + ba = ab + ba$, we have $ab = -(ba)$ for arbitrary a, b, which implies

$$abc = a(bc) = -\big((bc)\big)a = -\big(b(ca)\big) = (ca)b = c(ab) = -\big((ab)c\big) = -abc.$$

2. We throw a dice (which selects one of the numbers $1, 2, \ldots, 6$ with equal probability) n times. What is the probability that the sum of the values is divisible by 5? (20 points)

Solution 1. For all nonnegative integers n and modulo 5 residue class r, denote by $p_n^{(r)}$ the probability that after n throwing the sum of values is congruent to r modulo n. It is obvious that $p_0^{(0)} = 1$ and $p_0^{(1)} = p_0^{(2)} = p_0^{(3)} = p_0^{(4)} = 0$.

Moreover, for any $n > 0$ we have

$$p_n^{(r)} = \sum_{i=1}^{6} \frac{1}{6} p_{n-1}^{(r-i)}. \tag{1}$$

From this recursion we can compute the probabilities for small values of n and can conjecture that $p_n^{(r)} = \frac{1}{5} + \frac{4}{5\cdot 6^n}$ if $n \equiv r \pmod 5$ and $p_n^{(r)} = \frac{1}{5} - \frac{1}{5\cdot 6^n}$ otherwise. From (1), this conjecture can be proved by induction.

Solution 2. Let S be the set of all sequences consisting of digits $1, \ldots, 6$ of length n. We create collections of these sequences.

Let a collection contain sequences of the form

$$\underbrace{66\ldots6}_{k} XY_1 \ldots Y_{n-k-1},$$

where $X \in \{1, 2, 3, 4, 5\}$ and k and the digits $Y_1, \ldots, Y_{n-k-1}$ are fixed. Then each collection consists of 5 sequences, and the sums of the digits of sequences give a whole residue system mod 5.

Except for the sequence $66\ldots6$, each sequence is the element of one collection. This means that the number of the sequences, which have a sum of digits divisible by 5, is $\frac{1}{5}(6^n - 1) + 1$ if n is divisible by 5, otherwise $\frac{1}{5}(6^n - 1)$.

Thus, the probability is $\frac{1}{5} + \frac{4}{5\cdot 6^n}$ if n is divisible by 5, otherwise it is $\frac{1}{5} - \frac{1}{5\cdot 6^n}$.

Solution 3. For arbitrary positive integer k denote by p_k the probability that the sum of values is k. Define the generating function

$$f(x) = \sum_{k=1}^{\infty} p_k x^k = \left(\frac{x + x^2 + x^3 + x^4 + x^5 + x^6}{6}\right)^n.$$

(The last equality can be easily proved by induction.)

Our goal is to compute the sum $\sum_{k=1}^{\infty} p_{5k}$. Let $\varepsilon = \cos\frac{2\pi}{5} + i \sin\frac{2\pi}{5}$ be the first 5th root of unity. Then

$$\sum_{k=1}^{\infty} p_{5k} = \frac{f(1) + f(\varepsilon) + f(\varepsilon^2) + f(\varepsilon^3) + f(\varepsilon^4)}{5}.$$

Obviously $f(1) = 1$, and $f(\varepsilon^j) = \frac{\varepsilon^{jn}}{6^n}$ for $j = 1, 2, 3, 4$. This implies that $f(\varepsilon) + f(\varepsilon^2) + f(\varepsilon^3) + f(\varepsilon^4)$ is $\frac{4}{6^n}$ if n is divisible by 5, otherwise it is $\frac{-1}{6^n}$. Thus, $\sum_{k=1}^{\infty} p_{5k}$ is $\frac{1}{5} + \frac{4}{5 \cdot 6^n}$ if n is divisible by 5, otherwise it is $\frac{1}{5} - \frac{1}{5 \cdot 6^n}$.

3. Assume that $x_1, \ldots, x_n \geq -1$ and $\sum_{i=1}^{n} x_i^3 = 0$. Prove that $\sum_{i=1}^{n} x_i \leq \frac{n}{3}$. (20 points)

Solution. The inequality

$$0 \leq x^3 - \frac{3}{4}x + \frac{1}{4} = (x+1)\left(x - \frac{1}{2}\right)^2$$

holds for $x \geq -1$.

Substituting $x_1, \ldots, x_n$, we obtain

$$0 \leq \sum_{i=1}^{n} \left(x_i^3 - \frac{3}{4}x_i + \frac{1}{4}\right) = \sum_{i=1}^{n} x_i^3 - \frac{3}{4}\sum_{i=1}^{n} x_i + \frac{n}{4} = 0 - \frac{3}{4}\sum_{i=1}^{n} x_i + \frac{n}{4},$$

so $\sum_{i=1}^{n} x_i \leq \frac{n}{3}$.

Remark. Equailty holds only in the case when $n = 9k$, k of the $x_1, \ldots, x_n$ are -1, and $8k$ of them are $\frac{1}{2}$.

4. Prove that there exists no function $f : (0, +\infty) \to (0, +\infty)$ such that $f^2(x) \geq f(x+y)\big(f(x) + y\big)$ for any $x, y > 0$. (20 points)

Solution. Assume that such a function exists. The initial inequality can be written in the form $f(x) - f(x+y) \geq f(x) - \frac{f^2(x)}{f(x)+y} = \frac{f(x)y}{f(x)+y}$. Obviously, f is a decreasing function. Fix $x > 0$ and choose $n \in \mathbf{N}$ such that $nf(x+1) \geq 1$. For $k = 0, 1, \ldots, n-1$ we have

$$f\left(x + \frac{k}{n}\right) - f\left(x + \frac{k+1}{n}\right) \geq \frac{f\big(x + \frac{k}{n}\big)}{nf\big(x + \frac{k}{n}\big) + 1} \geq \frac{1}{2n}.$$

The additon of these inequalities gives $f(x+1) \leq f(x) - \frac{1}{2}$. From this it follows that $f(x+2m) \leq f(x) - m$ for all $m \in \mathbf{N}$. Taking $m \geq f(x)$, we get a contradiction with the conditon $f(x) > 0$.

5. Let S be the set of all words consisting of the letters x, y, z, and consider an equivalence relation $\sim$ on S satisfying the following conditions: for arbitrary words $u, v, w \in S$

(i) $uu \sim u$;

(ii) if $v \sim w$, then $uv \sim uw$ and $vu \sim wu$.

Show that every word in S is equivalent to a word of length at most 8. (20 points)

Solution. First we prove the following lemma: If a word $u \in S$ contains at least one of each letter, and $v \in S$ is an arbitrary word, then there exists a word $w \in S$ such that $uvw \sim u$.

If v contains a single letter, say x, write u in the form $u = u_1xu_2$, and choose $w = u_2$. Then $uvw = (u_1xu_2)xu_2 = u_1((xu_2)(xu_2)) \sim u_1(xu_2) = u$.

In the general case, let the letters of v be $a_1, \ldots, a_k$. Then one can choose some words $w_1, \ldots, w_k$ such that $(ua_1)w_1 \sim u$, $(ua_1a_2)w_2 \sim ua_1$, $\ldots$, $(ua_1 \ldots a_k)w_k \sim ua_1 \ldots a_{k-1}$. Then $u \sim ua_1w_1 \sim ua_1a_2w_2w_1 \sim \ldots \sim ua_1 \ldots a_kw_k \ldots w_1 = uv(w_k \ldots w_1)$, so $w = w_k \ldots w_1$ is a good choice.

Consider now an arbitrary word a, which contains more than 8 digits. We shall prove that there is a shorter word which is equivalent to a. If a can be written in the form $uvvw$, its length can be reduced by $uvvw \sim uvw$. So we can assume that a does not have this form.

Write a in the form $a = bcd$, where b and d are the first and last four letter of a, respectively. We prove that $a \sim bd$.

It is easy to check that b and d contains all the three letters x, y and z, otherwise their length could be reduced. By the lemma there is a word e such that $b(cd)e \sim b$, and there is a word f such that $def \sim d$. Then we can write

$$a = bcd \sim bc(def) \sim bc(dedef) = (bcde)(def) \sim bd.$$

Remark. Of course, it is enough to give for every word of length 9 an shortest shorter word. Assuming that the first letter is x and the second is y, it is easy (but a little long) to check that there are 18 words of length 9 which cannot be written in the form $uvvw$.

For five of these words there is a 2-step solution, for example

$$xyxzyzx\,\underline{zy} \sim xy\,\underline{xzyz\,xzyzy} \sim xyx\,\underline{zy\,zy} \sim xyxzy.$$

In the remaining 13 cases we need more steps. The general algorithm given by the Solution works for these cases as well, but needs also very long words. For example, to reduce the length of the word $a = xyzyxzxyz$, we have set $b = xyzy$, $c = x$, $d = zxyz$, $e = xyxzxzyxyzy$, $f = zyxyxzyxzxzxzxyxyzxyz$. The longest word in the algorithm was

$$bcdedef = xyzyxzxyzxyxzxzyxyzyzxyzxyxzxzyxyzyzyxyxzyxzxzxzxyxyzxyz,$$

which is of length 46. This is not the shortest way: reducing the length of word a can be done for example by the following steps:

$$xyzyxzx\,\underline{yz} \sim xyzyxz\,\underline{xyzy}\,z \sim xyzyxzxy\,\underline{zyx}\,yzyz \sim \underline{xyzyxz\,xyzyxz}\,yx\,\underline{yz\,yz} \sim xy\,\underline{zyx\,zyx}\,yz \sim xyzyxyz.$$

(The last example is due to Nayden Kambouchev from Sofia University.)

6. Let A be a subset of $\mathbf{Z}_n = \mathbf{Z}/n\mathbf{Z}$ containing at most $\frac{1}{100}\ln n$ elements. Define the rth Fourier coefficient of A for $r \in \mathbf{Z}_n$ by

$$f(r) = \sum_{s\in A} \exp\left(\frac{2\pi i}{n} sr\right).$$

Prove that there exists an $r \neq 0$, such that $\left|f(r)\right| \geq \frac{|A|}{2}$. (20 points)

Solution. Let $A = \{a_1, \ldots, a_k\}$. Consider the k-tuples

$$\left(\exp\frac{2\pi i a_1 t}{n}, \ldots, \exp\frac{2\pi i a_k t}{n}\right) \in \mathbf{C}^k, \qquad t = 0, 1, \ldots, n-1.$$

Each component is in the unit circle $|z| = 1$. Split the circle into 6 equal arcs. This induces a decomposition of the k-tuples into 6^k classes. By the condition $k \leq \frac{1}{100}\ln n$ we have $n > 6^k$, so there are two k-tuples in the same class say for $t_1 < t_2$. Set $r = t_2 - t_1$. Then

$$\operatorname{Re}\exp\frac{2\pi i a_j r}{n} = \cos\left(\frac{2\pi a_j t_2}{n} - \frac{2\pi a_j t_1}{n}\right) \geq \cos\frac{\pi}{3} = \frac{1}{2}$$

for all j, so

$$|f(r)| \geq \operatorname{Re}\, f(r) \geq \frac{k}{2}.$$

Solutions for the first day problems at the IMC 2000

Problem 1.

Is it true that if $f:[0,1]\to[0,1]$ *is*

a) monotone increasing

b) monotone decreasing

then there exists an $x\in[0,1]$ *for which* $f(x)=x$*?*

Solution.

a) Yes.

Proof: Let $A=\{x\in[0,1]:f(x)>x\}$. If $f(0)=0$ we are done, if not then A is non-empty (0 is in A) bounded, so it has supremum, say a. Let $b=f(a)$.

I. case: $a<b$. Then, using that f is monotone and a was the sup, we get $b=f(a)\le f((a+b)/2)\le(a+b)/2$, which contradicts $a<b$.

II. case: $a>b$. Then we get $b=f(a)\ge f((a+b)/2)>(a+b)/2$ contradiction. Therefore we must have $a=b$.

b) No. Let, for example,

$$f(x)=1-x/2 \quad \text{if} \quad x\le 1/2$$

and

$$f(x)=1/2-x/2 \quad \text{if} \quad x>1/2$$

This is clearly a good counter-example.

Problem 2.

Let $p(x)=x^5+x$ *and* $q(x)=x^5+x^2$*. Find all pairs* (w,z) *of complex numbers with* $w\ne z$ *for which* $p(w)=p(z)$ *and* $q(w)=q(z)$*.*

Short solution. Let

$$P(x,y)=\frac{p(x)-p(y)}{x-y}=x^4+x^3y+x^2y^2+xy^3+y^4+1$$

and

$$Q(x,y)=\frac{q(x)-q(y)}{x-y}=x^4+x^3y+x^2y^2+xy^3+y^4+x+y.$$

We need those pairs (w,z) which satisfy $P(w,z)=Q(w,z)=0$.

From $P-Q=0$ we have $w+z=1$. Let $c=wz$. After a short calculation we obtain $c^2-3c+2=0$, which has the solutions $c=1$ and $c=2$. From the system $w+z=1$, $wz=c$ we obtain the following pairs:

$$\left(\frac{1\pm\sqrt{3}i}{2},\frac{1\mp\sqrt{3}i}{2}\right) \quad \text{and} \quad \left(\frac{1\pm\sqrt{7}i}{2},\frac{1\mp\sqrt{7}i}{2}\right).$$

Problem 3.

A and B are square complex matrices of the same size and

$$\operatorname{rank}(AB - BA) = 1\ .$$

Show that $(AB - BA)^2 = 0$.

Let $C = AB - BA$. Since $\operatorname{rank} C = 1$, at most one eigenvalue of C is different from 0. Also $\operatorname{tr} C = 0$, so all the eigevalues are zero. In the Jordan canonical form there can only be one 2×2 cage and thus $C^2 = 0$.

Problem 4.

a) Show that if (x_i) *is a decreasing sequence of positive numbers then*

$$\left(\sum_{i=1}^{n} x_i^2\right)^{1/2} \leq \sum_{i=1}^{n} \frac{x_i}{\sqrt{i}}.$$

b) Show that there is a constant C *so that if* (x_i) *is a decreasing sequence of positive numbers then*

$$\sum_{m=1}^{\infty} \frac{1}{\sqrt{m}} \left(\sum_{i=m}^{\infty} x_i^2\right)^{1/2} \leq C \sum_{i=1}^{\infty} x_i.$$

Solution.

a)

$$(\sum_{i=1}^{n} \frac{x_i}{\sqrt{i}})^2 = \sum_{i,j}^{n} \frac{x_i x_j}{\sqrt{i}\sqrt{j}} \geq \sum_{i=1}^{n} \frac{x_i}{\sqrt{i}} \sum_{j=1}^{i} \frac{x_i}{\sqrt{j}} \geq \sum_{i=1}^{n} \frac{x_i}{\sqrt{i}} i \frac{x_i}{\sqrt{i}} = \sum_{i=1}^{n} x_i^2$$

b)

$$\sum_{m=1}^{\infty} \frac{1}{\sqrt{m}} (\sum_{i=m}^{\infty} x_i^2)^{1/2} \leq \sum_{m=1}^{\infty} \frac{1}{\sqrt{m}} \sum_{i=m}^{\infty} \frac{x_i}{\sqrt{i-m+1}}$$

by a)

$$= \sum_{i=1}^{\infty} x_i \sum_{m=1}^{i} \frac{1}{\sqrt{m}\sqrt{i-m+1}}$$

You can get a sharp bound on

$$\sup_i \sum_{m=1}^{i} \frac{1}{\sqrt{m}\sqrt{i-m+1}}$$

by checking that it is at most

$$\int_0^{i+1} \frac{1}{\sqrt{x}\sqrt{i+1-x}} dx = \pi$$

Alternatively you can observe that

$$\sum_{m=1}^{i} \frac{1}{\sqrt{m}\sqrt{i+1-m}} = 2\sum_{m=1}^{i/2} \frac{1}{\sqrt{m}\sqrt{i+1-m}} \leq$$

$$\leq 2\frac{1}{\sqrt{i/2}} \sum_{m=1}^{i/2} \frac{1}{\sqrt{m}} \leq 2\frac{1}{\sqrt{i/2}}.2\sqrt{i/2} = 4$$

Problem 5.

Let R be a ring of characteristic zero (not necessarily commutative). Let e, f and g be idempotent elements of R satisfying $e+f+g=0$. Show that $e=f=g=0$.

(R is of characteristic zero means that, if $a \in R$ and n is a positive integer, then $na \neq 0$ unless $a=0$. An idempotent x is an element satisfying $x=x^2$.)

Solution. Suppose that $e+f+g=0$ for given idempotents $e, f, g \in R$. Then

$$g = g^2 = (-(e+f))^2 = e + (ef+fe) + f = (ef+fe) - g,$$

i.e. ef+fe=2g, whence the additive commutator

$$[e,f] = ef - fe = [e, ef+fe] = 2[e,g] = 2[e,-e-f] = -2[e,f],$$

i.e. $ef = fe$ (since R has zero characteristic). Thus $ef+fe=2g$ becomes $ef=g$, so that $e+f+ef=0$. On multiplying by e, this yields $e+2ef=0$, and similarly $f+2ef=0$, so that $f=-2ef=e$, hence $e=f=g$ by symmetry. Hence, finaly, $3e=e+f+g=0$, i.e. $e=f=g=0$.

For part (i) just omit some of this.

Problem 6.

Let $f: \mathbb{R} \to (0,\infty)$ be an increasing differentiable function for which $\lim_{x\to\infty} f(x) = \infty$ and f' is bounded.

Let $F(x) = \int_0^x f$. Define the sequence (a_n) inductively by

$$a_0 = 1, \quad a_{n+1} = a_n + \frac{1}{f(a_n)},$$

and the sequence (b_n) simply by $b_n = F^{-1}(n)$. Prove that $\lim_{n\to\infty}(a_n - b_n) = 0$.

Solution. From the conditions it is obvious that F is increasing and $\lim_{n\to\infty} b_n = \infty$.

By Lagrange's theorem and the recursion in (1), for all $k \geq 0$ integers there exists a real number $\xi \in (a_k, a_{k+1})$ such that

$$F(a_{k+1}) - F(a_k) = f(\xi)(a_{k+1} - a_k) = \frac{f(\xi)}{f(a_k)}. \tag{2}$$

By the monotonity, $f(a_k) \le f(\xi) \le f(a_{k+1})$, thus

$$1 \le F(a_{k+1}) - F(a_k) \le \frac{f(a_{k+1})}{f(a_k)} = 1 + \frac{f(a_{k+1}) - f(a_k)}{f(a_k)}. \tag{3}$$

Summing (3) for $k = 0, \dots, n-1$ and substituting $F(b_n) = n$, we have

$$F(b_n) < n + F(a_0) \le F(a_n) \le F(b_n) + F(a_0) + \sum_{k=0}^{n-1} \frac{f(a_{k+1}) - f(a_k)}{f(a_k)}. \tag{4}$$

From the first two inequalities we already have $a_n > b_n$ and $\lim_{n\to\infty} a_n = \infty$.

Let ε be an arbitrary positive number. Choose an integer K_ε such that $f(a_{K_\varepsilon}) > \frac{2}{\varepsilon}$. If n is sufficiently large, then

$$F(a_0) + \sum_{k=0}^{n-1} \frac{f(a_{k+1}) - f(a_k)}{f(a_k)} =$$

$$= \left(F(a_0) + \sum_{k=0}^{K_\varepsilon - 1} \frac{f(a_{k+1}) - f(a_k)}{f(a_k)} \right) + \sum_{k=K_\varepsilon}^{n-1} \frac{f(a_{k+1}) - f(a_k)}{f(a_k)} < \tag{5}$$

$$< O_\varepsilon(1) + \frac{1}{f(a_{K_\varepsilon})} \sum_{k=K_\varepsilon}^{n-1} \big(f(a_{k+1}) - f(a_k) \big) <$$

$$< O_\varepsilon(1) + \frac{\varepsilon}{2} \big(f(a_n) - f(a_{K_\varepsilon}) \big) < \varepsilon f(a_n).$$

Inequalities (4) and (5) together say that for any positive ε, if n is sufficiently large,

$$F(a_n) - F(b_n) < \varepsilon f(a_n).$$

Again, by Lagrange's theorem, there is a real number $\zeta \in (b_n, a_n)$ such that

$$F(a_n) - F(b_n) = f(\zeta)(a_n - b_n) > f(b_n)(a_n - b_n), \tag{6}$$

thus

$$f(b_n)(a_n - b_n) < \varepsilon f(a_n). \tag{7}$$

Let B be an upper bound for f'. Apply $f(a_n) < f(b_n) + B(a_n - b_n)$ in (7):

$$f(b_n)(a_n - b_n) < \varepsilon \big(f(b_n) + B(a_n - b_n) \big),$$

$$\big(f(b_n) - \varepsilon B \big)(a_n - b_n) < \varepsilon f(b_n). \tag{8}$$

Due to $\lim_{n\to\infty} f(b_n) = \infty$, the first factor is positive, and we have

$$a_n - b_n < \varepsilon \frac{f(b_n)}{f(b_n) - \varepsilon B} < 2\varepsilon \tag{9}$$

for sufficiently large n.

Thus, for arbitrary positive ε we proved that $0 < a_n - b_n < 2\varepsilon$ if n is sufficiently large.

Solutions for the second day problems at the IMC 2000

Problem 1.

a) Show that the unit square can be partitioned into n smaller squares if n is large enough.

b) Let $d \geq 2$. Show that there is a constant $N(d)$ such that, whenever $n \geq N(d)$, a d-dimensional unit cube can be partitioned into n smaller cubes.

Solution. We start with the following lemma: If a and b be coprime positive integers then every sufficiently large positive integer m can be expressed in the form $ax + by$ with x, y non-negative integers.

Proof of the lemma. The numbers $0, a, 2a, \ldots, (b-1)a$ give a complete residue system modulo b. Consequently, for any m there exists a $0 \leq x \leq b-1$ so that $ax \equiv m \pmod{b}$. If $m \geq (b-1)a$, then $y = (m - ax)/b$, for which $x + by = m$, is a non-negative integer, too.

Now observe that any dissection of a cube into n smaller cubes may be refined to give a dissection into $n + (a^d - 1)$ cubes, for any $a \geq 1$. This refinement is achieved by picking an arbitrary cube in the dissection, and cutting it into a^d smaller cubes. To prove the required result, then, it suffices to exhibit two relatively prime integers of form $a^d - 1$. In the 2-dimensional case, $a_1 = 2$ and $a_2 = 3$ give the coprime numbers $2^2 - 1 = 3$ and $3^2 - 1 = 8$. In the general case, two such integers are $2^d - 1$ and $(2^d - 1)^d - 1$, as is easy to check.

Problem 2. *Let f be continuous and nowhere monotone on $[0, 1]$. Show that the set of points on which f attains local minima is dense in $[0, 1]$.*

(A function is nowhere monotone if there exists no interval where the function is monotone. A set is dense if each non-empty open interval contains at least one element of the set.)

Solution. Let $(x - \alpha, x + \alpha) \subset [0, 1]$ be an arbitrary non-empty open interval. The function f is not monoton in the intervals $[x - \alpha, x]$ and $[x, x + \alpha]$, thus there exist some real numbers $x - \alpha \leq p < q \leq x$, $x \leq r < s \leq x + \alpha$ so that $f(p) > f(q)$ and $f(r) < f(s)$.

By Weierstrass' theorem, f has a global minimum in the interval $[p, s]$. The values $f(p)$ and $f(s)$ are not the minimum, because they are greater than $f(q)$ and $f(s)$, respectively. Thus the minimum is in the interior of the interval, it is a local minimum. So each non-empty interval $(x - \alpha, x + \alpha) \subset [0, 1]$ contains at least one local minimum.

Problem 3. *Let $p(z)$ be a polynomial of degree n with complex coefficients. Prove that there exist at least $n + 1$ complex numbers z for which $p(z)$ is 0 or 1.*

Solution. The statement is not true if p is a constant polynomial. We prove it only in the case if n is positive.

For an arbitrary polynomial $q(z)$ and complex number c, denote by $\mu(q, c)$ the largest exponent α for which $q(z)$ is divisible by $(z - c)^\alpha$. (With other words, if c is a root of q, then $\mu(q, c)$ is the root's multiplicity. Otherwise 0.)

Denote by S_0 and S_1 the sets of complex numbers z for which $p(z)$ is 0 or 1, respectively. These sets contain all roots of the polynomials $p(z)$ and $p(z)-1$, thus

$$\sum_{c\in S_0}\mu(p,c)=\sum_{c\in S_1}\mu(p-1,c)=n. \tag{1}$$

The polynomial p' has at most $n-1$ roots ($n>0$ is used here). This implies that

$$\sum_{c\in S_0\cup S_1}\mu(p',c)\le n-1. \tag{2}$$

If $p(c)=0$ or $p(c)-1=0$, then

$$\mu(p,c)-\mu(p'c)=1 \quad \text{or} \quad \mu(p-1,c)-\mu(p'c)=1, \tag{3}$$

respectively. Putting (1), (2) and (3) together we obtain

$$\big|S_0\big|+\big|S_1\big|=\sum_{c\in S_0}\big(\mu(p,c)-\mu(p',c)\big)+\sum_{c\in S_1}\big(\mu(p-1,c)-\mu(p',c)\big)=$$

$$=\sum_{c\in S_0}\mu(p,c)+\sum_{c\in S_1}\mu(p-1,c)-\sum_{c\in S_0\cup S_1}\mu(p',c)\ge n+n-(n-1)=n+1.$$

Problem 4. *Suppose the graph of a polynomial of degree 6 is tangent to a straight line at 3 points A_1, A_2, A_3, where A_2 lies between A_1 and A_3.*

a) Prove that if the lengths of the segments A_1A_2 and A_2A_3 are equal, then the areas of the figures bounded by these segments and the graph of the polynomial are equal as well.

b) Let $k=\dfrac{A_2A_3}{A_1A_2}$, and let K be the ratio of the areas of the appropriate figures. Prove that

$$\frac{2}{7}k^5<K<\frac{7}{2}k^5.$$

Solution. a) Without loss of generality, we can assume that the point A_2 is the origin of system of coordinates. Then the polynomial can be presented in the form

$$y=\big(a_0x^4+a_1x^3+a_2x^2+a_3x+a_4\big)x^2+a_5x,$$

where the equation $y=a_5x$ determines the straight line A_1A_3. The abscissas of the points A_1 and A_3 are $-a$ and a, $a>0$, respectively. Since $-a$ and a are points of tangency, the numbers $-a$ and a must be double roots of the polynomial $a_0x^4+a_1x^3+a_2x^2+a_3x+a_4$. It follows that the polynomial is of the form

$$y=a_0(x^2-a^2)^2+a_5x.$$

The equality follows from the equality of the integrals

$$\int_{-a}^{0} a_0(x^2 - a^2)x^2 \mathrm{d}x = \int_{0}^{a} a_0(x^2 - a^2)x^2 \mathrm{d}x$$

due to the fact that the function $y = a_0(x^2 - a^2)$ is even.

b) Without loss of generality, we can assume that $a_0 = 1$. Then the function is of the form

$$y = (x + a)^2(x - b)^2x^2 + a_5x,$$

where a and b are positive numbers and $b = ka$, $0 < k < \infty$. The areas of the figures at the segments A_1A_2 and A_2A_3 are equal respectively to

$$\int_{-a}^{0} (x + a)^2(x - b)^2x^2 \mathrm{d}x = \frac{a^7}{210}(7k^2 + 7k + 2)$$

and

$$\int_{0}^{b} (x + a)^2(x - b)^2x^2 \mathrm{d}x = \frac{a^7}{210}(2k^2 + 7k + 7)$$

Then

$$K = k^5 \frac{2k^2 + 7k + 7}{7k^2 + 7k + 2}.$$

The derivative of the function $f(k) = \frac{2k^2+7k+7}{7k^2+7k+2}$ is negative for $0 < k < \infty$. Therefore $f(k)$ decreases from $\frac{7}{2}$ to $\frac{2}{7}$ when k increases from 0 to ∞. Inequalities $\frac{2}{7} < \frac{2k^2+7k+7}{7k^2+7k+2} < \frac{7}{2}$ imply the desired inequalities.

Problem 5. *Let $\mathbb{R}^+$ be the set of positive real numbers. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $x, y \in \mathbb{R}^+$*

$$f(x)f(yf(x)) = f(x + y).$$

First solution. First, if we assume that $f(x) > 1$ for some $x \in \mathbb{R}^+$, setting $y = \dfrac{x}{f(x) - 1}$ gives the contradiction $f(x) = 1$. Hence $f(x) \leq 1$ for each $x \in \mathbb{R}^+$, which implies that f is a decreasing function.

If $f(x) = 1$ for some $x \in \mathbb{R}^+$, then $f(x + y) = f(y)$ for each $y \in \mathbb{R}^+$, and by the monotonicity of f it follows that $f \equiv 1$.

Let now $f(x) < 1$ for each $x \in \mathbb{R}^+$. Then f is strictly decreasing function, in particular injective. By the equalities

$$f(x)f(yf(x)) = f(x + y) =$$

$$= f\big(yf(x) + x + y(1 - f(x))\big) = f(yf(x))f\Big(\big(x + y(1 - f(x))\big)f(yf(x))\Big)$$

we obtain that $x = (x + y(1 - f(x)))f(yf(x))$. Setting $x = 1,\ z = xf(1)$ and $a = \dfrac{1 - f(1)}{f(1)}$, we get $f(z) = \dfrac{1}{1 + az}$.

Combining the two cases, we conclude that $f(x) = \dfrac{1}{1 + ax}$ for each $x \in \mathbb{R}^+$, where $a \geq 0$. Conversely, a direct verification shows that the functions of this form satisfy the initial equality.

Second solution. As in the first solution we get that f is a decreasing function, in particular differentiable almost everywhere. Write the initial equality in the form

$$\frac{f(x + y) - f(x)}{y} = f^2(x)\frac{f(yf(x)) - 1}{yf(x)}.$$

It follows that if f is differentiable at the point $x \in \mathbb{R}^+$, then there exists the limit $\lim\limits_{z \to 0+} \dfrac{f(z) - 1}{z} =: -a$. Therefore $f'(x) = -af^2(x)$ for each $x \in \mathbb{R}^+$, i.e. $\left(\dfrac{1}{f(x)}\right)' = a$, which means that $f(x) = \dfrac{1}{ax + b}$. Substituting in the initial relaton, we find that $b = 1$ and $a \geq 0$.

Problem 6. *For an $m \times m$ real matrix A, e^A is defined as $\sum\limits_{n=0}^{\infty} \frac{1}{n!}A^n$. (The sum is convergent for all matrices.) Prove or disprove, that for all real polynomials p and $m \times m$ real matrices A and B, $p(e^{AB})$ is nilpotent if and only if $p(e^{BA})$ is nilpotent. (A matrix A is nilpotent if $A^k = 0$ for some positive integer k.)*

Solution. First we prove that for any polynomial q and $m \times m$ matrices A and B, the characteristic polinomials of $q(e^{AB})$ and $q(e^{BA})$ are the same. It is easy to check that for any matrix X, $q(e^X) = \sum\limits_{n=0}^{\infty} c_n X^n$ with some real numbers c_n which depend on q. Let

$$C = \sum_{n=1}^{\infty} c_n \cdot (BA)^{n-1}B = \sum_{n=1}^{\infty} c_n \cdot B(AB)^{n-1}.$$

Then $q(e^{AB}) = c_0 I + AC$ and $q(e^{BA}) = c_0 I + CA$. It is well-known that the characteristic polynomials of AC and CA are the same; denote this polynomial by $f(x)$. Then the characteristic polynomials of matrices $q(e^{AB})$ and $q(e^{BA})$ are both $f(x - c_0)$.

Now assume that the matrix $p(e^{AB})$ is nilpotent, i.e. $\big(p(e^{AB})\big)^k = 0$ for some positive integer k. Chose $q = p^k$. The characteristic polynomial of the matrix $q(e^{AB}) = 0$ is x^m, so the same holds for the matrix $q(e^{BA})$. By the theorem of Cayley and Hamilton, this implies that $\big(q(e^{BA})\big)^m = \big(p(e^{BA})\big)^{km} = 0$. Thus the matrix $q(e^{BA})$ is nilpotent, too.

8^{th} **IMC 2001**
July 19 - July 25
Prague, Czech Republic

First day

Problem 1.
Let n be a positive integer. Consider an $n \times n$ matrix with entries $1, 2, \ldots, n^2$ written in order starting top left and moving along each row in turn left–to–right. We choose n entries of the matrix such that exactly one entry is chosen in each row and each column. What are the possible values of the sum of the selected entries?

Solution. Since there are exactly n rows and n columns, the choice is of the form

$$\{(j, \sigma(j)) : j = 1, \ldots, n\}$$

where $\sigma \in S_n$ is a permutation. Thus the corresponding sum is equal to

$$\sum_{j=1}^{n} n(j-1) + \sigma(j) = \sum_{j=1}^{n} nj - \sum_{j=1}^{n} n + \sum_{j=1}^{n} \sigma(j)$$

$$= n\sum_{j=1}^{n} j - \sum_{j=1}^{n} n + \sum_{j=1}^{n} j = (n+1)\frac{n(n+1)}{2} - n^2 = \frac{n(n^2+1)}{2},$$

which shows that the sum is independent of σ.

Problem 2.
Let r, s, t be positive integers which are pairwise relatively prime. If a and b are elements of a commutative multiplicative group with unity element e, and $a^r = b^s = (ab)^t = e$, prove that $a = b = e$.

Does the same conclusion hold if a and b are elements of an arbitrary non-commutative group?

Solution. 1. There exist integers u and v such that $us + vt = 1$. Since $ab = ba$, we obtain

$$ab = (ab)^{us+vt} = (ab)^{us} \left((ab)^t\right)^v = (ab)^{us} e = (ab)^{us} = a^{us} (b^s)^u = a^{us} e = a^{us}.$$

Therefore, $b^r = eb^r = a^r b^r = (ab)^r = a^{usr} = (a^r)^{us} = e$. Since $xr + ys = 1$ for suitable integers x and y,

$$b = b^{xr+ys} = (b^r)^x (b^s)^y = e.$$

It follows similarly that $a = e$ as well.

2. This is not true. Let $a = (123)$ and $b = (34567)$ be cycles of the permutation group S_7 of order 7. Then $ab = (1234567)$ and $a^3 = b^5 = (ab)^7 = e$.

Problem 3. Find $\lim\limits_{t \nearrow 1} (1-t) \sum\limits_{n=1}^{\infty} \frac{t^n}{1+t^n}$, where $t \nearrow 1$ means that t approaches 1 from below.

Solution.

$$\lim_{t\to 1-0}(1-t)\sum_{n=1}^{\infty}\frac{t^n}{1+t^n}=\lim_{t\to 1-0}\frac{1-t}{-\ln t}\cdot(-\ln t)\sum_{n=1}^{\infty}\frac{t^n}{1+t^n}=$$

$$=\lim_{t\to 1-0}(-\ln t)\sum_{n=1}^{\infty}\frac{1}{1+e^{-n\ln t}}=\lim_{h\to +0}h\sum_{n=1}^{\infty}\frac{1}{1+e^{nh}}=\int_0^{\infty}\frac{dx}{1+e^x}=\ln 2.$$

Problem 4.

Let k be a positive integer. Let $p(x)$ be a polynomial of degree n each of whose coefficients is -1, 1 or 0, and which is divisible by $(x-1)^k$. Let q be a prime such that $\frac{q}{\ln q}<\frac{k}{\ln(n+1)}$. Prove that the complex qth roots of unity are roots of the polynomial $p(x)$.

Solution. Let $p(x)=(x-1)^k\cdot r(x)$ and $\varepsilon_j=e^{2\pi i\cdot j/q}$ $(j=1,2,\dots,q-1)$. As is well-known, the polynomial $x^{q-1}+x^{q-2}+\ldots+x+1=(x-\varepsilon_1)\ldots(x-\varepsilon_{q-1})$ is irreducible, thus all $\varepsilon_1,\dots,\varepsilon_{q-1}$ are roots of $r(x)$, or none of them.

Suppose that none of $\varepsilon_1,\dots,\varepsilon_{q-1}$ is a root of $r(x)$. Then $\prod_{j=1}^{q-1}r(\varepsilon_j)$ is a rational integer, which is not 0 and

$$(n+1)^{q-1}\ge\prod_{j=1}^{q-1}\left|p(\varepsilon_j)\right|=\left|\prod_{j=1}^{q-1}(1-\varepsilon_j)^k\right|\cdot\left|\prod_{j=1}^{q-1}r(\varepsilon_j)\right|\ge$$

$$\ge\left|\prod_{j=1}^{q-1}(1-\varepsilon_j)\right|^k=(1^{q-1}+1^{q-2}+\ldots+1^1+1)^k=q^k.$$

This contradicts the condition $\frac{q}{\ln q}<\frac{k}{\ln(n+1)}$.

Problem 5.

Let A be an $n\times n$ complex matrix such that $A\ne\lambda I$ for all $\lambda\in\mathbf{C}$. Prove that A is similar to a matrix having at most one non-zero entry on the main diagonal.

Solution. The statement will be proved by induction on n. For $n=1$, there is nothing to do. In the case $n=2$, write $A=\begin{bmatrix}a&b\\c&d\end{bmatrix}$. If $b\ne 0$, and $c\ne 0$ or $b=c=0$ then A is similar to

$$\begin{bmatrix}1&0\\a/b&1\end{bmatrix}\begin{bmatrix}a&b\\c&d\end{bmatrix}\begin{bmatrix}1&0\\-a/b&1\end{bmatrix}=\begin{bmatrix}0&b\\c-ad/b&a+d\end{bmatrix}$$

or

$$\begin{bmatrix}1&-a/c\\0&1\end{bmatrix}\begin{bmatrix}a&b\\c&d\end{bmatrix}\begin{bmatrix}1&a/c\\0&1\end{bmatrix}=\begin{bmatrix}0&b-ad/c\\c&a+d\end{bmatrix},$$

respectively. If $b=c=0$ and $a\ne d$, then A is similar to

$$\begin{bmatrix}1&1\\0&1\end{bmatrix}\begin{bmatrix}a&0\\0&d\end{bmatrix}\begin{bmatrix}1&-1\\0&1\end{bmatrix}=\begin{bmatrix}a&d-a\\0&d\end{bmatrix},$$

and we can perform the step seen in the case $b \neq 0$ again.

Assume now that $n > 3$ and the problem has been solved for all $n' < n$. Let $A = \begin{bmatrix} A' & * \\ * & \beta \end{bmatrix}_n$, where A' is $(n-1) \times (n-1)$ matrix. Clearly we may assume that $A' \neq \lambda' I$, so the induction provides a P with, say, $P^{-1}A'P = \begin{bmatrix} 0 & * \\ * & \alpha \end{bmatrix}_{n-1}$. But then the matrix

$$B = \begin{bmatrix} P^{-1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A' & * \\ * & \beta \end{bmatrix} \begin{bmatrix} P & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} P^{-1}A'P & * \\ * & \beta \end{bmatrix}$$

is similar to A and its diagonal is $(0, 0, \ldots, 0, \alpha, \beta)$. On the other hand, we may also view B as $\begin{bmatrix} 0 & * \\ * & C \end{bmatrix}_n$, where C is an $(n-1) \times (n-1)$ matrix with diagonal $(0, \ldots, 0, \alpha, \beta)$. If the inductive hypothesis is applicable to C, we would have $Q^{-1}CQ = D$, with $D = \begin{bmatrix} 0 & * \\ * & \gamma \end{bmatrix}_{n-1}$ so that finally the matrix

$$E = \begin{bmatrix} 1 & 0 \\ 0 & Q^{-1} \end{bmatrix} \cdot B \cdot \begin{bmatrix} 1 & 0 \\ 0 & Q \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & Q^{-1} \end{bmatrix} \begin{bmatrix} 0 & * \\ * & C \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & Q \end{bmatrix} = \begin{bmatrix} 0 & * \\ * & D \end{bmatrix}$$

is similar to A and its diagonal is $(0, 0, \ldots, 0, \gamma)$, as required.

The inductive argument can fail only when $n - 1 = 2$ and the resulting matrix applying P has the form

$$P^{-1}AP = \begin{bmatrix} 0 & a & b \\ c & d & 0 \\ e & 0 & d \end{bmatrix}$$

where $d \neq 0$. The numbers a, b, c, e cannot be 0 at the same time. If, say, $b \neq 0$, A is similar to

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & a & b \\ c & d & 0 \\ e & 0 & d \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -b & a & b \\ c & d & 0 \\ e-b-d & a & b+d \end{bmatrix}.$$

Performing half of the induction step again, the diagonal of the resulting matrix will be $(0, d-b, d+b)$ (the trace is the same) and the induction step can be finished. The cases $a \neq 0$, $c \neq 0$ and $e \neq 0$ are similar.

Problem 6.

Suppose that the differentiable functions $a, b, f, g : \mathbb{R} \to \mathbb{R}$ satisfy

$$f(x) \geq 0, f'(x) \geq 0, g(x) > 0, g'(x) > 0 \text{ for all } x \in \mathbb{R},$$

$$\lim_{x\to\infty} a(x) = A > 0, \quad \lim_{x\to\infty} b(x) = B > 0, \quad \lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty,$$

and

$$\frac{f'(x)}{g'(x)} + a(x)\frac{f(x)}{g(x)} = b(x).$$

Prove that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{B}{A+1}.$$

Solution. Let $0<\varepsilon<A$ be an arbitrary real number. If x is sufficiently large then $f(x)>0$, $g(x)>0$, $|a(x)-A|<\varepsilon$, $|b(x)-B|<\varepsilon$ and

$$B-\varepsilon<b(x)=\frac{f'(x)}{g'(x)}+a(x)\frac{f(x)}{g(x)}<\frac{f'(x)}{g'(x)}+(A+\varepsilon)\frac{f(x)}{g(x)}< \tag{1}$$

$$<\frac{(A+\varepsilon)(A+1)}{A}\cdot\frac{f'(x)\big(g(x)\big)^A+A\cdot f(x)\cdot\big(g(x)\big)^{A-1}\cdot g'(x)}{(A+1)\cdot\big(g(x)\big)^A\cdot g'(x)}=$$

$$=\frac{(A+\varepsilon)(A+1)}{A}\cdot\frac{\Big(f(x)\cdot\big(g(x)\big)^A\Big)'}{\Big(\big(g(x)\big)^{A+1}\Big)'},$$

thus

$$\frac{\Big(f(x)\cdot\big(g(x)\big)^A\Big)'}{\Big(\big(g(x)\big)^{A+1}\Big)'}>\frac{A(B-\varepsilon)}{(A+\varepsilon)(A+1)}. \tag{2}$$

It can be similarly obtained that, for sufficiently large x,

$$\frac{\Big(f(x)\cdot\big(g(x)\big)^A\Big)'}{\Big(\big(g(x)\big)^{A+1}\Big)'}<\frac{A(B+\varepsilon)}{(A-\varepsilon)(A+1)}. \tag{3}$$

From $\varepsilon\to 0$, we have

$$\lim_{x\to\infty}\frac{\Big(f(x)\cdot\big(g(x)\big)^A\Big)'}{\Big(\big(g(x)\big)^{A+1}\Big)'}=\frac{B}{A+1}.$$

By l'Hospital's rule this implies

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{f(x)\cdot\big(g(x)\big)^A}{\big(g(x)\big)^{A+1}}=\frac{B}{A+1}.$$

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Second day

Problem 1.
Let $r, s \geq 1$ be integers and $a_0, a_1, \ldots, a_{r-1}, b_0, b_1, \ldots, b_{s-1}$ be real non-negative numbers such that

$$(a_0 + a_1x + a_2x^2 + \ldots + a_{r-1}x^{r-1} + x^r)(b_0 + b_1x + b_2x^2 + \ldots + b_{s-1}x^{s-1} + x^s) =$$

$$1 + x + x^2 + \ldots + x^{r+s-1} + x^{r+s}.$$

Prove that each a_i and each b_j equals either 0 or 1.

Solution. Multiply the left hand side polynomials. We obtain the following equalities:

$$a_0b_0 = 1, \quad a_0b_1 + a_1b_0 = 1, \quad \ldots$$

Among them one can find equations

$$a_0 + a_1b_{s-1} + a_2b_{s-2} + \ldots = 1$$

and

$$b_0 + b_1a_{r-1} + b_2a_{r-2} + \ldots = 1.$$

From these equations it follows that $a_0, b_0 \leq 1$. Taking into account that $a_0b_0 = 1$ we can see that $a_0 = b_0 = 1$.

Now looking at the following equations we notice that all a's must be less than or equal to 1. The same statement holds for the b's. It follows from $a_0b_1 + a_1b_0 = 1$ that one of the numbers a_1, b_1 equals 0 while the other one must be 1. Follow by induction.

Problem 2.
Let $a_0 = \sqrt{2}$, $b_0 = 2$, $a_{n+1} = \sqrt{2 - \sqrt{4 - a_n^2}}$, $b_{n+1} = \dfrac{2b_n}{2 + \sqrt{4 + b_n^2}}$.

a) Prove that the sequences (a_n), (b_n) are decreasing and converge to 0.

b) Prove that the sequence $(2^n a_n)$ is increasing, the sequence $(2^n b_n)$ is decreasing and that these two sequences converge to the same limit.

c) Prove that there is a positive constant C such that for all n the following inequality holds: $0 < b_n - a_n < \dfrac{C}{8^n}$.

Solution. Obviously $a_2 = \sqrt{2 - \sqrt{2}} < \sqrt{2}$. Since the function $f(x) = \sqrt{2 - \sqrt{4 - x^2}}$ is increasing on the interval $[0, 2]$ the inequality $a_1 > a_2$ implies that $a_2 > a_3$. Simple induction ends the proof of monotonicity of (a_n). In the same way we prove that (b_n) decreases $\left(\text{just notice that } g(x) = \dfrac{2x}{2 + \sqrt{4 + x^2}} = 2/\left(2/x + \sqrt{1 + 4/x^2}\right)\right)$. It is a matter of simple manipulation to prove that $2f(x) > x$ for all $x \in (0, 2)$, this implies that the sequence $(2^n a_n)$ is strictly increasing. The inequality $2g(x) < x$ for $x \in (0,2)$ implies that the sequence $(2^n b_n)$ strictly decreases. By an easy induction one can show that $a_n^2 = \frac{4b_n^2}{4+b_n^2}$ for positive integers n. Since the limit of the *decreasing* sequence $(2^n b_n)$ of *positive* numbers is finite we have

$$\lim 4^n a_n^2 = \lim \frac{4 \cdot 4^n b_n^2}{4+b_n^2} = \lim 4^n b_n^2 .$$

We know already that the limits $\lim 2^n a_n$ and $\lim 2^n b_n$ are equal. The first of the two is positive because the sequence $(2^n a_n)$ is strictly increasing. The existence of a number C follows easily from the equalities

$$2^n b_n - 2^n a_n = \left(4^n b_n^2 - \frac{4^{n+1} b_n^2}{4+b_n^2}\right) / \left(2^n b_n + 2^n a_n\right) = \frac{(2^n b_n)^4}{4+b_n^2} \cdot \frac{1}{4^n} \cdot \frac{1}{2^n(b_n+a_n)}$$

and from the existence of positive limits $\lim 2^n b_n$ and $\lim 2^n a_n$.

Remark. The last problem may be solved in a much simpler way by someone who is able to make use of sine and cosine. It is enough to notice that $a_n = 2\sin\frac{\pi}{2^{n+1}}$ and $b_n = 2\tan\frac{\pi}{2^{n+1}}$.

Problem 3.

Find the maximum number of points on a sphere of radius 1 in $\mathbb{R}^n$ such that the distance between any two of these points is strictly greater than $\sqrt{2}$.

Solution. The unit sphere in $\mathbb{R}^n$ is defined by

$$S_{n-1} = \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n | \sum_{k=1}^{n} x_k^2 = 1 \right\}.$$

The distance between the points $X = (x_1, \dots, x_n)$ and $Y = (y_1, \dots, y_n)$ is:

$$d^2(X,Y) = \sum_{k=1}^{n} (x_k - y_k)^2.$$

We have

$$\begin{aligned} d(X,Y) > \sqrt{2} \quad &\Leftrightarrow \quad d^2(X,Y) > 2 \\ &\Leftrightarrow \quad \sum_{k=1}^{n} x_k^2 + \sum_{k=1}^{n} y_k^2 + 2\sum_{k=1}^{n} x_k y_k > 2 \\ &\Leftrightarrow \quad \sum_{k=1}^{n} x_k y_k < 0 \end{aligned}$$

Taking account of the symmetry of the sphere, we can suppose that

$$A_1 = (-1, 0, \dots, 0).$$

For $X = A_1$, $\sum\limits_{k=1}^{n} x_k y_k < 0$ implies $y_1 > 0$, $\forall\, Y \in M_n$.

Let $X = (x_1, \overline{X})$, $Y = (y_1, \overline{Y}) \in M_n \backslash \{A_1\}$, $\overline{X}, \overline{Y} \in \mathbb{R}^{n-1}$.

We have

$$\sum_{k=1}^{n} x_k y_k < 0 \Rightarrow x_1 y_1 + \sum_{k=1}^{n-1} \overline{x}_k \overline{y}_k < 0 \Leftrightarrow \sum_{k=1}^{n-1} x'_k y'_k < 0,$$

where

$$x'_k = \frac{\overline{x}_k}{\sqrt{\sum \overline{x}_k^2}}, \quad y'_k = \frac{\overline{y}_k}{\sqrt{\sum \overline{y}_k^2}}.$$

therefore

$$(x'_1, \ldots, x'_{n-1}), (y'_1, \ldots, y'_{n-1}) \in S_{n-2}$$

and verifies $\sum\limits_{k=1}^{n} x_k y_k < 0$.

If a_n is the search number of points in $\mathbb{R}^n$ we obtain $a_n \leq 1 + a_{n-1}$ and $a_1 = 2$ implies that $a_n \leq n + 1$.

We show that $a_n = n + 1$, giving an example of a set M_n with $(n+1)$ elements satisfying the conditions of the problem.

$$\begin{aligned}
A_1 &= (-1, 0, 0, 0, \ldots, 0, 0) \\
A_2 &= \left(\tfrac{1}{n}, -c_1, 0, 0, \ldots, 0, 0\right) \\
A_3 &= \left(\tfrac{1}{n}, \tfrac{1}{n-1} \cdot c_1, -c_2, 0, \ldots, 0, 0\right) \\
A_4 &= \left(\tfrac{1}{n}, \tfrac{1}{n-1} \cdot c_1, \tfrac{1}{n-1} \cdot c_2, -c_3, \ldots, 0, 0\right)
\end{aligned}$$

$$\begin{aligned}
A_{n-1} &= \left(\tfrac{1}{n}, \tfrac{1}{n-1} \cdot c_1, \tfrac{1}{n-2} \cdot c_2, \tfrac{1}{n-3} \cdot c_3, \ldots, -c_{n-2}, 0\right) \\
A_n &= \left(\tfrac{1}{n}, \tfrac{1}{n-1} \cdot c_1, \tfrac{1}{n-2} \cdot c_1, \tfrac{1}{n-3} \cdot c_3, \ldots, \tfrac{1}{2} \cdot c_{n-2}, -c_{n-1}\right) \\
A_{n+1} &= \left(\tfrac{1}{n}, \tfrac{1}{n-1} \cdot c_1, \tfrac{1}{n-2} \cdot c_2, \tfrac{1}{n-3} \cdot c_3, \ldots, \tfrac{1}{2} \cdot c_{n-2}, c_{n-1}\right)
\end{aligned}$$

where

$$c_k = \sqrt{\left(1 + \frac{1}{n}\right)\left(1 - \frac{1}{n-k+1}\right)}, \quad k = \overline{1, n-1}.$$

We have $\sum\limits_{k=1}^{n} x_k y_k = -\frac{1}{n} < 0$ and $\sum\limits_{k-=1}^{n} x_k^2 = 1, \quad \forall X, Y \in \{A_1, \ldots, A_{n+1}\}$.

These points are on the unit sphere in $\mathbb{R}^n$ and the distance between any two points is equal to

$$d = \sqrt{2}\sqrt{1 + \frac{1}{n}} > \sqrt{2}.$$

Remark. For $n = 2$ the points form an equilateral triangle in the unit circle; for $n = 3$ the four points from a regular tetrahedron and in $\mathbb{R}^n$ the points from an n dimensional regular simplex.

Problem 4.

Let $A = (a_{k,\ell})_{k,\ell=1,\ldots,n}$ be an $n \times n$ complex matrix such that for each $m \in \{1, \ldots, n\}$ and $1 \leq j_1 < \ldots < j_m \leq n$ the determinant of the matrix $(a_{j_k, j_\ell})_{k,\ell=1,\ldots,m}$ is zero. Prove that $A^n = 0$ and that there exists a permutation $\sigma \in S_n$ such that the matrix

$$(a_{\sigma(k),\sigma(\ell)})_{k,\ell=1,\ldots,n}$$

has all of its nonzero elements above the diagonal.

Solution. We will only prove (2), since it implies (1). Consider a directed graph G with n vertices $V_1, \dots, V_n$ and a directed edge from V_k to V_ℓ when $a_{k,\ell} \neq 0$. We shall prove that it is acyclic.

Assume that there exists a cycle and take one of minimum length m. Let $j_1 < \dots < j_m$ be the vertices the cycle goes through and let $\sigma_0 \in S_n$ be a permutation such that $a_{j_k, j_{\sigma_0(k)}} \neq 0$ for $k = 1, \dots, m$. Observe that for any other $\sigma \in S_n$ we have $a_{j_k, j_{\sigma(k)}} = 0$ for some $k \in \{1, \dots, m\}$, otherwise we would obtain a different cycle through the same set of vertices and, consequently, a shorter cycle. Finally

$$0 = \det(a_{j_k, j_\ell})_{k,\ell=1,\dots,m}$$

$$= (-1)^{\text{sign } \sigma_0} \prod_{k=1}^{m} a_{j_k, j_{\sigma_0(k)}} + \sum_{\sigma \neq \sigma_0} (-1)^{\text{sign } \sigma} \prod_{k=1}^{m} a_{j_k, j_{\sigma(k)}} \neq 0,$$

which is a contradiction.

Since G is acyclic there exists a topological ordering i.e. a permutation $\sigma \in S_n$ such that $k < \ell$ whenever there is an edge from $V_{\sigma(k)}$ to $V_{\sigma(\ell)}$. It is easy to see that this permutation solves the problem.

Problem 5. Let $\mathbb{R}$ be the set of real numbers. Prove that there is no function $f : \mathbb{R} \to \mathbb{R}$ with $f(0) > 0$, and such that

$$f(x+y) \geq f(x) + yf(f(x)) \quad \text{for all } x, y \in \mathbb{R}.$$

Solution. Suppose that there exists a function satisfying the inequality. If $f(f(x)) \leq 0$ for all x, then f is a decreasing function in view of the inequalities $f(x+y) \geq f(x) + yf(f(x)) \geq f(x)$ for any $y \leq 0$. Since $f(0) > 0 \geq f(f(x))$, it implies $f(x) > 0$ for all x, which is a contradiction. Hence there is a z such that $f(f(z)) > 0$. Then the inequality $f(z+x) \geq f(z) + xf(f(z))$ shows that $\lim\limits_{x\to\infty} f(x) = +\infty$ and therefore $\lim\limits_{x\to\infty} f(f(x)) = +\infty$. In particular, there exist $x, y > 0$ such that $f(x) \geq 0$, $f(f(x)) > 1$, $y \geq \frac{x+1}{f(f(x))-1}$ and $f(f(x+y+1)) \geq 0$. Then $f(x+y) \geq f(x) + yf(f(x)) \geq x+y+1$ and hence

$$\begin{aligned} f(f(x+y)) &\geq f(x+y+1) + \big(f(x+y) - (x+y+1)\big)f(f(x+y+1)) \geq \\ &\geq f(x+y+1) \geq f(x+y) + f(f(x+y)) \geq \\ &\geq f(x) + yf(f(x)) + f(f(x+y)) > f(f(x+y)). \end{aligned}$$

This contradiction completes the solution of the problem.

Problem 6.
For each positive integer n, let $f_n(\vartheta) = \sin\vartheta \cdot \sin(2\vartheta) \cdot \sin(4\vartheta) \cdots \sin(2^n\vartheta)$.
For all real ϑ and all n, prove that

$$|f_n(\vartheta)| \le \frac{2}{\sqrt{3}} |f_n(\pi/3)|.$$

Solution. We prove that $g(\vartheta) = |\sin\vartheta||\sin(2\vartheta)|^{1/2}$ attains its maximum value $(\sqrt{3}/2)^{3/2}$ at points $2^k\pi/3$ (where k is a positive integer). This can be seen by using derivatives or a classical bound like

$$|g(\vartheta)| = |\sin\vartheta||\sin(2\vartheta)|^{1/2} = \frac{\sqrt{2}}{\sqrt[4]{3}} \left(\sqrt[4]{|\sin\vartheta| \cdot |\sin\vartheta| \cdot |\sin\vartheta| \cdot |\sqrt{3}\cos\vartheta|} \right)^2$$

$$\le \frac{\sqrt{2}}{\sqrt[4]{3}} \cdot \frac{3\sin^2\vartheta + 3\cos^2\vartheta}{4} = \left(\frac{\sqrt{3}}{2} \right)^{3/2}.$$

Hence

$$\left| \frac{f_n(\vartheta)}{f_n(\pi/3)} \right| = \left| \frac{g(\vartheta) \cdot g(2\vartheta)^{1/2} \cdot g(4\vartheta)^{3/4} \cdots g(2^{n-1}\vartheta)^E}{g(\pi/3) \cdot g(2\pi/3)^{1/2} \cdot g(4\pi/3)^{3/4} \cdots g(2^{n-1}\pi/3)^E} \right| \cdot \left| \frac{\sin(2^n\vartheta)}{\sin(2^n\pi/3)} \right|^{1-E/2}$$

$$\le \left| \frac{\sin(2^n\vartheta)}{\sin(2^n\pi/3)} \right|^{1-E/2} \le \left(\frac{1}{\sqrt{3}/2} \right)^{1-E/2} \le \frac{2}{\sqrt{3}}.$$

where $E = \frac{2}{3}(1 - (-1/2)^n)$. This is exactly the bound we had to prove.

Solutions for problems in the 9^{th} International Mathematics Competition for University Students

Warsaw, July 19 - July 25, 2002

First Day

Problem 1. A standard parabola is the graph of a quadratic polynomial $y = x^2 + ax + b$ with leading coefficient 1. Three standard parabolas with vertices V_1, V_2, V_3 intersect pairwise at points A_1, A_2, A_3. Let $A \mapsto s(A)$ be the reflection of the plane with respect to the x axis.

Prove that standard parabolas with vertices $s(A_1)$, $s(A_2)$, $s(A_3)$ intersect pairwise at the points $s(V_1)$, $s(V_2)$, $s(V_3)$.

Solution. First we show that the standard parabola with vertex V contains point A if and only if the standard parabola with vertex $s(A)$ contains point $s(V)$.

Let $A = (a, b)$ and $V = (v, w)$. The equation of the standard parabola with vertex $V = (v, w)$ is $y = (x - v)^2 + w$, so it contains point A if and only if $b = (a - v)^2 + w$. Similarly, the equation of the parabola with vertex $s(A) = (a, -b)$ is $y = (x - a)^2 - b$; it contains point $s(V) = (v, -w)$ if and only if $-w = (v - a)^2 - b$. The two conditions are equivalent.

Now assume that the standard parabolas with vertices V_1 and V_2, V_1 and V_3, V_2 and V_3 intersect each other at points A_3, A_2, A_1, respectively. Then, by the statement above, the standard parabolas with vertices $s(A_1)$ and $s(A_2)$, $s(A_1)$ and $s(A_3)$, $s(A_2)$ and $s(A_3)$ intersect each other at points V_3, V_2, V_1, respectively, because they contain these points.

Problem 2. Does there exist a continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that for every $x \in \mathbb{R}$ we have $f(x) > 0$ and $f'(x) = f(f(x))$?

Solution. Assume that there exists such a function. Since $f'(x) = f(f(x)) > 0$, the function is strictly monotone increasing.

By the monotonity, $f(x) > 0$ implies $f(f(x)) > f(0)$ for all x. Thus, $f(0)$ is a lower bound for $f'(x)$, and for all $x < 0$ we have $f(x) < f(0) + x \cdot f(0) = (1 + x)f(0)$. Hence, if $x \leq -1$ then $f(x) \leq 0$, contradicting the property $f(x) > 0$.

So such function does not exist.

Problem 3. Let n be a positive integer and let

$$a_k = \frac{1}{\binom{n}{k}}, \quad b_k = 2^{k-n}, \quad for \quad k = 1, 2, \ldots, n.$$

Show that

$$\frac{a_1 - b_1}{1} + \frac{a_2 - b_2}{2} + \cdots + \frac{a_n - b_n}{n} = 0. \tag{1}$$

Solution. Since $k\binom{n}{k} = n\binom{n-1}{k-1}$ for all $k \geq 1$, (1) is equivalent to

$$\frac{2^n}{n}\left[\frac{1}{\binom{n-1}{0}} + \frac{1}{\binom{n-1}{1}} + \cdots + \frac{1}{\binom{n-1}{n-1}}\right] = \frac{2^1}{1} + \frac{2^2}{2} + \cdots + \frac{2^n}{n}. \tag{2}$$

We prove (2) by induction. For $n = 1$, both sides are equal to 2.
Assume that (2) holds for some n. Let

$$x_n = \frac{2^n}{n}\left[\frac{1}{\binom{n-1}{0}} + \frac{1}{\binom{n-1}{1}} + \cdots + \frac{1}{\binom{n-1}{n-1}}\right];$$

then

$$x_{n+1} = \frac{2^{n+1}}{n+1}\sum_{k=0}^{n}\frac{1}{\binom{n}{k}} = \frac{2^n}{n+1}\left(1 + \sum_{k=0}^{n-1}\left(\frac{1}{\binom{n}{k}} + \frac{1}{\binom{n}{k+1}}\right) + 1\right) =$$

$$= \frac{2^n}{n+1}\sum_{k=0}^{n-1}\frac{\frac{n-k}{n} + \frac{k+1}{n}}{\binom{n-1}{k}} + \frac{2^{n+1}}{n+1} = \frac{2^n}{n}\sum_{k=0}^{n-1}\frac{1}{\binom{n-1}{k}} + \frac{2^{n+1}}{n+1} = x_n + \frac{2^{n+1}}{n+1}.$$

This implies (2) for $n + 1$.

Problem 4. Let $f\colon [a, b] \to [a, b]$ be a continuous function and let $p \in [a, b]$. Define $p_0 = p$ and $p_{n+1} = f(p_n)$ for $n = 0, 1, 2, \ldots$ Suppose that the set $T_p = \{p_n\colon n = 0, 1, 2, \ldots\}$ is closed, i.e., if $x \notin T_p$ then there is a $\delta > 0$ such that for all $x' \in T_p$ we have $|x' - x| \geq \delta$. Show that T_p has finitely many elements.

Solution. If for some $n > m$ the equality $p_m = p_n$ holds then T_p is a finite set. Thus we can assume that all points p_0, p_1, ... are distinct. There is a convergent subsequence p_{n_k} and its limit q is in T_p. Since f is continuous $p_{n_k+1} = f(p_{n_k}) \to f(q)$, so all, except for finitely many, points p_n are accumulation points of T_p. Hence we may assume that all of them are accumulation points of T_p. Let $d = \sup\{|p_m - p_n|\colon \quad m, n \geq 0\}$. Let δ_n be positive numbers such that $\sum_{n=0}^{\infty} \delta_n < \frac{d}{2}$. Let I_n be an interval of length less than δ_n centered at p_n such that there are there are infinitely many k's such that $p_k \notin \bigcup_{j=0}^{n} I_j$, this can be done by induction. Let $n_0 = 0$ and n_{m+1} be the smallest integer $k > n_m$ such that $p_k \notin \bigcup_{j=0}^{n_m} I_j$. Since T_p is closed the limit of the subsequence (p_{n_m}) must be in T_p but it is impossible because of the definition of I_n's, of course if the sequence (p_{n_m}) is not convergent we may replace it with its convergent subsequence. The proof is finished.

Remark. If $T_p = \{p_1, p_2, \dots\}$ and each p_n is an accumulation point of T_p, then T_p is the countable union of nowhere dense sets (i.e. the single-element sets $\{p_n\}$). If T is closed then this contradicts the Baire Category Theorem.

Problem 5. Prove or disprove the following statements:
(a) There exists a monotone function $f\colon [0,1] \to [0,1]$ such that for each $y \in [0,1]$ the equation $f(x) = y$ has uncountably many solutions x.
(b) There exists a continuously differentiable function $f\colon [0,1] \to [0,1]$ such that for each $y \in [0,1]$ the equation $f(x) = y$ has uncountably many solutions x.

Solution. a. It does not exist. For each y the set $\{x\colon \quad y = f(x)\}$ is either empty or consists of 1 point or is an interval. These sets are pairwise disjoint, so there are at most countably many of the third type.
b. Let f be such a map. Then for each value y of this map there is an x_0 such that $y = f(x)$ and $f'(x) = 0$, because an uncountable set $\{x\colon \quad y = f(x)\}$ contains an accumulation point x_0 and clearly $f'(x_0) = 0$. For every $\varepsilon > 0$ and every x_0 such that $f'(x_0) = 0$ there exists an open interval I_{x_0} such that if $x \in I_{x_0}$ then $|f'(x)| < \varepsilon$. The union of all these intervals I_{x_0} may be written as a union of pairwise disjoint open intervals J_n. The image of each J_n is an interval (or a point) of length $< \varepsilon \cdot \text{length}(J_n)$ due to Lagrange Mean Value Theorem. Thus the image of the interval $[0,1]$ may be covered with the intervals such that the sum of their lengths is $\varepsilon \cdot 1 = \varepsilon$. This is not possible for $\varepsilon < 1$.

Remarks. 1. The proof of part **b** is essentially the proof of the easy part of A. Sard's theorem about measure of the set of critical values of a smooth map.
2. If only continuity is required, there exists such a function, e.g. the first co-ordinate of the very well known Peano curve which is a continuous map from an interval onto a square.

Problem 6. For an $n \times n$ matrix M with real entries let $\|M\| = \sup_{x \in \mathbb{R}^n \setminus \{0\}} \frac{\|Mx\|_2}{\|x\|_2}$, where $\|\cdot\|_2$ denotes the Euclidean norm on $\mathbb{R}^n$. Assume that an $n \times n$ matrix A with real entries satisfies $\|A^k - A^{k-1}\| \leq \frac{1}{2002k}$ for all positive integers k. Prove that $\|A^k\| \leq 2002$ for all positive integers k.

Solution.

Lemma 1. Let $(a_n)_{n\geq 0}$ be a sequence of non-negative numbers such that $a_{2k} - a_{2k+1} \leq a_k^2$, $a_{2k+1} - a_{2k+2} \leq a_k a_{k+1}$ for any $k \geq 0$ and $\limsup n a_n < 1/4$. Then $\limsup \sqrt[n]{a_n} < 1$.

Proof. Let $c_l = \sup_{n \geq 2^l} (n+1) a_n$ for $l \geq 0$. We will show that $c_{l+1} \leq 4c_l^2$. Indeed, for any integer $n \geq 2^{l+1}$ there exists an integer $k \geq 2^l$ such that $n = 2k$ or $n = 2k+1$. In the first case there is $a_{2k} - a_{2k+1} \leq a_k^2 \leq \frac{c_l^2}{(k+1)^2} \leq \frac{4c_l^2}{2k+1} - \frac{4c_l^2}{2k+2}$, whereas in the second case there is $a_{2k+1} - a_{2k+2} \leq a_k a_{k+1} \leq \frac{c_l^2}{(k+1)(k+2)} \leq \frac{4c_l^2}{2k+2} - \frac{4c_l^2}{2k+3}$.

Hence a sequence $(a_n - \frac{4c_l^2}{n+1})_{n \geq 2^{l+1}}$ is non-decreasing and its terms are non-positive since it converges to zero. Therefore $a_n \leq \frac{4c_l^2}{n+1}$ for $n \geq 2^{l+1}$, meaning that $c_{l+1}^2 \leq 4c_l^2$. This implies that a sequence $((4c_l)^{2^{-l}})_{l \geq 0}$ is non-increasing and therefore bounded from above by some number $q \in (0,1)$ since all its terms except finitely many are less than 1. Hence $c_l \leq q^{2^l}$ for l large enough. For any n between 2^l and 2^{l+1} there is $a_n \leq \frac{c_l}{n+1} \leq q^{2^l} \leq (\sqrt{q})^n$ yielding $\limsup \sqrt[n]{a_n} \leq \sqrt{q} < 1$, yielding $\limsup \sqrt[n]{a_n} \leq \sqrt{q} < 1$, which ends the proof.

Lemma 2. Let T be a linear map from $\mathbb{R}^n$ into itself. Assume that $\limsup n \|T^{n+1} - T^n\| < 1/4$. Then $\limsup \|T^{n+1} - T^n\|^{1/n} < 1$. In particular T^n converges in the operator norm and T is power bounded.

Proof. Put $a_n = \|T^{n+1} - T^n\|$. Observe that

$$T^{k+m+1} - T^{k+m} = (T^{k+m+2} - T^{k+m+1}) - (T^{k+1} - T^k)(T^{m+1} - T^m)$$

implying that $a_{k+m} \leq a_{k+m+1} + a_k a_m$. Therefore the sequence $(a_m)_{m \geq 0}$ satisfies assumptions of Lemma 1 and the assertion of Proposition 1 follows.

Remarks. 1. The theorem proved above holds in the case of an operator T which maps a normed space X into itself, X does not have to be finite dimensional.

2. The constant $1/4$ in Lemma 1 cannot be replaced by any greater number since a sequence $a_n = \frac{1}{4n}$ satisfies the inequality $a_{k+m} - a_{k+m+1} \leq a_k a_m$ for any positive integers k and m whereas it does not have exponential decay.

3. The constant $1/4$ in Lemma 2 cannot be replaced by any number greater that $1/e$. Consider an operator $(Tf)(x) = xf(x)$ on $L^2([0,1])$. One can easily

check that $\limsup \|T^{n+1} - T^n\| = 1/e$, whereas T^n does not converge in the operator norm. The question whether in general $\limsup n\|T^{n+1} - T^n\| < \infty$ implies that T is power bounded remains open.

Remark The problem was incorrectly stated during the competition: instead of the inequality $\|A^k - A^{k-1}\| \le \frac{1}{2002k}$, the inequality $\|A^k - A^{k-1}\| \le \frac{1}{2002n}$ was assumed. If $A = \begin{pmatrix} 1 & \varepsilon \\ 0 & 1 \end{pmatrix}$ then $A^k = \begin{pmatrix} 1 & k\,\varepsilon \\ 0 & 1 \end{pmatrix}$. Therefore $A^k - A^{k-1} = \begin{pmatrix} 0 & \varepsilon \\ 0 & 0 \end{pmatrix}$, so for sufficiently small ε the condition is satisfied although the sequence $\big(\|A^k\|\big)$ is clearly unbounded.

Solutions for problems in the 9^{th} International Mathematics Competition for University Students

Warsaw, July 19 - July 25, 2002

Second Day

Problem 1. Compute the determinant of the $n \times n$ matrix $A = [a_{ij}]$,

$$a_{ij} = \begin{cases} (-1)^{|i-j|}, & if \quad i \neq j, \\ 2, & if \quad i = j. \end{cases}$$

Solution. Adding the second row to the first one, then adding the third row to the second one, ..., adding the nth row to the $(n-1)$th, the determinant does not change and we have

$$\det(A) = \begin{vmatrix} 2 & -1 & +1 & \dots & \pm1 & \mp1 \\ -1 & 2 & -1 & \dots & \mp1 & \pm1 \\ +1 & -1 & 2 & \dots & \pm1 & \mp1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mp1 & \pm1 & \mp1 & \dots & 2 & -1 \\ \pm1 & \mp1 & \pm1 & \dots & -1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 1 \\ \pm1 & \mp1 & \pm1 & \mp1 & \dots & -1 & 2 \end{vmatrix}.$$

Now subtract the first column from the second, then subtract the resulting second column from the third, ..., and at last, subtract the $(n-1)$th column from the nth column. This way we have

$$det(A) = \begin{vmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & n+1 \end{vmatrix} = n+1.$$

Problem 2. Two hundred students participated in a mathematical contest. They had 6 problems to solve. It is known that each problem was correctly solved by at least 120 participants. Prove that there must be two participants such that every problem was solved by at least one of these two students.

Solution. For each pair of students, consider the set of those problems which was not solved by them. There exist $\binom{200}{2} = 19900$ sets; we have to prove that at least one set is empty.

For each problem, there are at most 80 students who did not solve it. From these students at most $\binom{80}{2} = 3160$ pairs can be selected, so the problem can belong to at most 3160 sets. The 6 problems together can belong to at most $6 \cdot 3160 = 18960$ sets.

Hence, at least $19900 - 18960 = 940$ sets must be empty.

Problem 3. For each $n \geq 1$ let

$$a_n = \sum_{k=0}^{\infty} \frac{k^n}{k!}, \quad b_n = \sum_{k=0}^{\infty} (-1)^k \frac{k^n}{k!}.$$

Show that $a_n \cdot b_n$ is an integer.

Solution. We prove by induction on n that a_n/e and $b_n e$ are integers, we prove this for $n = 0$ as well. (For $n = 0$, the term 0^0 in the definition of the sequences must be replaced by 1.)

From the power series of e^x, $a_n = e^1 = e$ and $b_n = e^{-1} = 1/e$.

Suppose that for some $n \geq 0$, $a_0, a_1, \ldots, a_n$ and $b_0, b_1, ...b_n$ are all multipliers of e and $1/e$, respectively. Then, by the binomial theorem,

$$a_{n+1} = \sum_{k=0}^{n} \frac{(k+1)^{n+1}}{(k+1)!} = \sum_{k=0}^{\infty} \frac{(k+1)^n}{k!} = \sum_{k=0}^{\infty} \sum_{m=0}^{n} \binom{n}{m} \frac{k^m}{k!} =$$

$$= \sum_{m=0}^{n} \binom{n}{m} \sum_{k=0}^{\infty} \frac{k^m}{k!} = \sum_{m=0}^{n} \binom{n}{m} a_m$$

and similarly

$$b_{n+1} = \sum_{k=0}^{n} (-1)^{k+1} \frac{(k+1)^{n+1}}{(k+1)!} = -\sum_{k=0}^{\infty} (-1)^k \frac{(k+1)^n}{k!} =$$

$$= -\sum_{k=0}^{\infty} (-1)^k \sum_{m=0}^{n} \binom{n}{m} \frac{k^m}{k!} = -\sum_{m=0}^{n} \binom{n}{m} \sum_{k=0}^{\infty} (-1)^k \frac{k^m}{k!} = -\sum_{m=0}^{n} \binom{n}{m} b_m.$$

The numbers a_{n+1} and b_{n+1} are expressed as linear combinations of the previous elements with integer coefficients which finishes the proof.

Problem 4. In the tetrahedron $OABC$, let $\angle BOC = \alpha$, $\angle COA = \beta$ and $\angle AOB = \gamma$. Let σ be the angle between the faces OAB and OAC, and let τ be the angle between the faces OBA and OBC. Prove that

$$\gamma > \beta \cdot \cos\sigma + \alpha \cdot \cos\tau.$$

Solution. We can assume $OA = OB = OC = 1$. Intersect the unit sphere with center O with the angle domains AOB, BOC and COA; the intersections are "slices" and their areas are $\frac{1}{2}\gamma$, $\frac{1}{2}\alpha$ and $\frac{1}{2}\beta$, respectively.

Now project the slices AOC and COB to the plane OAB. Denote by C' the projection of vertex C, and denote by A' and B' the reflections of vertices A and B with center O, respectively. By the projection, $OC' < 1$.

The projections of arcs AC and BC are segments of ellipses with long axes AA' and BB', respectively. (The ellipses can be degenerate if σ or τ is right angle.) The two ellipses intersect each other in 4 points; both half ellipses connecting A and A' intersect both half ellipses connecting B and B'. There exist no more intersection, because two different conics cannot have more than 4 common points.

The signed areas of the projections of slices AOC and COB are $\frac{1}{2}\alpha \cdot \cos\tau$ and $\frac{1}{2}\beta \cdot \cos\sigma$, respectively. The statement says thet the sum of these signed areas is less than the area of slice BOA.

There are three significantly different cases with respect to the signs of $\cos\sigma$ and $\cos\tau$ (see Figure). If both signs are positive (case (a)), then the projections of slices OAC and OBC are subsets of slice OBC without common interior point, and they do not cover the whole slice OBC; this implies the statement. In cases (b) and (c) where at least one of the signs is negative, projections with positive sign are subsets of the slice OBC, so the statement is obvious again.

Problem 5. Let A be an $n \times n$ matrix with complex entries and suppose that $n > 1$. Prove that

$$A\overline{A} = I_n \iff \exists\, S \in GL_n(\mathbb{C}) \quad \text{such that} \quad A = S\overline{S}^{-1}.$$

(If $A = [a_{ij}]$ then $\overline{A} = [\overline{a_{ij}}]$, where $\overline{a_{ij}}$ is the complex conjugate of a_{ij}; $GL_n(\mathbb{C})$ denotes the set of all $n \times n$ invertible matrices with complex entries, and I_n is the identity matrix.)

Solution. The direction $\Leftarrow$ is trivial, since if $A = S\overline{S}^{-1}$, then $A\overline{A} = S\overline{S}^{-1} \cdot \overline{S}S^{-1} = I_n$.

For the direction $\Rightarrow$, we must prove that there exists an invertible matrix S such that $A\overline{S} = S$.

Let w be an arbitrary complex number which is not 0. Choosing $S = wA + \overline{w}I_n$, we have $A\overline{S} = A(\overline{w}\overline{A} + wI_n) = \overline{w}I_n + wA = S$. If S is singular, then $\frac{1}{w}S = A - (\overline{w}/w)I_n$ is singular as well, so $\overline{w}/w$ is an eigenvalue of A. Since A has finitely many eigenvalues and $\overline{w}/w$ can be any complex number on the unit circle, there exist such w that S is invertible.

Problem 6. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function whose gradient $\nabla f = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right)$ exists at every point of $\mathbb{R}^n$ and satisfies the condition

$$\exists L > 0 \quad \forall x_1, x_2 \in \mathbb{R}^n \quad \|\nabla f(x_1) - \nabla f(x_2)\| \le L\|x_1 - x_2\|.$$

Prove that

$$\forall x_1, x_2 \in \mathbb{R}^n \quad \|\nabla f(x_1) - \nabla f(x_2)\|^2 \le L\langle \nabla f(x_1) - \nabla f(x_2), x_1 - x_2\rangle. \quad (1)$$

In this formula $\langle a, b\rangle$ denotes the scalar product of the vectors a and b.

Solution. Let $g(x) = f(x) - f(x_1) - \langle \nabla f(x_1), x - x_1\rangle$. It is obvious that g has the same properties. Moreover, $g(x_1) = \nabla g(x_1) = 0$ and, due to convexity, g has 0 as the absolute minimum at x_1. Next we prove that

$$g(x_2) \ge \frac{1}{2L}\|\nabla g(x_2)\|^2. \quad (2)$$

Let $y_0 = x_2 - \frac{1}{L}\|\nabla g(x_2)\|$ and $y(t) = y_0 + t(x_2 - y_0)$. Then

$$g(x_2) = g(y_0) + \int_0^1 \langle \nabla g(y(t)), x_2 - y_0\rangle\, dt =$$

$$= g(y_0) + \langle \nabla g(x_2), x_2 - y_0\rangle - \int_0^1 \langle \nabla g(x_2) - \nabla g(y(t)), x_2 - y_0\rangle\, dt \ge$$

$$\ge 0 + \frac{1}{L}\|\nabla g(x_2)\|^2 - \int_0^1 \|\nabla g(x_2) - \nabla g(y(t))\| \cdot \|x_2 - y_0\|\, dt \ge$$

$$\ge \frac{1}{L}\|\nabla g(x_2)\|^2 - \|x_2 - y_0\| \int_0^1 L\|x_2 - g(y)\|\, dt =$$

$$= \frac{1}{L}\|\nabla g(x_2)\|^2 - L\|x_2 - y_0\|^2 \int_0^1 t\, dt = \frac{1}{2L}\|\nabla g(x_2)\|^2.$$

Substituting the definition of g into (2), we obtain

$$f(x_2) - f(x_1) - \langle \nabla f(x_1), x_2 - x_1\rangle \ge \frac{1}{2L}\|\nabla f(x_2) - \nabla f(x_1)\|^2,$$

$$\|\nabla f(x_2) - \nabla f(x_1)\|^2 \le 2L\langle \nabla f(x_1), x_1 - x_2\rangle + 2L(f(x_2) - f(x_1)). \quad (3)$$

Exchanging variables x_1 and x_2, we have

$$\|\nabla f(x_2) - \nabla f(x_1)\|^2 \le 2L\langle \nabla f(x_2), x_2 - x_1\rangle + 2L(f(x_1) - f(x_2)). \quad (4)$$

The statement (1) is the average of (3) and (4).

10^{th} **International Mathematical Competition for University Students**
Cluj-Napoca, July 2003

Day 1

1. (a) Let $a_1, a_2, \ldots$ be a sequence of real numbers such that $a_1 = 1$ and $a_{n+1} > \frac{3}{2}a_n$ for all n. Prove that the sequence

$$\frac{a_n}{\left(\frac{3}{2}\right)^{n-1}}$$

has a finite limit or tends to infinity. (10 points)

(b) Prove that for all $\alpha > 1$ there exists a sequence $a_1, a_2, \ldots$ with the same properties such that

$$\lim \frac{a_n}{\left(\frac{3}{2}\right)^{n-1}} = \alpha.$$

(10 points)

Solution. (a) Let $b_n = \dfrac{a_n}{\left(\frac{3}{2}\right)^{n-1}}$. Then $a_{n+1} > \frac{3}{2}a_n$ is equivalent to $b_{n+1} > b_n$, thus the sequence (b_n) is strictly increasing. Each increasing sequence has a finite limit or tends to infinity.

(b) For all $\alpha > 1$ there exists a sequence $1 = b_1 < b_2 < \ldots$ which converges to α. Choosing $a_n = \left(\frac{3}{2}\right)^{n-1} b_n$, we obtain the required sequence (a_n).

2. Let $a_1, a_2 \ldots, a_{51}$ be non-zero elements of a field. We simultaneously replace each element with the sum of the 50 remaining ones. In this way we get a sequence $b_1 \ldots, b_{51}$. If this new sequence is a permutation of the original one, what can be the characteristic of the field? (The characteristic of a field is p, if p is the smallest positive integer such that $\underbrace{x + x + \ldots + x}_{p} = 0$ for any element x of the field. If there exists no such p, the characteristic is 0.) (20 points)

Solution. Let $S = a_1 + a_2 + \cdots + a_{51}$. Then $b_1 + b_2 + \cdots + b_{51} = 50S$. Since $b_1, b_2, \cdots, b_{51}$ is a permutation of $a_1, a_2, \cdots, a_{51}$, we get $50S = S$, so $49S = 0$. Assume that the characteristic of the field is not equal to 7. Then $49S = 0$ implies that $S = 0$. Therefore $b_i = -a_i$ for $i = 1, 2, \cdots, 51$. On the other hand, $b_i = a_{\varphi(i)}$, where $\varphi \in S_{51}$. Therefore, if the characteristic is not 2, the sequence $a_1, a_2, \cdots, a_{51}$ can be partitioned into pairs $\{a_i, a_{\varphi(i)}\}$ of additive inverses. But this is impossible, since 51 is an odd number. It follows that the characteristic of the field is 7 or 2.

The characteristic can be either 2 or 7. For the case of 7, $x_1 = \ldots = x_{51} = 1$ is a possible choice. For the case of 2, any elements can be chosen such that $S = 0$, since then $b_i = -a_i = a_i$.

3. Let A be an $n \times n$ real matrix such that $3A^3 = A^2 + A + I$ (I is the identity matrix). Show that the sequence A^k converges to an idempotent matrix. (A matrix B is called idempotent if $B^2 = B$.) (20 points)

Solution. The minimal polynomial of A is a divisor of $3x^3 - x^2 - x - 1$. This polynomial has three different roots. This implies that A is diagonalizable: $A = C^{-1}DC$ where D is a diagonal matrix. The eigenvalues of the matrices A and D are all roots of polynomial $3x^3 - x^2 - x - 1$. One of the three roots is 1, the remaining two roots have smaller absolute value than 1. Hence, the diagonal elements of D^k, which are the kth powers of the eigenvalues, tend to either 0 or 1 and the limit $M = \lim D^k$ is idempotent. Then $\lim A^k = C^{-1}MC$ is idempotent as well.

4. Determine the set of all pairs (a, b) of positive integers for which the set of positive integers can be decomposed into two sets A and B such that $a \cdot A = b \cdot B$. (20 points)

Solution. Clearly a and b must be different since A and B are disjoint.

Let $\{a, b\}$ be a solution and consider the sets A, B such that $a \cdot A = b \cdot B$. Denoting $d = (a, b)$ the greatest common divisor of a and b, we have $a = d \cdot a_1,\ b = d \cdot b_1,\ (a_1, b_1) = 1$ and $a_1 \cdot A = b_1 \cdot B$. Thus $\{a_1, b_1\}$ is a solution and it is enough to determine the solutions $\{a, b\}$ with $(a, b) = 1$.

If $1 \in A$ then $a \in a \cdot A = b \cdot B$, thus b must be a divisor of a. Similarly, if $1 \in B$, then a is a divisor of b. Therefore, in all solutions, one of numbers a, b is a divisor of the other one.

Now we prove that if $n \geq 2$, then $(1, n)$ is a solution. For each positive integer k, let $f(k)$ be the largest non-negative integer for which $n^{f(k)} | k$. Then let $A = \{k :\ f(k) \text{ is odd}\}$ and $B = \{k :\ f(k) \text{ is even}\}$. This is a decomposition of all positive integers such that $A = n \cdot B$.

5. Let $g :\ [0, 1] \to \mathbb{R}$ be a continuous function and let $f_n :\ [0, 1] \to \mathbb{R}$ be a sequence of functions defined by $f_0(x) = g(x)$ and

$$f_{n+1}(x) = \frac{1}{x} \int_0^x f_n(t)dt \quad (x \in (0, 1],\ n = 0, 1, 2, \ldots).$$

Determine $\lim\limits_{n\to\infty} f_n(x)$ for every $x \in (0, 1]$. (20 points)

B. We shall prove in two different ways that $\lim_{n\to\infty} f_n(x) = g(0)$ for every $x \in (0, 1]$. (The second one is more lengthy but it tells us how to calculate f_n directly from g.)

Proof I. First we prove our claim for non-decreasing g. In this case, by induction, one can easily see that

1. each f_n is non-decrasing as well, and
2. $g(x) = f_0(x) \geq f_1(x) \geq f_2(x) \geq \ldots \geq g(0) \quad (x \in (0, 1])$.

Then (2) implies that there exists

$$h(x) = \lim_{n\to\infty} f_n(x) \qquad (x \in (0, 1]).$$

Clearly h is non-decreasing and $g(0) \leq h(x) \leq f_n(x)$ for any $x \in (0, 1], n = 0, 1, 2, \ldots$. Therefore to show that $h(x) = g(0)$ for any $x \in (0, 1]$, it is enough to prove that $h(1)$ cannot be greater than $g(0)$.

Suppose that $h(1) > g(0)$. Then there exists a $0 < \delta < 1$ such that $h(1) > g(\delta)$. Using the definition, (2) and (1) we get

$$f_{n+1}(1) = \int_0^1 f_n(t)dt \leq \int_0^\delta g(t)dt + \int_\delta^1 f_n(t)dt \leq \delta g(\delta) + (1 - \delta) f_n(1).$$

Hence

$$f_n(1) - f_{n+1}(1) \geq \delta(f_n(1) - g(\delta)) \geq \delta(h(1) - g(\delta)) > 0,$$

so $f_n(1) \to -\infty$, which is a contradiction.

Similarly, we can prove our claim for non-increasing continuous functions as well.

Now suppose that g is an arbitrary continuous function on $[0, 1]$. Let

$$M(x) = \sup_{t\in[0,x]} g(t), \qquad m(x) = \inf_{t\in[0,x]} g(t) \qquad\qquad (x \in [0, 1])$$

Then on $[0, 1]$ m is non-increasing, M is non-decreasing, both are continuous, $m(x) \leq g(x) \leq M(x)$ and $M(0) = m(0) = g(0)$. Define the sequences of functions $M_n(x)$ and $m_n(x)$ in the same way as f_n is defined but starting with $M_0 = M$ and $m_0 = m$.

Then one can easily see by induction that $m_n(x) \leq f_n(x) \leq M_n(x)$. By the first part of the proof, $\lim_n m_n(x) = m(0) = g(0) = M(0) = \lim_n M_n(x)$ for any $x \in (0, 1]$. Therefore we must have $\lim_n f_n(x) = g(0)$.

Proof II. To make the notation clearer we shall denote the variable of f_j by x_j. By definition (and Fubini theorem) we get that

$$\begin{aligned} f_{n+1}(x_{n+1}) =& \frac{1}{x_{n+1}} \int_0^{x_{n+1}} \frac{1}{x_n} \int_0^{x_n} \frac{1}{x_{n-1}} \int_0^{x_{n-1}} \cdots \int_0^{x_2} \frac{1}{x_1} \int_0^{x_1} g(x_0) dx_0 dx_1 \ldots dx_n \\ =& \frac{1}{x_{n+1}} \iint_{0 \le x_0 \le x_1 \le \ldots \le x_n \le x_{n+1}} g(x_0) \frac{dx_0 dx_1 \ldots dx_n}{x_1 \ldots x_n} \\ =& \frac{1}{x_{n+1}} \int_0^{x_{n+1}} g(x_0) \left(\iint_{x_0 \le x_1 \le \ldots \le x_n \le x_{n+1}} \frac{dx_1 \ldots dx_n}{x_1 \ldots x_n} \right) dx_0. \end{aligned}$$

Therefore with the notation

$$h_n(a,b) = \iint_{a \le x_1 \le \ldots \le x_n \le b} \frac{dx_1 \ldots dx_n}{x_1 \ldots x_n}$$

and $x = x_{n+1}, t = x_0$ we have

$$f_{n+1}(x) = \frac{1}{x} \int_0^x g(t) h_n(t,x) dt.$$

Using that $h_n(a,b)$ is the same for any permutation of $x_1, \ldots, x_n$ and the fact that the integral is 0 on any hyperplanes $(x_i = x_j)$ we get that

$$\begin{aligned} n!\ h_n(a,b) &= \iint_{a \le x_1, \ldots, x_n \le b} \frac{dx_1 \ldots dx_n}{x_1 \ldots x_n} = \int_a^b \cdots \int_a^b \frac{dx_1 \ldots dx_n}{x_1 \ldots x_n} \\ &= \left(\int_a^b \frac{dx}{x} \right)^n = (\log(b/a))^n. \end{aligned}$$

Therefore

$$f_{n+1}(x) = \frac{1}{x} \int_0^x g(t) \frac{(\log(x/t))^n}{n!} dt.$$

Note that if g is constant then the definition gives $f_n = g$. This implies on one hand that we must have

$$\frac{1}{x} \int_0^x \frac{(\log(x/t))^n}{n!} dt = 1$$

and on the other hand that, by replacing g by $g - g(0)$, we can suppose that $g(0) = 0$.

Let $x \in (0,1]$ and $\varepsilon > 0$ be fixed. By continuity there exists a $0 < \delta < x$ and an M such that $|g(t)| < \varepsilon$ on $[0,\delta]$ and $|g(t)| \le M$ on $[0,1]$. Since

$$\lim_{n \to \infty} \frac{(\log(x/\delta))^n}{n!} = 0$$

there exists an n_0 sucht that $\log(x/\delta))^n/n! < \varepsilon$ whenever $n \ge n_0$. Then, for any $n \ge n_0$, we have

$$\begin{aligned} |f_{n+1}(x)| &\le \frac{1}{x} \int_0^x |g(t)| \frac{(\log(x/t))^n}{n!} dt \\ &\le \frac{1}{x} \int_0^\delta \varepsilon \frac{(\log(x/t))^n}{n!} dt + \frac{1}{x} \int_\delta^x |g(t)| \frac{(\log(x/\delta))^n}{n!} dt \\ &\le \frac{1}{x} \int_0^x \varepsilon \frac{(\log(x/t))^n}{n!} dt + \frac{1}{x} \int_\delta^x M \varepsilon dt \\ &\le \varepsilon + M\varepsilon. \end{aligned}$$

Therefore $\lim_n f(x) = 0 = g(0)$.

6. Let $f(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$ be a polynomial with real coefficients. Prove that if all roots of f lie in the left half-plane $\{z \in \mathbb{C} : \text{ Re } z < 0\}$ then

$$a_k a_{k+3} < a_{k+1} a_{k+2}$$

holds for every $k = 0, 1, \ldots, n-3$. (20 points)

Solution. The polynomial f is a product of linear and quadratic factors, $f(z) = \prod_i (k_i z + l_i) \cdot \prod_j (p_j z^2 + q_j z + r_j)$, with $k_i, l_i, p_j, q_j, r_j \in \mathbb{R}$. Since all roots are in the left half-plane, for each i, k_i and l_i are of the same sign, and for each j, p_j, q_j, r_j are of the same sign, too. Hence, multiplying f by -1 if necessary, the roots of f don't change and f becomes the polynomial with all positive coefficients.

For the simplicity, we extend the sequence of coefficients by $a_{n+1} = a_{n+2} = \ldots = 0$ and $a_{-1} = a_{-2} = \ldots = 0$ and prove the same statement for $-1 \le k \le n-2$ by induction.

For $n \le 2$ the statement is obvious: a_{k+1} and a_{k+2} are positive and at least one of a_{k-1} and a_{k+3} is 0; hence, $a_{k+1} a_{k+2} > a_k a_{k+3} = 0$.

Now assume that $n \ge 3$ and the statement is true for all smaller values of n. Take a divisor of $f(z)$ which has the form $z^2 + pz + q$ where p and q are positive real numbers. (Such a divisor can be obtained from a conjugate pair of roots or two real roots.) Then we can write

$$f(z) = (z^2 + pz + q)(b_{n-2} z^{n-2} + \ldots + b_1 z + b_0) = (z^2 + pz + q) g(x). \qquad (1)$$

The roots polynomial $g(z)$ are in the left half-plane, so we have $b_{k+1} b_{k+2} < b_k b_{k+3}$ for all $-1 \le k \le n-4$. Defining $b_{n-1} = b_n = \ldots = 0$ and $b_{-1} = b_{-2} = \ldots = 0$ as well, we also have $b_{k+1} b_{k+2} \le b_k b_{k+3}$ for all integer k.

Now we prove $a_{k+1} a_{k+2} > a_k a_{k+3}$. If $k = -1$ or $k = n-2$ then this is obvious since $a_{k+1} a_{k+2}$ is positive and $a_k a_{k+3} = 0$. Thus, assume $0 \le k \le n-3$. By an easy computation,

$$a_{k+1} a_{k+2} - a_k a_{k+3} =$$

$$= (qb_{k+1} + pb_k + b_{k-1})(qb_{k+2} + pb_{k+1} + b_k) - (qb_k + pb_{k-1} + b_{k-2})(qb_{k+3} + pb_{k+2} + b_{k+1}) =$$
$$= (b_{k-1} b_k - b_{k-2} b_{k+1}) + p(b_k^2 - b_{k-2} b_{k+2}) + q(b_{k-1} b_{k+2} - b_{k-2} b_{k+3}) +$$
$$+ p^2 (b_k b_{k+1} - b_{k-1} b_{k+2}) + q^2 (b_{k+1} b_{k+2} - b_k b_{k+3}) + pq(b_{k+1}^2 - b_{k-1} b_{k+3}).$$

We prove that all the six terms are non-negative and at least one is positive. Term $p^2(b_k b_{k+1} - b_{k-1} b_{k+2})$ is positive since $0 \le k \le n-3$. Also terms $b_{k-1} b_k - b_{k-2} b_{k+1}$ and $q^2(b_{k+1} b_{k+2} - b_k b_{k+3})$ are non-negative by the induction hypothesis.

To check the sign of $p(b_k^2 - b_{k-2} b_{k+2})$ consider

$$b_{k-1}(b_k^2 - b_{k-2} b_{k+2}) = b_{k-2}(b_k b_{k+1} - b_{k-1} b_{k+2}) + b_k (b_{k-1} b_k - b_{k-2} b_{k+1}) \ge 0.$$

If $b_{k-1} > 0$ we can divide by it to obtain $b_k^2 - b_{k-2} b_{k+2} \ge 0$. Otherwise, if $b_{k-1} = 0$, either $b_{k-2} = 0$ or $b_{k+2} = 0$ and thus $b_k^2 - b_{k-2} b_{k+2} = b_k^2 \ge 0$. Therefore, $p(b_k^2 - b_{k-2} b_{k+2}) \ge 0$ for all k. Similarly, $pq(b_{k+1}^2 - b_{k-1} b_{k+3}) \ge 0$.

The sign of $q(b_{k-1} b_{k+2} - b_{k-2} b_{k+3})$ can be checked in a similar way. Consider

$$b_{k+1}(b_{k-1} b_{k+2} - b_{k-2} b_{k+3}) = b_{k-1}(b_{k+1} b_{k+2} - b_k b_{k+3}) + b_{k+3}(b_{k-1} b_k - b_{k-2} b_{k+1}) \ge 0.$$

If $b_{k+1} > 0$, we can divide by it. Otherwise either $b_{k-2} = 0$ or $b_{k+3} = 0$. In all cases, we obtain $b_{k-1} b_{k+2} - b_{k-2} b_{k+3} \ge 0$.

Now the signs of all terms are checked and the proof is complete.

10^{th} **International Mathematical Competition for University Students**
Cluj-Napoca, July 2003

Day 2

1. Let A and B be $n \times n$ real matrices such that $AB + A + B = 0$. Prove that $AB = BA$.

Solution. Since $(A+I)(B+I) = AB + A + B + I = I$ (I is the identity matrix), matrices $A + I$ and $B + I$ are inverses of each other. Then $(A + I)(B + I) = (B + I)(A + I)$ and $AB + BA$.

2. Evaluate the limit

$$\lim_{x\to 0+} \int_x^{2x} \frac{\sin^m t}{t^n} dt \quad (m, n \in \mathbb{N}).$$

Solution. We use the fact that $\dfrac{\sin t}{t}$ is decreasing in the interval $(0, \pi)$ and $\lim\limits_{t\to 0+0} \dfrac{\sin t}{t} = 1$. For all $x \in (0, \frac{\pi}{2})$ and $t \in [x, 2x]$ we have $\dfrac{\sin 2x}{2} x < \dfrac{\sin t}{t} < 1$, thus

$$\left(\frac{\sin 2x}{2x}\right)^m \int_x^{2x} \frac{t^m}{t^n} < \int_x^{2x} \frac{\sin^m t}{t^n} dt < \int_x^{2x} \frac{t^m}{t^n} dt,$$

$$\int_x^{2x} \frac{t^m}{t^n} dt = x^{m-n+1} \int_1^2 u^{m-n} du.$$

The factor $\left(\dfrac{\sin 2x}{2x}\right)^m$ tends to 1. If $m - n + 1 < 0$, the limit of x^{m-n+1} is infinity; if $m - n + 1 > 0$ then 0. If $m - n + 1 = 0$ then $x^{m-n+1} \int_1^2 u^{m-n} du = \ln 2$. Hence,

$$\lim_{x\to 0+0} \int_x^{2x} \frac{\sin^m t}{t^n} dt = \begin{cases} 0, & m \geq n \\ \ln 2, & n - m = 1 \\ +\infty, & n - m > 1. \end{cases}.$$

3. Let A be a closed subset of $\mathbb{R}^n$ and let B be the set of all those points $b \in \mathbb{R}^n$ for which there exists exactly one point $a_0 \in A$ such that

$$|a_0 - b| = \inf_{a\in A} |a - b|.$$

Prove that B is dense in $\mathbb{R}^n$; that is, the closure of B is $\mathbb{R}^n$.

Solution. Let $b_0 \notin A$ (otherwise $b_0 \in A \subset B$), $\varrho = \inf\limits_{a\in A} |a - b_0|$. The intersection of the ball of radius $\varrho + 1$ with centre b_0 with set A is compact and there exists $a_0 \in A$: $|a_0 - b_0| = \varrho$.

Denote by $\mathbf{B}_r(a) = \{x \in R^n : \ |x-a| \le r\}$ and $\partial\mathbf{B}_r(a) = \{x \in R^n : \ |x-a| = r\}$ the ball and the sphere of center a and radius r, respectively.

If a_0 is not the unique nearest point then for any point a on the open line segment (a_0, b_0) we have $\mathbf{B}_{|a-a_0|}(a) \subset \mathbf{B}_\varrho(b_0)$ and $\partial\mathbf{B}_{|a-a_0|}(a) \bigcap \partial\mathbf{B}_\varrho(b_0) = \{a_0\}$, therefore $(a_0, b_0) \subset B$ and b_0 is an accumulation point of set B.

4. Find all positive integers n for which there exists a family $\mathcal{F}$ of three-element subsets of $S = \{1, 2, \ldots, n\}$ satisfying the following two conditions:

(i) for any two different elements $a, b \in S$, there exists exactly one $A \in \mathcal{F}$ containing both a, b;

(ii) if a, b, c, x, y, z are elements of S such that if $\{a, b, x\}, \{a, c, y\}, \{b, c, z\} \in \mathcal{F}$, then $\{x, y, z\} \in \mathcal{F}$.

Solution. The condition (i) of the problem allows us to define a (well-defined) operation $*$ on the set S given by

$$a * b = c \text{ if and only if } \{a, b, c\} \in F, \text{ where } a \neq b.$$

We note that this operation is still not defined completely (we need to define $a * a$), but nevertheless let us investigate its features. At first, due to (i), for $a \neq b$ the operation obviously satisfies the following three conditions:

(a) $a \neq a * b \neq b$;

(b) $a * b = b * a$;

(c) $a * (a * b) = b$.

What does the condition (ii) give? It claims that

(e') $x * (a * c) = x * y = z = b * c = (x * a) * c$

for any three different x, a, c, i.e. that the operation is associative if the arguments are different. Now we can complete the definition of $*$. In order to save associativity for non-different arguments, i.e. to make $b = a * (a * b) = (a * a) * b$ hold, we will add to S an extra element, call it 0, and define

(d) $a * a = 0$ and $a * 0 = 0 * a = a$.

Now it is easy to check that, for any $a, b, c \in S \cup \{0\}$, (a),(b),(c) and (d), still hold, and

(e) $a * b * c := (a * b) * c = a * (b * c)$.

We have thus obtained that $(S \cup \{0\}, *)$ has the structure of a finite Abelian group, whose elements are all of order two. Since the order of every such group is a power of 2, we conclude that $|S \cup \{0\}| = n + 1 = 2^m$ and $n = 2^m - 1$ for some integer $m \ge 1$.

Given $n = 2^m - 1$, according to what we have proven till now, we will construct a family of three-element subsets of S satisfying (i) and (ii). Let us define the operation $*$ in the following manner:

if $a = a_0 + 2a_1 + \ldots + 2^{m-1}a_{m-1}$ and $b = b_0 + 2b_1 + \ldots + 2^{m-1}b_{m-1}$, where a_i, b_i are either 0 or 1, we put $a * b = |a_0 - b_0| + 2|a_1 - b_1| + \ldots + 2^{m-1}|a_{m-1} - b_{m-1}|$.

It is simple to check that this $*$ satisfies (a),(b),(c) and (e'). Therefore, if we include in F all possible triples $a, b, a*b$, the condition (i) follows from (a),(b) and (c), whereas the condition (ii) follows from (e')

The answer is: $n = 2^m - 1$.

5. (a) Show that for each function $f : \mathbb{Q} \times \mathbb{Q} \to \mathbb{R}$ there exists a function $g : \mathbb{Q} \to \mathbb{R}$ such that $f(x, y) \le g(x) + g(y)$ for all $x, y \in \mathbb{Q}$.

(b) Find a function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ for which there is no function $g : \mathbb{R} \to \mathbb{R}$ such that $f(x, y) \le g(x) + g(y)$ for all $x, y \in \mathbb{R}$.

Solution. a) Let $\varphi : \mathbb{Q} \to \mathbb{N}$ be a bijection. Define $g(x) = \max\{|f(s,t)| : s, t \in \mathbb{Q}, \varphi(s) \le \varphi(x), \varphi(t) \le \varphi(x)\}$. We have $f(x, y) \le \max\{g(x), g(y)\} \le g(x) + g(y)$.

b) We shall show that the function defined by $f(x, y) = \frac{1}{|x-y|}$ for $x \ne y$ and $f(x, x) = 0$ satisfies the problem. If, by contradiction there exists a function g as above, it results, that $g(y) \ge \frac{1}{|x-y|} - f(x)$ for $x, y \in \mathbb{R}$, $x \ne y$; one obtains that for each $x \in \mathbb{R}$, $\lim\limits_{y \to x} g(y) = \infty$. We show, that there exists no function g having an infinite limit at each point of a bounded and closed interval $[a, b]$.

For each $k \in \mathbb{N}^+$ denote $A_k = \{x \in [a, b] : |g(x)| \le k\}$.

We have obviously $[a, b] = \cup_{k=1}^{\infty} A_k$. The set $[a, b]$ is uncountable, so at least one of the sets A_k is infinite (in fact uncountable). This set A_k being infinite, there exists a sequence in A_k having distinct terms. This sequence will contain a convergent subsequence $(x_n)_{n \in \mathbb{N}}$ convergent to a point $x \in [a, b]$. But $\lim\limits_{y \to x} g(y) = \infty$ implies that $g(x_n) \to \infty$, a contradiction because $|g(x_n)| \le k$, $\forall n \in \mathbb{N}$.

Second solution for part (b). Let S be the set of all sequences of real numbers. The cardinality of S is $|S| = |\mathbb{R}|^{\aleph_0} = 2^{\aleph_0^2} = 2^{\aleph_0} = |\mathbb{R}|$. Thus, there exists a bijection $h : \mathbb{R} \to S$. Now define the function f in the following way. For any real x and positive integer n, let $f(x, n)$ be the nth element of sequence $h(x)$. If y is not a positive integer then let $f(x, y) = 0$. We prove that this function has the required property.

Let g be an arbitrary $\mathbb{R} \to \mathbb{R}$ function. We show that there exist real numbers x, y such that $f(x, y) > g(x) + g(y)$. Consider the sequence $(n + g(n))_{n=1}^{\infty}$. This sequence is an element of S, thus $(n + g(n))_{n=1}^{\infty} = h(x)$ for a certain real x. Then for an arbitrary positive integer n, $f(x, n)$ is the nth element, $f(x, n) = n + g(n)$. Choosing n such that $n > g(x)$, we obtain $f(x, n) = n + g(n) > g(x) + g(n)$.

6. Let $(a_n)_{n \in \mathbb{N}}$ be the sequence defined by

$$a_0 = 1, \;\; a_{n+1} = \frac{1}{n+1} \sum_{k=0}^{n} \frac{a_k}{n-k+2}.$$

Find the limit

$$\lim_{n \to \infty} \sum_{k=0}^{n} \frac{a_k}{2^k},$$

if it exists.

Solution. Consider the generating function $f(x) = \sum_{n=0}^{\infty} a_n x^n$. By induction $0 < a_n \le 1$, thus this series is absolutely convergent for $|x| < 1$, $f(0) = 1$ and the function is positive in the interval $[0, 1)$. The goal is to compute $f(\frac{1}{2})$.

By the recurrence formula,

$$f'(x) = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{a_k}{n-k+2}x^n =$$

$$= \sum_{k=0}^{\infty} a_k x^k \sum_{n=k}^{\infty}\frac{x^{n-k}}{n-k+2} = f(x)\sum_{m=0}^{\infty}\frac{x^m}{m+2}.$$

Then

$$\ln f(x) = \ln f(x) - \ln f(0) = \int_0^x \frac{f'}{f} = \sum_{m=0}^{\infty}\frac{x^{m+1}}{(m+1)(m+2)} =$$

$$= \sum_{m=0}^{\infty}\left(\frac{x^{m+1}}{(m+1)} - \frac{x^{m+1}}{(m+2)}\right) = 1 + \left(1 - \frac{1}{x}\right)\sum_{m=0}^{\infty}\frac{x^{m+1}}{(m+1)} = 1 + \left(1 - \frac{1}{x}\right)\ln\frac{1}{1-x},$$

$$\ln f\left(\frac{1}{2}\right) = 1 - \ln 2,$$

and thus $f(\frac{1}{2}) = \frac{e}{2}$.

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Solutions for problems on Day 1

Problem 1. Let S be an infinite set of real numbers such that $|s_1 + s_2 + \cdots + s_k| < 1$ for every finite subset $\{s_1, s_2, \ldots, s_k\} \subset S$. Show that S is countable. [20 points]

Solution. Let $S_n = S \cap (\frac{1}{n}, \infty)$ for any integer $n > 0$. It follows from the inequality that $|S_n| < n$. Similarly, if we define $S_{-n} = S \cap (-\infty, -\frac{1}{n})$, then $|S_{-n}| < n$. Any nonzero $x \in S$ is an element of some S_n or S_{-n}, because there exists an n such that $x > \frac{1}{n}$, or $x < -\frac{1}{n}$. Then $S \subset \{0\} \cup \bigcup\limits_{n \in N} (S_n \cup S_{-n})$, S is a countable union of finite sets, and hence countable.

Problem 2. Let $P(x) = x^2 - 1$. How many distinct real solutions does the following equation have:

$$\underbrace{P(P(\ldots(P}_{2004}(x))\ldots)) = 0 \ ? \qquad \text{[20 points]}$$

Solution. Put $P_n(x) = \underbrace{P(P(...(P}_{n}(x))...))$. As $P_1(x) \geq -1$, for each $x \in R$, it must be that $P_{n+1}(x) = P_1(P_n(x)) \geq -1$, for each $n \in N$ and each $x \in R$. Therefore the equation $P_n(x) = a$, where $a < -1$ has no real solutions. Let us prove that the equation $P_n(x) = a$, where $a > 0$, has exactly two distinct real solutions. To this end we use mathematical induction by n. If $n = 1$ the assertion follows directly. Assuming that the assertion holds for a $n \in N$ we prove that it must also hold for $n + 1$. Since $P_{n+1}(x) = a$ is equivalent to $P_1(P_n(x)) = a$, we conclude that $P_n(x) = \sqrt{a+1}$ or $P_n(x) = -\sqrt{a+1}$. The equation $P_n(x) = \sqrt{a+1}$, as $\sqrt{a+1} > 1$, has exactly two distinct real solutions by the inductive hypothesis, while the equation $P_n(x) = -\sqrt{a+1}$ has no real solutions (because $-\sqrt{a+1} < -1$). Hence the equation $P_{n+1}(x) = a$, has exactly two distinct real solutions.

Let us prove now that the equation $P_n(x) = 0$ has exactly $n + 1$ distinct real solutions. Again we use mathematical induction. If $n = 1$ the solutions are $x = \pm 1$, and if $n = 2$ the solutions are $x = 0$ and $x = \pm\sqrt{2}$, so in both cases the number of solutions is equal to $n + 1$. Suppose that the assertion holds for some $n \in N$. Note that $P_{n+2}(x) = P_2(P_n(x)) = P_n^2(x)(P_n^2(x) - 2)$, so the set of all real solutions of the equation $P_{n+2} = 0$ is exactly the union of the sets of all real solutions of the equations $P_n(x) = 0$, $P_n(x) = \sqrt{2}$ and $P_n(x) = -\sqrt{2}$. By the inductive hypothesis the equation $P_n(x) = 0$ has exactly $n + 1$ distinct real solutions, while the equations $P_n(x) = \sqrt{2}$ and $P_n(x) = -\sqrt{2}$ have two and no distinct real solutions, respectively. Hence, the sets above being pairwise disjoint, the equation $P_{n+2}(x) = 0$ has exactly $n + 3$ distinct real solutions. Thus we have proved that, for each $n \in N$, the equation $P_n(x) = 0$ has exactly $n + 1$ distinct real solutions, so the answer to the question posed in this problem is 2005.

Problem 3. Let S_n be the set of all sums $\sum\limits_{k=1}^{n} x_k$, where $n \geq 2$, $0 \leq x_1, x_2, \ldots, x_n \leq \frac{\pi}{2}$ and

$$\sum_{k=1}^{n} \sin x_k = 1\,.$$

a) Show that S_n is an interval. [10 points]
b) Let l_n be the length of S_n. Find $\lim\limits_{n\to\infty} l_n$. [10 points]

Solution. (a) Equivalently, we consider the set

$$Y = \{y = (y_1, y_2, ..., y_n) |\ 0 \leq y_1, y_2, ..., y_n \leq 1,\ y_1 + y_2 + ... + y_n = 1\} \subset R^n$$

and the image $f(Y)$ of Y under

$$f(y) = \arcsin y_1 + \arcsin y_2 + ... + \arcsin y_n.$$

Note that $f(Y) = S_n$. Since Y is a connected subspace of R^n and f is a continuous function, the image $f(Y)$ is also connected, and we know that the only connected subspaces of R are intervals. Thus S_n is an interval.

(b) We prove that

$$n \ \arcsin \frac{1}{n} \leq x_1 + x_2 + ... + x_n \leq \frac{\pi}{2}.$$

Since the graph of $\sin x$ is concave down for $x \in [0, \frac{\pi}{2}]$, the chord joining the points $(0, 0)$ and $(\frac{\pi}{2}, 1)$ lies below the graph. Hence

$$\frac{2x}{\pi} \leq \sin x \text{ for all } x \in [0, \frac{\pi}{2}]$$

and we can deduce the right-hand side of the claim:

$$\frac{2}{\pi}(x_1 + x_2 + ... + x_n) \leq \sin x_1 + \sin x_2 + ... + \sin x_n = 1.$$

The value 1 can be reached choosing $x_1 = \frac{\pi}{2}$ and $x_2 = \cdots = x_n = 0$.

The left-hand side follows immediately from Jensen's inequality, since $\sin x$ is concave down for $x \in [0, \frac{\pi}{2}]$ and $0 \leq \frac{x_1+x_2+...+x_n}{n} < \frac{\pi}{2}$

$$\frac{1}{n} = \frac{\sin x_1 + \sin x_2 + ... + \sin x_n}{n} \leq \sin \frac{x_1 + x_2 + ... + x_n}{n}.$$

Equality holds if $x_1 = \cdots = x_n = \arcsin \frac{1}{n}$.

Now we have computed the minimum and maximum of interval S_n; we can conclude that $S_n = [n \ \arcsin \frac{1}{n}, \frac{\pi}{2}]$. Thus $l_n = \frac{\pi}{2} - n \ \arcsin \frac{1}{n}$ and

$$\lim_{n\to\infty} l_n = \frac{\pi}{2} - \lim_{n\to\infty} \frac{\arcsin(1/n)}{1/n} = \frac{\pi}{2} - 1.$$

Problem 4. Suppose $n \geq 4$ and let M be a finite set of n points in $\mathbb{R}^3$, no four of which lie in a plane. Assume that the points can be coloured black or white so that any sphere which intersects M in at least four points has the property that exactly half of the points in the intersection of M and the sphere are white. Prove that all of the points in M lie on one sphere. [20 points]

Solution. Define $f : M \to \{-1, 1\}$, $f(X) = \begin{cases} -1, & \text{if } X \text{ is white} \\ 1, & \text{if } X \text{ is black} \end{cases}$. The given condition becomes $\sum_{X \in S} f(X) = 0$ for any sphere S which passes through at least 4 points of M. For any 3 given points A, B, C in M, denote by $S(A, B, C)$ the set of all spheres which pass through A, B, C and at least one other point of M and by $|S(A, B, C)|$ the number of these spheres. Also, denote by $\sum$ the sum $\sum_{X \in M} f(X)$.

We have

$$0 = \sum_{S \in S(A,B,C)} \sum_{X \in S} f(X) = (|S(A, B, C)| - 1)(f(A) + f(B) + f(C)) + \sum \qquad (1)$$

since the values of A, B, C appear $|S(A, B, C)|$ times each and the other values appear only once.

If there are 3 points A, B, C such that $|S(A, B, C)| = 1$, the proof is finished.

If $|S(A, B, C)| > 1$ for any distinct points A, B, C in M, we will prove at first that $\sum = 0$.

Assume that $\sum > 0$. From (1) it follows that $f(A) + f(B) + f(C) < 0$ and summing by all $\binom{n}{3}$ possible choices of (A, B, C) we obtain that $\binom{n}{3} \sum < 0$, which means $\sum < 0$ (contradicts the starting assumption). The same reasoning is applied when assuming $\sum < 0$.

Now, from $\sum = 0$ and (1), it follows that $f(A) + f(B) + f(C) = 0$ for any distinct points A, B, C in M. Taking another point $D \in M$, the following equalities take place

$$f(A) + f(B) + f(C) = 0$$
$$f(A) + f(B) + f(D) = 0$$
$$f(A) + f(C) + f(D) = 0$$
$$f(B) + f(C) + f(D) = 0$$

which easily leads to $f(A) = f(B) = f(C) = f(D) = 0$, which contradicts the definition of f.

Problem 5. Let X be a set of $\binom{2k-4}{k-2} + 1$ real numbers, $k \geq 2$. Prove that there exists a monotone sequence $\{x_i\}_{i=1}^{k} \subseteq X$ such that

$$|x_{i+1} - x_1| \geq 2|x_i - x_1|$$

for all $i = 2, \ldots, k - 1$. [20 points]

Solution. We prove a more general statement:

Lemma. Let $k, l \geq 2$, let X be a set of $\binom{k+l-4}{k-2}+1$ real numbers. Then either X contains an increasing sequence $\{x_i\}_{i=1}^k \subseteq X$ of length k and

$$|x_{i+1} - x_1| \geq 2|x_i - x_1| \quad \forall i = 2, \ldots, k-1,$$

or X contains a decreasing sequence $\{x_i\}_{i=1}^l \subseteq X$ of length l and

$$|x_{i+1} - x_1| \geq 2|x_i - x_1| \quad \forall i = 2, \ldots, l-1.$$

Proof of the lemma. We use induction on $k + l$. In case $k = 2$ or $l = 2$ the lemma is obviously true.

Now let us make the induction step. Let m be the minimal element of X, M be its maximal element. Let

$$X_m = \{x \in X : x \leq \frac{m+M}{2}\}, \quad X_M = \{x \in X : x > \frac{m+M}{2}\}.$$

Since $\binom{k+l-4}{k-2} = \binom{k+(l-1)-4}{k-2} + \binom{(k-1)+l-4}{(k-1)-2}$, we can see that either

$$|X_m| \geq \binom{(k-1)+l-4}{(k-1)-2} + 1, \quad \text{or} \quad |X_M| \geq \binom{k+(l-1)-4}{k-2} + 1.$$

In the first case we apply the inductive assumption to X_m and either obtain a decreasing sequence of length l with the required properties (in this case the inductive step is made), or obtain an increasing sequence $\{x_i\}_{i=1}^{k-1} \subseteq X_m$ of length $k-1$. Then we note that the sequence $\{x_1, x_2, \ldots, x_{k-1}, M\} \subseteq X$ has length k and all the required properties.

In the case $|X_M| \geq \binom{k+(l-1)-4}{k-2} + 1$ the inductive step is made in a similar way. Thus the lemma is proved.

The reader may check that the number $\binom{k+l-4}{k-2} + 1$ cannot be smaller in the lemma.

Problem 6. For every complex number $z \notin \{0, 1\}$ define

$$f(z) := \sum (\log z)^{-4},$$

where the sum is over all branches of the complex logarithm.

a) Show that there are two polynomials P and Q such that $f(z) = P(z)/Q(z)$ for all $z \in \mathbb{C} \setminus \{0, 1\}$. [10 points]

b) Show that for all $z \in \mathbb{C} \setminus \{0, 1\}$

$$f(z) = z\frac{z^2+4z+1}{6(z-1)^4}. \qquad \text{[10 points]}$$

Solution 1. It is clear that the left hand side is well defined and independent of the order of summation, because we have a sum of the type $\sum n^{-4}$, and the branches of the logarithms do not matter because all branches are taken. It is easy to check that the convergence is locally uniform on $\mathbb{C} \setminus \{0, 1\}$; therefore, f is a holomorphic function on the complex plane, except possibly for isolated singularities at 0 and 1. (We omit the detailed estimates here.)

The function log has its only (simple) zero at $z = 1$, so f has a quadruple pole at $z = 1$.

Now we investigate the behavior near infinity. We have $\mathrm{Re}(\log(z)) = \log|z|$, hence (with $c := \log|z|$)

$$\begin{aligned}
|\sum (\log z)^{-4}| &\leq \sum |\log z|^{-4} = \sum (\log|z| + 2\pi i n)^{-4} + O(1) \\
&= \int_{-\infty}^{\infty} (c + 2\pi i x)^{-4}\, dx + O(1) \\
&= c^{-4} \int_{-\infty}^{\infty} (1 + 2\pi i x/c)^{-4}\, dx + O(1) \\
&= c^{-3} \int_{-\infty}^{\infty} (1 + 2\pi i t)^{-4}\, dt + O(1) \\
&\leq \alpha(\log|z|)^{-3}
\end{aligned}$$

for a universal constant α. Therefore, the infinite sum tends to 0 as $|z| \to \infty$. In particular, the isolated singularity at ∞ is not essential, but rather has (at least a single) zero at ∞.

The remaining singularity is at $z=0$. It is readily verified that $f(1/z)=f(z)$ (because $\log(1/z)=-\log(z)$); this implies that f has a zero at $z=0$.

We conclude that the infinite sum is holomorphic on $\mathbb{C}$ with at most one pole and without an essential singularity at ∞, so it is a rational function, i.e. we can write $f(z)=P(z)/Q(z)$ for some polynomials P and Q which we may as well assume coprime. This solves the first part.

Since f has a quadruple pole at $z=1$ and no other poles, we have $Q(z)=(z-1)^4$ up to a constant factor which we can as well set equal to 1, and this determines P uniquely. Since $f(z)\to 0$ as $z\to\infty$, the degree of P is at most 3, and since $P(0)=0$, it follows that $P(z)=z(az^2+bz+c)$ for yet undetermined complex constants a,b,c.

There are a number of ways to compute the coefficients a,b,c, which turn out to be $a=c=1/6$, $b=2/3$. Since $f(z)=f(1/z)$, it follows easily that $a=c$. Moreover, the fact $\lim_{z\to 1}(z-1)^4 f(z)=1$ implies $a+b+c=1$ (this fact follows from the observation that at $z=1$, all summands cancel pairwise, except the principal branch which contributes a quadruple pole). Finally, we can calculate

$$f(-1)=\pi^{-4}\sum_{n odd} n^{-4}=2\pi^{-4}\sum_{n\geq 1 odd} n^{-4}=2\pi^{-4}\left(\sum_{n\geq 1} n^{-4}-\sum_{n\geq 1 even} n^{-4}\right)=\frac{1}{48}\,.$$

This implies $a-b+c=-1/3$. These three equations easily yield a,b,c.

Moreover, the function f satisfies $f(z)+f(-z)=16f(z^2)$: this follows because the branches of $\log(z^2)=\log((-z)^2)$ are the numbers $2\log(z)$ and $2\log(-z)$. This observation supplies the two equations $b=4a$ and $a=c$, which can be used instead of some of the considerations above.

Another way is to compute $g(z)=\sum\frac{1}{(\log z)^2}$ first. In the same way, $g(z)=\frac{dz}{(z-1)^2}$. The unknown coefficient d can be computed from $\lim_{z\to 1}(z-1)^2 g(z)=1$; it is $d=1$. Then the exponent 2 in the denominator can be increased by taking derivatives (see Solution 2). Similarly, one can start with exponent 3 directly.

A more straightforward, though tedious way to find the constants is computing the first four terms of the Laurent series of f around $z=1$. For that branch of the logarithm which vanishes at 1, for all $|w|<\frac{1}{2}$ we have

$$\log(1+w)=w-\frac{w^2}{2}+\frac{w^3}{3}-\frac{w^4}{4}+O(|w|^5);$$

after some computation, one can obtain

$$\frac{1}{\log(1+w)^4}=w^{-4}+2w^{-2}+\frac{7}{6}w^{-2}+\frac{1}{6}w^{-1}+O(1).$$

The remaining branches of logarithm give a bounded function. So

$$f(1+w)=w^{-4}+2w^{-2}+\frac{7}{6}w^{-2}+\frac{1}{6}w^{-1}$$

(the remainder vanishes) and

$$f(z)=\frac{1+2(z-1)+\frac{7}{6}(z-1)^2+\frac{1}{6}(z-1)^3}{(z-1)^4}=\frac{z(z^2+4z+1)}{6(z-1)^4}.$$

Solution 2. ¿From the well-known series for the cotangent function,

$$\lim_{N\to\infty}\sum_{k=-N}^{N}\frac{1}{w+2\pi i\cdot k}=\frac{i}{2}\cot\frac{iw}{2}$$

and

$$\lim_{N\to\infty}\sum_{k=-N}^{N}\frac{1}{\log z+2\pi i\cdot k}=\frac{i}{2}\cot\frac{i\log z}{2}=\frac{i}{2}\cdot i\frac{e^{2i\cdot\frac{i\log z}{2}}+1}{e^{2i\cdot\frac{i\log z}{2}}-1}=\frac{1}{2}+\frac{1}{z-1}.$$

Taking derivatives we obtain

$$\sum\frac{1}{(\log z)^2}=-z\cdot\left(\frac{1}{2}+\frac{1}{z-1}\right)'=\frac{z}{(z-1)^2},$$

$$\sum\frac{1}{(\log z)^3}=-\frac{z}{2}\cdot\left(\frac{z}{(z-1)^2}\right)'=\frac{z(z+1)}{2(z-1)^3}$$

and

$$\sum\frac{1}{(\log z)^4}=-\frac{z}{3}\cdot\left(\frac{z(z+1)}{2(z-1)^3}\right)'=\frac{z(z^2+4z+1)}{2(z-1)^4}.$$

11^{th} International Mathematical Competition for University Students
Skopje, 25–26 July 2004

Solutions for problems on Day 2

1. Let A be a real 4×2 matrix and B be a real 2×4 matrix such that

$$AB = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}.$$

Find BA. [20 points]

Solution. Let $A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$ and $B = \begin{pmatrix} B_1 & B_2 \end{pmatrix}$ where A_1, A_2, B_1, B_2 are 2×2 matrices. Then

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \begin{pmatrix} B_1 & B_2 \end{pmatrix} = \begin{pmatrix} A_1B_1 & A_1B_2 \\ A_2B_1 & A_2B_2 \end{pmatrix}$$

therefore, $A_1B_1 = A_2B_2 = I_2$ and $A_1B_2 = A_2B_1 = -I_2$. Then $B_1 = A_1^{-1}$, $B_2 = -A_1^{-1}$ and $A_2 = B_2^{-1} = -A_1$. Finally,

$$BA = \begin{pmatrix} B_1 & B_2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = B_1A_1 + B_2A_2 = 2I_2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

2. Let $f, g \colon [a,b] \to [0,\infty)$ be continuous and non-decreasing functions such that for each $x \in [a,b]$ we have

$$\int_a^x \sqrt{f(t)}\,dt \le \int_a^x \sqrt{g(t)}\,dt$$

and $\int_a^b \sqrt{f(t)}\,dt = \int_a^b \sqrt{g(t)}\,dt$.

Prove that $\int_a^b \sqrt{1+f(t)}\,dt \ge \int_a^b \sqrt{1+g(t)}\,dt$. [20 points]

Solution. Let $F(x) = \int_a^x \sqrt{f(t)}\,dt$ and $G(x) = \int_a^x \sqrt{g(t)}\,dt$. The functions F, G are convex, $F(a) = 0 = G(a)$ and $F(b) = G(b)$ by the hypothesis. We are supposed to show that

$$\int_a^b \sqrt{1 + \left(F'(t)\right)^2}\,dt \ge \int_a^b \sqrt{1 + \left(G'(t)\right)^2}\,dt$$

i.e. The length ot the graph of F is $\ge$ the length of the graph of G. This is clear since both functions are convex, their graphs have common ends and the graph of F is *below* the graph of G — the length of the graph of F is the least upper bound of the lengths of the graphs of piecewise linear functions whose values at the points of non-differentiability coincide with the values of F, if a convex polygon P_1 is contained in a polygon P_2 then the perimeter of P_1 is $\le$ the perimeter of P_2.

3. Let D be the closed unit disk in the plane, and let $p_1, p_2, \ldots, p_n$ be fixed points in D. Show that there exists a point p in D such that the sum of the distances of p to each of $p_1, p_2, \ldots, p_n$ is greater than or equal to 1. [20 points]

Solution. considering as vectors, thoose p to be the unit vector which points into the opposite direction as $\sum_{i=1}^n p_i$. Then, by the triangle inequality,

$$\sum_{i=1}^n |p - p_i| \ge \left| np - \sum_{i=1}^n p_i \right| = n + \left| \sum_{i=1}^n p_i \right| \ge n..$$

4. For $n \geq 1$ let M be an $n \times n$ complex matrix with distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_k$, with multiplicities $m_1, m_2, \dots, m_k$, respectively. Consider the linear operator L_M defined by $L_M(X) = MX + XM^T$, for any complex $n \times n$ matrix X. Find its eigenvalues and their multiplicities. (M^T denotes the transpose of M; that is, if $M = (m_{k,l})$, then $M^T = (m_{l,k})$.) [20 points]

Solution. We first solve the problem for the special case when the eigenvalues of M are distinct and all sums $\lambda_r + \lambda_s$ are different. Let λ_r and λ_s be two eigenvalues of M and $\vec{v}_r$, $\vec{v}_s$ eigenvectors associated to them, i.e. $M\vec{v}_j = \lambda\vec{v}_j$ for $j = r, s$. We have $M\vec{v}_r(\vec{v}_s)^T + \vec{v}_r(\vec{v}_s)^T M^T = (M\vec{v}_r)(\vec{v}_s)^T + \vec{v}_r\big(M\vec{v}_s\big)^T = \lambda_r\vec{v}_r(\vec{v}_s)^T + \lambda_s\vec{v}_r(\vec{v}_s)^T$, so $\vec{v}_r(\vec{v}_s)$ is an eigenmatrix of L_M with the eigenvalue $\lambda_r + \lambda_s$.

Notice that if $\lambda_r \neq \lambda_s$ then vectors $\vec{u}, \vec{w}$ are linearly independent and matrices $\vec{u}(\vec{w})^T$ and $\vec{w}(\vec{u})^T$ are linearly independent, too. This implies that the eigenvalue $\lambda_r + \lambda_s$ is double if $r \neq s$.

The map L_M maps n^2–dimensional linear space into itself, so it has at most n^2 eigenvalues. We already found n^2 eigenvalues, so there exists no more and the problem is solved for the special case.

In the general case, matrix M is a limit of matrices $M_1, M_2, \dots$ such that each of them belongs to the special case above. By the continuity of the eigenvalues we obtain that the eigenvalues of L_M are

- $2\lambda_r$ with multiplicity m_r^2 ($r = 1, \dots, k$);
- $\lambda_r + \lambda_s$ with multiplicity $2m_r m_s$ ($1 \leq r < s \leq k$).

(It can happen that the sums $\lambda_r + \lambda_s$ are not pairwise different; for those multiple values the multiplicities should be summed up.)

5. Prove that

$$\int_0^1 \int_0^1 \frac{dx\, dy}{x^{-1} + |\ln y| - 1} \leq 1. \quad \text{[20 points]}$$

Solution 1. First we use the inequality

$$x^{-1} - 1 \geq |\ln x|, \ x \in (0, 1],$$

which follows from

$$(x^{-1} - 1)\big|_{x=1} = |\ln x|\big|_{x=1} = 0,$$
$$(x^{-1} - 1)' = -\frac{1}{x^2} \leq -\frac{1}{x} = |\ln x|', \ x \in (0, 1].$$

Therefore

$$\int_0^1 \int_0^1 \frac{dx\, dy}{x^{-1} + |\ln y| - 1} \leq \int_0^1 \int_0^1 \frac{dx\, dy}{|\ln x| + |\ln y|} = \int_0^1 \int_0^1 \frac{dx\, dy}{|\ln(x \cdot y)|}.$$

Substituting $y = u/x$, we obtain

$$\int_0^1 \int_0^1 \frac{dx\, dy}{|\ln(x \cdot y)|} = \int_0^1 \left(\int_u^1 \frac{dx}{x} \right) \frac{du}{|\ln u|} = \int_0^1 |\ln u| \cdot \frac{du}{|\ln u|} = 1.$$

Solution 2. Substituting $s = x^{-1} - 1$ and $u = s - \ln y$,

$$\int_0^1 \int_0^1 \frac{dx\, dy}{x^{-1} + |\ln y| - 1} = \int_0^\infty \int_s^\infty \frac{e^{s-u}}{(s+1)^2 u} du ds = \int_0^\infty \left(\int_0^u \frac{e^s}{(s+1)^2} ds \right) \frac{e^{-u}}{u} ds du.$$

Since the function $\frac{e^s}{(s+1)^2}$ is convex,

$$\int_0^u \frac{e^s}{(s+1)^2} ds \leq \frac{u}{2} \left(\frac{e^u}{(u+1)^2} + 1 \right)$$

so

$$\int_0^1 \int_0^1 \frac{dx\, dy}{x^{-1} + |\ln y| - 1} \leq \int_0^\infty \frac{u}{2} \left(\frac{e^u}{(u+1)^2} + 1 \right) \frac{e^{-u}}{u} du = \frac{1}{2} \left(\int_0^\infty \frac{du}{(u+1)^2} + \int_0^\infty e^{-u} du \right) = 1.$$

6. For $n \geq 0$ define matrices A_n and B_n as follows: $A_0 = B_0 = (1)$ and for every $n > 0$

$$A_n = \begin{pmatrix} A_{n-1} & A_{n-1} \\ A_{n-1} & B_{n-1} \end{pmatrix} \text{ and } B_n = \begin{pmatrix} A_{n-1} & A_{n-1} \\ A_{n-1} & 0 \end{pmatrix}.$$

Denote the sum of all elements of a matrix M by $S(M)$. Prove that $S(A_n^{k-1}) = S(A_k^{n-1})$ for every $n, k \geq 1$. [20 points]

Solution. The quantity $S(A_n^{k-1})$ has a special combinatorical meaning. Consider an $n \times k$ table filled with 0's and 1's such that no 2×2 contains only 1's. Denote the number of such fillings by F_{nk}. The filling of each row of the table corresponds to some integer ranging from 0 to $2^n - 1$ written in base 2. F_{nk} equals to the number of k-tuples of integers such that every two consecutive integers correspond to the filling of $n \times 2$ table without 2×2 squares filled with 1's.

Consider binary expansions of integers i and j $\overline{i_n i_{n-1} \dots i_1}$ and $\overline{j_n j_{n-1} \dots j_1}$. There are two cases:

1. If $i_n j_n = 0$ then i and j can be consecutive iff $\overline{i_{n-1} \dots i_1}$ and $\overline{j_{n-1} \dots j_1}$ can be consequtive.
2. If $i_n = j_n = 1$ then i and j can be consecutive iff $i_{n-1} j_{n-1} = 0$ and $\overline{i_{n-2} \dots i_1}$ and $\overline{j_{n-2} \dots j_1}$ can be consecutive.

Hence i and j can be consecutive iff $(i+1, j+1)$-th entry of A_n is 1. Denoting this entry by $a_{i,j}$, the sum $S(A_n^{k-1}) = \sum_{i_1=0}^{2^n-1} \cdots \sum_{i_k=0}^{2^n-1} a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_{k-1} i_k}$ counts the possible fillings. Therefore $F_{nk} = S(A_n^{k-1})$.

The the obvious statement $F_{nk} = F_{kn}$ completes the proof.

12^{th} International Mathematics Competition for University Students

Blagoevgrad, July 22 - July 28, 2005

First Day

Problem 1. Let A be the $n\times n$ matrix, whose $(i,j)^{\text{th}}$ entry is $i+j$ for all $i,j=1,2,\ldots,n$. What is the rank of A?

Solution 1. For $n=1$ the rank is 1. Now assume $n\ge 2$. Since $A=(i)_{i,j=1}^n+(j)_{i,j=1}^n$, matrix A is the sum of two matrixes of rank 1. Therefore, the rank of A is at most 2. The determinant of the top-left 2×2 minor is -1, so the rank is exactly 2.

Therefore, the rank of A is 1 for $n=1$ and 2 for $n\ge 2$.

Solution 2. Consider the case $n\ge 2$. For $i=n,n-1,\ldots,2$, subtract the $(i-1)^{\text{th}}$ row from the n^{th} row. Then subtract the second row from all lower rows.

$$rank\begin{pmatrix} 2 & 3 & \ldots & n+1\\ 3 & 4 & \ldots & n+2\\ \vdots & & \ddots & \vdots\\ n+1 & n+2 & \ldots & 2n\end{pmatrix} = rank\begin{pmatrix} 2 & 3 & \ldots & n+1\\ 1 & 1 & \ldots & 1\\ \vdots & & \ddots & \vdots\\ 1 & 1 & \ldots & 1\end{pmatrix} = rank\begin{pmatrix} 1 & 2 & \ldots & n\\ 1 & 1 & \ldots & 1\\ 0 & 0 & \ldots & 0\\ \vdots & & \ddots & \vdots\\ 0 & 0 & \ldots & 0\end{pmatrix} = 2.$$

Problem 2. For an integer $n\ge 3$ consider the sets

$$S_n=\{(x_1,x_2,\ldots,x_n):\forall i\ x_i\in\{0,1,2\}\}$$

$$A_n=\{(x_1,x_2,\ldots,x_n)\in S_n:\forall i\le n-2\ \ |\{x_i,x_{i+1},x_{i+2}\}|\ne 1\}$$

and

$$B_n=\{(x_1,x_2,\ldots,x_n)\in S_n:\forall i\le n-1\ (x_i=x_{i+1}\Rightarrow x_i\ne 0)\}.$$

Prove that $|A_{n+1}|=3\cdot|B_n|$.

($|A|$ denotes the number of elements of the set A.)

Solution 1. Extend the definitions also for $n=1,2$. Consider the following sets

$$A'_n=\{(x_1,x_2,\ldots,x_n)\in A_n:x_{n-1}=x_n\},\quad A''_n=A_n\setminus A'_n,$$

$$B'_n=\{(x_1,x_2,\ldots,x_n)\in B_n:x_n=0\},\quad B''_n=B_n\setminus B'_n$$

and denote $a_n=|A_n|$, $a'_n=|A'_n|$, $a''_n=|A''_n|$, $b_n=|B_n|$, $b'_n=|B'_n|$, $b''_n=|B''_n|$.

It is easy to observe the following relations between the a–sequences

$$\begin{cases} a_n &= a'_n+a''_n\\ a'_{n+1} &= a''_n\\ a''_{n+1} &= 2a'_n+2a''_n\end{cases},$$

which lead to $a_{n+1}=2a_n+2a_{n-1}$.

For the b–sequences we have the same relations

$$\begin{cases} b_n &= b'_n+b''_n\\ b'_{n+1} &= b''_n\\ b''_{n+1} &= 2b'_n+2b''_n\end{cases},$$

therefore $b_{n+1}=2b_n+2b_{n-1}$.

By computing the first values of (a_n) and (b_n) we obtain

$$\begin{cases} a_1=3,\quad a_2=9,\quad a_3=24\\ \qquad\quad b_1=3,\quad b_2=8\end{cases}$$

which leads to

$$\begin{cases} a_2 = 3b_1 \\ a_3 = 3b_2 \end{cases}$$

Now, reasoning by induction, it is easy to prove that $a_{n+1} = 3b_n$ for every $n \geq 1$.
Solution 2. Regarding x_i to be elements of $\mathbb{Z}_3$ and working "modulo 3", we have that

$$(x_1, x_2, \ldots, x_n) \in A_n \Rightarrow (x_1 + 1, x_2 + 1, \ldots, x_n + 1) \in A_n,\ (x_1 + 2, x_2 + 2, \ldots, x_n + 2) \in A_n$$

which means that 1/3 of the elements of A_n start with 0. We establish a bijection between the subset of all the vectors in A_{n+1} which start with 0 and the set B_n by

$$(0, x_1, x_2, \ldots, x_n) \in A_{n+1} \longmapsto (y_1, y_2, \ldots, y_n) \in B_n$$
$$y_1 = x_1, y_2 = x_2 - x_1, y_3 = x_3 - x_2, \ldots, y_n = x_n - x_{n-1}$$

(if $y_k = y_{k+1} = 0$ then $x_k - x_{k-1} = x_{k+1} - x_k = 0$ (where $x_0 = 0$), which gives $x_{k-1} = x_k = x_{k+1}$, which is not possible because of the definition of the sets A_p; therefore, the definition of the above function is correct).

The inverse is defined by

$$(y_1, y_2, \ldots, y_n) \in B_n \longmapsto (0, x_1, x_2, \ldots, x_n) \in A_{n+1}$$
$$x_1 = y_1, x_2 = y_1 + y_2, \ldots, x_n = y_1 + y_2 + \cdots + y_n$$

Problem 3. Let $f : \mathbb{R} \to [0, \infty)$ be a continuously differentiable function. Prove that

$$\left| \int_0^1 f^3(x)\,dx - f^2(0) \int_0^1 f(x)\,dx \right| \leq \max_{0 \leq x \leq 1} |f'(x)| \left(\int_0^1 f(x)\,dx \right)^2.$$

Solution 1. Let $M = \max\limits_{0 \leq x \leq 1} |f'(x)|$. By the inequality $-M \leq f'(x) \leq M$, $x \in [0, 1]$ it follows:

$$-Mf(x) \leq f(x) f'(x) \leq Mf(x),\ x \in [0, 1].$$

By integration

$$-M \int_0^x f(t)\,dt \leq \frac{1}{2} f^2(x) - \frac{1}{2} f^2(0) \leq M \int_0^x f(t)\,dt,\ x \in [0, 1]$$
$$-Mf(x) \int_0^x f(t)\,dt \leq \frac{1}{2} f^3(x) - \frac{1}{2} f^2(0) f(x) \leq Mf(x) \int_0^x f(t)\,dt,\ x \in [0, 1].$$

Integrating the last inequality on $[0, 1]$ it follows that

$$-M \left(\int_0^1 f(x)dx \right)^2 \leq \int_0^1 f^3(x)\,dx - f^2(0) \int_0^1 f(x)\,dx \leq M \left(\int_0^1 f(x)dx \right)^2 \Leftrightarrow$$
$$\left| \int_0^1 f^3(x)\,dx - f^2(0) \int_0^1 f(x)\,dx \right| \leq M \left(\int_0^1 f(x)\,dx \right)^2.$$

Solution 2. Let $M = \max\limits_{0 \leq x \leq 1} |f'(x)|$ and $F(x) = -\int_x^1 f$; then $F' = f$, $F(0) = -\int_0^1 f$ and $F(1) = 0$. Integrating by parts,

$$\int_0^1 f^3 = \int_0^1 f^2 \cdot F' = [f^2 F]_0^1 - \int_0^1 (f^2)' F =$$
$$= f^2(1)F(1) - f^2(0)F(0) - \int_0^1 2Fff' = f^2(0) \int_0^1 f - \int_0^1 2Fff'.$$

Then

$$\left| \int_0^1 f^3(x)\,dx - f^2(0) \int_0^1 f(x)\,dx \right| = \left| \int_0^1 2Fff' \right| \leq \int_0^1 2Ff|f'| \leq M \int_0^1 2Ff = M \cdot [F^2]_0^1 = M \left(\int_0^1 f \right)^2.$$

Problem 4. Find all polynomials $P(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$ $(a_n \neq 0)$ satisfying the following two conditions:

(i) $(a_0, a_1, ..., a_n)$ is a permutation of the numbers $(0, 1, ..., n)$

and

(ii) all roots of $P(x)$ are rational numbers.

Solution 1. Note that $P(x)$ does not have any positive root because $P(x) > 0$ for every $x > 0$. Thus, we can represent them in the form $-\alpha_i$, $i = 1, 2, \ldots, n$, where $\alpha_i \geq 0$. If $a_0 \neq 0$ then there is a $k \in \mathbb{N}, 1 \leq k \leq n-1$, with $a_k = 0$, so using Viete's formulae we get

$$\alpha_1\alpha_2...\alpha_{n-k-1}\alpha_{n-k} + \alpha_1\alpha_2...\alpha_{n-k-1}\alpha_{n-k+1} + ... + \alpha_{k+1}\alpha_{k+2}...\alpha_{n-1}\alpha_n = \frac{a_k}{a_n} = 0,$$

which is impossible because the left side of the equality is positive. Therefore $a_0 = 0$ and one of the roots of the polynomial, say α_n, must be equal to zero. Consider the polynomial $Q(x) = a_n x^{n-1} + a_{n-1}x^{n-2} + ... + a_1$. It has zeros $-\alpha_i$, $i = 1, 2, \ldots, n-1$. Again, Viete's formulae, for $n \geq 3$, yield:

$$\alpha_1\alpha_2...\alpha_{n-1} = \frac{a_1}{a_n} \tag{1}$$

$$\alpha_1\alpha_2...\alpha_{n-2} + \alpha_1\alpha_2...\alpha_{n-3}\alpha_{n-1} + ... + \alpha_2\alpha_3...\alpha_{n-1} = \frac{a_2}{a_n} \tag{2}$$

$$\alpha_1 + \alpha_2 + ... + \alpha_{n-1} = \frac{a_{n-1}}{a_n}. \tag{3}$$

Dividing (2) by (1) we get

$$\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + ... + \frac{1}{\alpha_{n-1}} = \frac{a_2}{a_1}. \tag{4}$$

From (3) and (4), applying the AM-HM inequality we obtain

$$\frac{a_{n-1}}{(n-1)a_n} = \frac{\alpha_1 + \alpha_2 + ... + \alpha_{n-1}}{n-1} \geq \frac{n-1}{\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + ... + \frac{1}{\alpha_{n-1}}} = \frac{(n-1)a_1}{a_2},$$

therefore $\frac{a_2 a_{n-1}}{a_1 a_n} \geq (n-1)^2$. Hence $\frac{n^2}{2} \geq \frac{a_2 a_{n-1}}{a_1 a_n} \geq (n-1)^2$, implying $n \leq 3$. So, the only polynomials possibly satisfying (i) and (ii) are those of degree at most three. These polynomials can easily be found and they are $P(x) = x$, $P(x) = x^2 + 2x$, $P(x) = 2x^2 + x$, $P(x) = x^3 + 3x^2 + 2x$ and $P(x) = 2x^3 + 3x^2 + x$. $\square$

Solution 2. Consider the prime factorization of P in the ring $\mathbb{Z}[x]$. Since all roots of P are rational, P can be written as a product of n linear polynomials with rational coefficients. Therefore, all prime factor of P are linear and P can be written as

$$P(x) = \prod_{k=1}^{n}(b_k x + c_k)$$

where the coefficients b_k, c_k are integers. Since the leading coefficient of P is positive, we can assume $b_k > 0$ for all k. The coefficients of P are nonnegative, so P cannot have a positive root. This implies $c_k \geq 0$. It is not possible that $c_k = 0$ for two different values of k, because it would imply $a_0 = a_1 = 0$. So $c_k > 0$ in at least $n-1$ cases.

Now substitute $x = 1$.

$$P(1) = a_n + \cdots + a_0 = 0 + 1 + \cdots + n = \frac{n(n+1)}{2} = \prod_{k=1}^{n}(b_k + c_k) \geq 2^{n-1};$$

therefore it is necessary that $2^{n-1} \leq \frac{n(n+1)}{2}$, therefore $n \leq 4$. Moreover, the number $\frac{n(n+1)}{2}$ can be written as a product of $n-1$ integers greater than 1.

If $n = 1$, the only solution is $P(x) = 1x + 0$.

If $n = 2$, we have $P(1) = 3 = 1 \cdot 3$, so one factor must be x, the other one is $x + 2$ or $2x + 1$. Both $x(x+2) = 1x^2 + 2x + 0$ and $x(2x+1) = 2x^2 + 1x + 0$ are solutions.

If $n=3$, then $P(1)=6=1\cdot2\cdot3$, so one factor must be x, another one is $x+1$, the third one is again $x+2$ or $2x+1$. The two polynomials are $x(x+1)(x+2)=1x^3+3x^2+2x+0$ and $x(x+1)(2x+1)=2x^3+3x^2+1x+0$, both have the proper set of coefficients.

In the case $n=4$, there is no solution because $\frac{n(n+1)}{2}=10$ cannot be written as a product of 3 integers greater than 1.

Altogether we found 5 solutions: $1x+0$, $1x^2+2x+0$, $2x^2+1x+0$, $1x^3+3x^2+2x+0$ and $2x^3+3x^2+1x+0$.

Problem 5. Let $f:\ (0,\infty)\to\mathbb{R}$ be a twice continuously differentiable function such that

$$|f''(x)+2xf'(x)+(x^2+1)f(x)|\le 1$$

for all x. Prove that $\lim\limits_{x\to\infty} f(x)=0$.

Solution 1. Let $g(x)=f'(x)+xf(x)$; then $f''(x)+2xf'(x)+(x^2+1)f(x)=g'(x)+xg(x)$.

We prove that if h is a continuously differentiable function such that $h'(x)+xh(x)$ is bounded then $\lim\limits_{\infty} h=0$. Applying this lemma for $h=g$ then for $h=f$, the statement follows.

Let M be an upper bound for $|h'(x)+xh(x)|$ and let $p(x)=h(x)e^{x^2/2}$. (The function $e^{-x^2/2}$ is a solution of the differential equation $u'(x)+xu(x)=0$.) Then

$$|p'(x)|=|h'(x)+xh(x)|e^{x^2/2}\le Me^{x^2/2}$$

and

$$|h(x)|=\left|\frac{p(x)}{e^{x^2/2}}\right|=\left|\frac{p(0)+\int_0^x p'}{e^{x^2/2}}\right|\le\frac{|p(0)|+M\int_0^x e^{x^2/2}dx}{e^{x^2/2}}.$$

Since $\lim\limits_{x\to\infty} e^{x^2/2}=\infty$ and $\lim\dfrac{\int_0^x e^{x^2/2}dx}{e^{x^2/2}}=0$ (by L'Hospital's rule), this implies $\lim_{x\to\infty} h(x)=0$.

Solution 2. Apply L'Hospital rule twice on the fraction $\dfrac{f(x)e^{x^2/2}}{e^{x^2/2}}$. (Note that L'Hospital rule is valid if the denominator converges to infinity, without any assumption on the numerator.)

$$\lim_{x\to\infty} f(x)=\lim_{x\to\infty}\frac{f(x)e^{x^2/2}}{e^{x^2/2}}=\lim_{x\to\infty}\frac{(f'(x)+xf(x))e^{x^2/2}}{xe^{x^2/2}}=\lim_{x\to\infty}\frac{(f''(x)+2xf'(x)+(x^2+1)f(x))e^{x^2/2}}{(x^2+1)e^{x^2/2}}=$$

$$=\lim_{x\to\infty}\frac{f''(x)+2xf'(x)+(x^2+1)f(x)}{x^2+1}=0.$$

Problem 6. Given a group G, denote by $G(m)$ the subgroup generated by the m^{th} powers of elements of G. If $G(m)$ and $G(n)$ are commutative, prove that $G(\gcd(m,n))$ is also commutative. ($\gcd(m,n)$ denotes the greatest common divisor of m and n.)

Solution. Write $d=\gcd(m,n)$. It is easy to see that $\langle G(m),G(n)\rangle=G(d)$; hence, it will suffice to check commutativity for any two elements in $G(m)\cup G(n)$, and so for any two generators a^m and b^n. Consider their commutator $z=a^{-m}b^{-n}a^mb^n$; then the relations

$$z=(a^{-m}ba^m)^{-n}b^n=a^{-m}(b^{-n}ab^n)^m$$

show that $z\in G(m)\cap G(n)$. But then z is in the center of $G(d)$. Now, from the relation $a^mb^n=b^na^mz$, it easily follows by induction that

$$a^{ml}b^{nl}=b^{nl}a^{ml}z^{l^2}.$$

Setting $l=m/d$ and $l=n/d$ we obtain $z^{(m/d)^2}=z^{(n/d)^2}=e$, but this implies that $z=e$ as well.

12^{th} International Mathematics Competition for University Students

Blagoevgrad, July 22 - July 28, 2005

Second Day

Problem 1. Let $f(x) = x^2 + bx + c$, where b and c are real numbers, and let

$$M = \{x \in \mathbb{R} : \ |f(x)| < 1\}.$$

Clearly the set M is either empty or consists of disjoint open intervals. Denote the sum of their lengths by $|M|$. Prove that

$$|M| \le 2\sqrt{2}.$$

Solution. Write $f(x) = \left(x + \frac{b}{2}\right)^2 + d$ where $d = c - \frac{b^2}{4}$. The absolute minimum of f is d.

If $d \ge 1$ then $f(x) \ge 1$ for all x, $M = \emptyset$ and $|M| = 0$.

If $-1 < d < 1$ then $f(x) > -1$ for all x,

$$-1 < \left(x + \frac{b}{2}\right)^2 + d < 1 \quad \iff \quad \left|x + \frac{b}{2}\right| < \sqrt{1-d}$$

so

$$M = \left(-\frac{b}{2} - \sqrt{1-d}, -\frac{b}{2} + \sqrt{1-d}\right)$$

and

$$|M| = 2\sqrt{1-d} < 2\sqrt{2}.$$

If $d \le -1$ then

$$-1 < \left(x + \frac{b}{2}\right)^2 + d < 1 \quad \iff \quad \sqrt{|d|-1} < \left|x + \frac{b}{2}\right| < \sqrt{|d|+1}$$

so

$$M = \left(-\sqrt{|d|+1}, -\sqrt{|d|-1}\right) \cup \left(\sqrt{|d|-1}, \sqrt{|d|+1}\right)$$

and

$$|M| = 2\left(\sqrt{|d|+1} - \sqrt{|d|-1}\right) = 2\frac{(|d|+1) - (|d|-1)}{\sqrt{|d|+1} + \sqrt{|d|-1}} \le 2\frac{2}{\sqrt{1+1} + \sqrt{1-0}} = 2\sqrt{2}.$$

Problem 2. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $(f(x))^n$ is a polynomial for every $n = 2, 3, \ldots$. Does it follow that f is a polynomial?

Solution 1. Yes, it is even enough to assume that f^2 and f^3 are polynomials.

Let $p = f^2$ and $q = f^3$. Write these polynomials in the form of

$$p = a \cdot p_1^{a_1} \cdot \ldots \cdot p_k^{a_k}, \qquad q = b \cdot q_1^{b_1} \cdot \ldots \cdot q_l^{b_l},$$

where $a, b \in \mathbb{R}$, $a_1, \dots, a_k, b_1, \dots b_l$ are positive integers and $p_1, \dots, p_k, q_1, \dots, q_l$ are irreducible polynomials with leading coefficients 1. For $p^3 = q^2$ and the factorisation of $p^3 = q^2$ is unique we get that $a^3 = b^2$, $k = l$ and for some $(i_1, \dots, i_k)$ permutation of $(1, \dots, k)$ we have $p_1 = q_{i_1}, \dots, p_k = q_{i_k}$ and $3a_1 = 2b_{i_1}, \dots, 3a_k = 2b_{i_k}$. Hence $b_1, \dots, b_l$ are divisible by 3 let $r = b^{1/3} \cdot q_1^{b_1/3} \cdot \dots \cdot q_l^{b_l/3}$ be a polynomial. Since $r^3 = q = f^3$ we have $f = r$.

Solution 2. Let $\frac{p}{q}$ be the simplest form of the rational function $\frac{f^3}{f^2}$. Then the simplest form of its square is $\frac{p^2}{q^2}$. On the other hand $\frac{p^2}{q^2} = \left(\frac{f^3}{f^2}\right)^2 = f^2$ is a polynomial therefore q must be a constant and so $f = \frac{f^3}{f^2} = \frac{p}{q}$ is a polynomial.

Problem 3. In the linear space of all real $n \times n$ matrices, find the maximum possible dimension of a linear subspace V such that

$$\forall X, Y \in V \quad \text{trace}(XY) = 0.$$

(The trace of a matrix is the sum of the diagonal entries.)

Solution. If A is a nonzero symmetric matrix, then $\text{trace}(A^2) = \text{trace}(A^t A)$ is the sum of the squared entries of A which is positive. So V cannot contain any symmetric matrix but 0.

Denote by S the linear space of all real $n \times n$ symmetric matrices; $\dim V = \frac{n(n+1)}{2}$. Since $V \cap S = \{0\}$, we have $\dim V + \dim S \le n^2$ and thus $\dim V \le n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$.

The space of strictly upper triangular matrices has dimension $\frac{n(n-1)}{2}$ and satisfies the condition of the problem.

Therefore the maximum dimension of V is $\frac{n(n-1)}{2}$.

Problem 4. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is three times differentiable, then there exists a real number $\xi \in (-1, 1)$ such that

$$\frac{f'''(\xi)}{6} = \frac{f(1) - f(-1)}{2} - f'(0).$$

Solution 1. Let

$$g(x) = -\frac{f(-1)}{2} x^2(x-1) - f(0)(x^2-1) + \frac{f(1)}{2} x^2(x+1) - f'(0)x(x-1)(x+1).$$

It is easy to check that $g(\pm 1) = f(\pm 1)$, $g(0) = f(0)$ and $g'(0) = f'(0)$.

Apply Rolle's theorem for the function $h(x) = f(x) - g(x)$ and its derivatives. Since $h(-1) = h(0) = h(1) = 0$, there exist $\eta \in (-1, 0)$ and $\vartheta \in (0, 1)$ such that $h'(\eta) = h'(\vartheta) = 0$. We also have $h'(0) = 0$, so there exist $\varrho \in (\eta, 0)$ and $\sigma \in (0, \vartheta)$ such that $h''(\varrho) = h''(\sigma) = 0$. Finally, there exists a $\xi \in (\varrho, \sigma) \subset (-1, 1)$ where $h'''(\xi) = 0$. Then

$$f'''(\xi) = g'''(\xi) = -\frac{f(-1)}{2} \cdot 6 - f(0) \cdot 0 + \frac{f(1)}{2} \cdot 6 - f'(0) \cdot 6 = \frac{f(1) - f(-1)}{2} - f'(0).$$

Solution 2. The expression $\dfrac{f(1)-f(-1)}{2} - f'(0)$ is the divided difference $f[-1,0,0,1]$ and there exists a number $\xi \in (-1,1)$ such that $f[-1,0,0,1] = \dfrac{f'''(\xi)}{3!}$.

Problem 5. Find all $r > 0$ such that whenever $f : \mathbb{R}^2 \to \mathbb{R}$ is a differentiable function such that $|\operatorname{grad} f(0,0)| = 1$ and $|\operatorname{grad} f(u) - \operatorname{grad} f(v)| \le |u - v|$ for all $u, v \in \mathbb{R}^2$, then the maximum of f on the disk $\{u \in \mathbb{R}^2 : |u| \le r\}$ is attained at exactly one point. ($\operatorname{grad} f(u) = (\partial_1 f(u), \partial_2 f(u))$ is the gradient vector of f at the point u. For a vector $u = (a,b)$, $|u| = \sqrt{a^2+b^2}$.)

Solution. To get an upper bound for r, set $f(x,y) = x - \dfrac{x^2}{2} + \dfrac{y^2}{2}$. This function satisfies the conditions, since $\operatorname{grad} f(x,y) = (1-x, y)$, $\operatorname{grad} f(0,0) = (1,0)$ and $|\operatorname{grad} f(x_1,y_1) - \operatorname{grad} f(x_2,y_2)| = |(x_2-x_1, y_1-y_2)| = |(x_1,y_1)-(x_2,y_2)|$.

In the disk $D_r = \{(x,y) : x^2+y^2 \le r^2\}$

$$f(x,y) = \frac{x^2+y^2}{2} - \left(x - \frac{1}{2}\right)^2 + \frac{1}{4} \le \frac{r^2}{2} + \frac{1}{4}.$$

If $r > \frac{1}{2}$ then the absolute maximum is $\frac{r^2}{2} + \frac{1}{4}$, attained at the points $\left(\frac{1}{2}, \pm\sqrt{r^2 - \frac{1}{4}}\right)$. Therefore, it is necessary that $r \le \frac{1}{2}$ because if $r > \frac{1}{2}$ then the maximum is attained twice.

Suppose now that $r \le 1/2$ and that f attains its maximum on D_r at u, v, $u \ne v$. Since $|\operatorname{grad} f(z) - \operatorname{grad} f(0)| \le r$, $|\operatorname{grad} f(z)| \ge 1 - r > 0$ for all $z \in D_r$. Hence f may attain its maximum only at the boundary of D_r, so we must have $|u| = |v| = r$ and $\operatorname{grad} f(u) = au$ and $\operatorname{grad} f(v) = bv$, where $a, b \ge 0$. Since $au = \operatorname{grad} f(u)$ and $bv = \operatorname{grad} f(v)$ belong to the disk D with centre $\operatorname{grad} f(0)$ and radius r, they do not belong to the interior of D_r. Hence $|\operatorname{grad} f(u) - \operatorname{grad} f(v)| = |au - bv| \ge |u - v|$ and this inequality is strict since $D \cap D_r$ contains no more than one point. But this contradicts the assumption that $|\operatorname{grad} f(u) - \operatorname{grad} f(v)| \le |u - v|$. So all $r \le \frac{1}{2}$ satisfies the condition.

Problem 6. Prove that if p and q are rational numbers and $r = p + q\sqrt{7}$, then there exists a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \ne \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ with integer entries and with $ad - bc = 1$ such that

$$\frac{ar+b}{cr+d} = r.$$

Solution. First consider the case when $q = 0$ and r is rational. Choose a positive integer t such that $r^2 t$ is an integer and set

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 + rt & -r^2 t \\ t & 1 - rt \end{pmatrix}.$$

Then

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1 \quad \text{and} \quad \frac{ar+b}{cr+d} = \frac{(1+rt)r - r^2 t}{tr + (1-rt)} = r.$$

Now assume $q \neq 0$. Let the minimal polynomial of r in $\mathbb{Z}[x]$ be ux^2+vx+w. The other root of this polynomial is $\overline{r} = p - q\sqrt{7}$, so $v = -u(r+\overline{r}) = -2up$ and $w = ur\overline{r} = u(p^2-7q^2)$. The discriminant is $v^2 - 4uw = 7 \cdot (2uq)^2$. The left-hand side is an integer, implying that also $\Delta = 2uq$ is an integer.

The equation $\frac{ar+b}{cr+d} = r$ is equivalent to $cr^2 + (d-a)r - b = 0$. This must be a multiple of the minimal polynomial, so we need

$$c = ut, \quad d - a = vt, \quad -b = wt$$

for some integer $t \neq 0$. Putting together these equalities with $ad - bc = 1$ we obtain that

$$(a+d)^2 = (a-d)^2 + 4ad = 4 + (v^2 - 4uw)t^2 = 4 + 7\Delta^2 t^2.$$

Therefore $4 + 7\Delta^2 t^2$ must be a perfect square. Introducing $s = a + d$, we need an integer solution (s, t) for the Diophantine equation

$$s^2 - 7\Delta^2 t^2 = 4 \tag{1}$$

such that $t \neq 0$.

The numbers s and t will be even. Then $a + d = s$ and $d - a = vt$ will be even as well and a and d will be really integers.

Let $(8 \pm 3\sqrt{7})^n = k_n \pm l_n\sqrt{7}$ for each integer n. Then $k_n^2 - 7l_n^2 = (k_n + l_n\sqrt{7})(k_n - l_n\sqrt{7}) = ((8+3\sqrt{7})^n(8-3\sqrt{7}))^n = 1$ and the sequence (l_n) also satisfies the linear recurrence $l_{n+1} = 16l_n - l_{n-1}$. Consider the residue of l_n modulo Δ. There are Δ^2 possible residue pairs for (l_n, l_{n+1}) so some are the same. Starting from such two positions, the recurrence shows that the sequence of residues is periodic in both directions. Then there are infinitely many indices such that $l_n \equiv l_0 = 0 \pmod{\Delta}$.

Taking such an index n, we can set $s = 2k_n$ and $t = 2l_n/\Delta$.

Remarks. 1. It is well-known that if $D > 0$ is not a perfect square then the Pell-like Diophantine equation

$$x^2 - Dy^2 = 1$$

has infinitely many solutions. Using this fact the solution can be generalized to all quadratic algebraic numbers.

2. It is also known that the continued fraction of a real number r is periodic from a certain point if and only if r is a root of a quadratic equation. This fact can lead to another solution.

13^{th} International Mathematics Competition for University Students
Odessa, July 20-26, 2006
First Day

Problem 1. Let $f:\mathbb{R}\to\mathbb{R}$ be a real function. Prove or disprove each of the following statements.

(a) If f is continuous and range$(f)=\mathbb{R}$ then f is monotonic.

(b) If f is monotonic and range$(f)=\mathbb{R}$ then f is continuous.

(c) If f is monotonic and f is continuous then range$(f)=\mathbb{R}$.

(20 points)

Solution. (a) False. Consider function $f(x)=x^3-x$. It is continuous, range$(f)=\mathbb{R}$ but, for example, $f(0)=0$, $f(\frac{1}{2})=-\frac{3}{8}$ and $f(1)=0$, therefore $f(0)>f(\frac{1}{2})$, $f(\frac{1}{2})<f(1)$ and f is not monotonic.

(b) True. Assume first that f is non-decreasing. For an arbitrary number a, the limits $\lim\limits_{a-} f$ and $\lim\limits_{a+} f$ exist and $\lim\limits_{a-} f \le \lim\limits_{a+} f$. If the two limits are equal, the function is continuous at a. Otherwise, if $\lim\limits_{a-} f = b < \lim\limits_{a+} f = c$, we have $f(x)\le b$ for all $x<a$ and $f(x)\ge c$ for all $x>a$; therefore range$(f)\subset(-\infty,b)\cup(c,\infty)\cup\{f(a)\}$ cannot be the complete $\mathbb{R}$.

For non-increasing f the same can be applied writing reverse relations or $g(x)=-f(x)$.

(c) False. The function $g(x)=\arctan x$ is monotonic and continuous, but range$(g)=(-\pi/2,\pi/2)\neq\mathbb{R}$.

Problem 2. Find the number of positive integers x satisfying the following two conditions:

1. $x<10^{2006}$;

2. x^2-x is divisible by 10^{2006}.

(20 points)

Solution 1. Let $S_k=\left\{0<x<10^k \,\middle|\, x^2-x \text{ is divisible by } 10^k\right\}$ and $s(k)=|S_k|, k\ge 1$. Let $x=a_{k+1}a_k\ldots a_1$ be the decimal writing of an integer $x\in S_{k+1}$, $k\ge 1$. Then obviously $y=a_k\ldots a_1\in S_k$. Now, let $y=a_k\ldots a_1\in S_k$ be fixed. Considering a_{k+1} as a variable digit, we have $x^2-x=\left(a_{k+1}10^k+y\right)^2-\left(a_{k+1}10^k+y\right)=(y^2-y)+a_{k+1}10^k(2y-1)+a_{k+1}^2 10^{2k}$. Since $y^2-y=10^k z$ for an iteger z, it follows that x^2-x is divisible by 10^{k+1} if and only if $z+a_{k+1}(2y-1)\equiv 0 \pmod{10}$. Since $y\equiv 3 \pmod{10}$ is obviously impossible, the congruence has exactly one solution. Hence we obtain a one-to-one correspondence between the sets S_{k+1} and S_k for every $k\ge 1$. Therefore $s(2006)=s(1)=3$, because $S_1=\{1,5,6\}$.

Solution 2. Since $x^2-x=x(x-1)$ and the numbers x and $x-1$ are relatively prime, one of them must be divisible by 2^{2006} and one of them (may be the same) must be divisible by 5^{2006}. Therefore, x must satisfy the following two conditions:

$$x\equiv 0 \text{ or } 1 \pmod{2^{2006}};$$

$$x\equiv 0 \text{ or } 1 \pmod{5^{2006}}.$$

Altogether we have 4 cases. The Chinese remainder theorem yields that in each case there is a unique solution among the numbers $0,1,\ldots,10^{2006}-1$. These four numbers are different because each two gives different residues modulo 2^{2006} or 5^{2006}. Moreover, one of the numbers is 0 which is not allowed.

Therefore there exist 3 solutions.

Problem 3. Let A be an $n\times n$-matrix with integer entries and $b_1,\ldots,b_k$ be integers satisfying $\det A=b_1\cdot\ldots\cdot b_k$. Prove that there exist $n\times n$-matrices $B_1,\ldots,B_k$ with integer entries such that $A=B_1\cdot\ldots\cdot B_k$ and $\det B_i=b_i$ for all $i=1,\ldots,k$.

(20 points)

Solution. By induction, it is enough to consider the case $m=2$. Furthermore, we can multiply A with any integral matrix with determinant 1 from the right or from the left, without changing the problem. Hence we can assume A to be upper triangular.

Lemma. Let A be an integral upper triangular matrix, and let b, c be integers satisfying $\det A = bc$. Then there exist integral upper triangular matrices B, C such that $\det B = b$, $\det C = c$, $A = BC$.

Proof. The proof is done by induction on n, the case $n = 1$ being obvious. Assume the statement is true for $n-1$. Let A, b, c as in the statement of the lemma. Define B_{nn} to be the greatest common divisor of b and A_{nn}, and put $C_{nn} = \frac{A_{nn}}{B_{nn}}$. Since A_{nn} divides bc, C_{nn} divides $\frac{b}{B_{nn}}c$, which divides c. Hence C_{nn} divides c. Therefore, $b' = \frac{b}{B_{nn}}$ and $c' = \frac{c}{C_{nn}}$ are integers. Define A' to be the upper-left $(n-1)\times(n-1)$-submatrix of A; then $\det A' = b'c'$. By induction we can find the upper-left $(n-1)\times(n-1)$-part of B and C in such a way that $\det B = b$, $\det C = c$ and $A = BC$ holds on the upper-left $(n-1)\times(n-1)$-submatrix of A. It remains to define $B_{i,n}$ and $C_{i,n}$ such that $A = BC$ also holds for the (i,n)-th entry for all $i < n$.

First we check that B_{ii} and C_{nn} are relatively prime for all $i < n$. Since B_{ii} divides b', it is certainly enough to prove that b' and C_{nn} are relatively prime, i.e.

$$\gcd\left(\frac{b}{\gcd(b, A_{nn})}, \frac{A_{nn}}{\gcd(b, A_{nn})}\right) = 1,$$

which is obvious. Now we define $B_{j,n}$ and $C_{j,n}$ inductively: Suppose we have defined $B_{i,n}$ and $C_{i,n}$ for all $i = j+1, j+2, \ldots, n-1$. Then $B_{j,n}$ and $C_{j,n}$ have to satisfy

$$A_{j,n} = B_{j,j}C_{j,n} + B_{j,j+1}C_{j+1,n} + \cdots + B_{j,n}C_{n,n}$$

Since $B_{j,j}$ and $C_{n,n}$ are relatively prime, we can choose integers $C_{j,n}$ and $B_{j,n}$ such that this equation is satisfied. Doing this step by step for all $j = n-1, n-2, \ldots, 1$, we finally get B and C such that $A = BC$. $\square$

Problem 4. Let f be a rational function (i.e. the quotient of two real polynomials) and suppose that $f(n)$ is an integer for infinitely many integers n. Prove that f is a polynomial.
(20 points)

Solution. Let S be an infinite set of integers such that rational function $f(x)$ is integral for all $x \in S$.

Suppose that $f(x) = p(x)/q(x)$ where p is a polynomial of degree k and q is a polynomial of degree n. Then p, q are solutions to the simultaneous equations $p(x) = q(x)f(x)$ for all $x \in S$ that are not roots of q. These are linear simultaneous equations in the coefficients of p, q with rational coefficients. Since they have a solution, they have a rational solution.

Thus there are polynomials p', q' with rational coefficients such that $p'(x) = q'(x)f(x)$ for all $x \in S$ that are not roots of q. Multiplying this with the previous equation, we see that $p'(x)q(x)f(x) = p(x)q'(x)f(x)$ for all $x \in S$ that are not roots of q. If x is not a root of p or q, then $f(x) \neq 0$, and hence $p'(x)q(x) = p(x)q'(x)$ for all $x \in S$ except for finitely many roots of p and q. Thus the two polynomials $p'q$ and pq' are equal for infinitely many choices of value. Thus $p'(x)q(x) = p(x)q'(x)$. Dividing by $q(x)q'(x)$, we see that $p'(x)/q'(x) = p(x)/q(x) = f(x)$. Thus $f(x)$ can be written as the quotient of two polynomials with rational coefficients. Multiplying up by some integer, it can be written as the quotient of two polynomials with integer coefficients.

Suppose $f(x) = p''(x)/q''(x)$ where p'' and q'' both have integer coefficients. Then by Euler's division algorithm for polynomials, there exist polynomials s and r, both of which have rational coefficients such that $p''(x) = q''(x)s(x) + r(x)$ and the degree of r is less than the degree of q''. Dividing by $q''(x)$, we get that $f(x) = s(x) + r(x)/q''(x)$. Now there exists an integer N such that $Ns(x)$ has integral coefficients. Then $Nf(x) - Ns(x)$ is an integer for all $x \in S$. However, this is equal to the rational function Nr/q'', which has a higher degree denominator than numerator, so tends to 0 as x tends to ∞. Thus for all sufficiently large $x \in S$, $Nf(x) - Ns(x) = 0$ and hence $r(x) = 0$. Thus r has infinitely many roots, and is 0. Thus $f(x) = s(x)$, so f is a polynomial.

Problem 5. Let $a, b, c, d, e > 0$ be real numbers such that $a^2 + b^2 + c^2 = d^2 + e^2$ and $a^4 + b^4 + c^4 = d^4 + e^4$. Compare the numbers $a^3 + b^3 + c^3$ and $d^3 + e^3$.
(20 points)

Solution. Without loss of generality $a \geq b \geq c$, $d \geq e$. Let $c^2 = e^2 + \Delta$, $\Delta \in \mathbb{R}$. Then $d^2 = a^2 + b^2 + \Delta$ and the second equation implies

$$a^4 + b^4 + (e^2 + \Delta)^2 = (a^2 + b^2 + \Delta)^2 + e^4, \quad \Delta = -\tfrac{a^2b^2}{a^2+b^2-e^2}.$$

(Here $a^2 + b^2 - e^2 \geq \frac{2}{3}(a^2 + b^2 + c^2) - \frac{1}{2}(d^2 + e^2) = \frac{1}{6}(d^2 + e^2) > 0$.)
Since $c^2 = e^2 - \frac{a^2b^2}{a^2+b^2-e^2} = \frac{(a^2-e^2)(e^2-b^2)}{a^2+b^2-e^2} > 0$ then $a > e > b$.
Therefore $d^2 = a^2 + b^2 - \frac{a^2b^2}{a^2+b^2-e^2} < a^2$ and $a > d \geq e > b \geq c$.

Consider a function $f(x) = a^x + b^x + c^x - d^x - e^x$, $x \in \mathbb{R}$. We shall prove that $f(x)$ has only two zeroes $x = 2$ and $x = 4$ and changes the sign at these points. Suppose the contrary. Then Rolle's theorem implies that $f'(x)$ has at least two distinct zeroes. Without loss of generality $a = 1$. Then $f'(x) = \ln b \cdot b^x + \ln c \cdot c^x - \ln d \cdot d^x - \ln e \cdot e^x$, $x \in \mathbb{R}$. If $f'(x_1) = f'(x_2) = 0$, $x_1 < x_2$, then

$$\ln b \cdot b^{x_i} + \ln c \cdot c^{x_i} = \ln d \cdot d^{x_i} + \ln e \cdot e^{x_i}, \quad i = 1, 2,$$

but since $1 > d \geq e > b \geq c$ we have

$$\frac{(-\ln b) \cdot b^{x_2} + (-\ln c) \cdot c^{x_2}}{(-\ln b) \cdot b^{x_1} + (-\ln c) \cdot c^{x_1}} \leq b^{x_2 - x_1} < e^{x_2 - x_1} \leq \frac{(-\ln d) \cdot d^{x_2} + (-\ln e) \cdot e^{x_2}}{(-\ln d) \cdot d^{x_1} + (-\ln e) \cdot e^{x_1}},$$

a contradiction. Therefore $f(x)$ has a constant sign at each of the intervals $(-\infty, 2)$, $(2, 4)$ and $(4, \infty)$. Since $f(0) = 1$ then $f(x) > 0$, $x \in (-\infty, 2) \bigcup (4, \infty)$ and $f(x) < 0$, $x \in (2, 4)$. In particular, $f(3) = a^3 + b^3 + c^3 - d^3 - e^3 < 0$.

Problem 6. Find all sequences $a_0, a_1, \ldots, a_n$ of real numbers where $n \geq 1$ and $a_n \neq 0$, for which the following statement is true:

If $f : \mathbb{R} \to \mathbb{R}$ is an n times differentiable function and $x_0 < x_1 < \ldots < x_n$ are real numbers such that $f(x_0) = f(x_1) = \ldots = f(x_n) = 0$ then there exists an $h \in (x_0, x_n)$ for which

$$a_0 f(h) + a_1 f'(h) + \ldots + a_n f^{(n)}(h) = 0.$$

(20 points)

Solution. Let $A(x) = a_0 + a_1 x + \ldots + a_n x^n$. We prove that sequence $a_0, \ldots, a_n$ satisfies the required property if and only if all zeros of polynomial $A(x)$ are real.

(a) Assume that all roots of $A(x)$ are real. Let us use the following notations. Let I be the identity operator on $\mathbb{R} \to \mathbb{R}$ functions and D be differentiation operator. For an arbitrary polynomial $P(x) = p_0 + p_1 x + \ldots + p_n x^n$, write $P(D) = p_0 I + p_1 D + p_2 D^2 + \ldots + p_n D^n$. Then the statement can written as $(A(D)f)(\xi) = 0$.

First prove the statement for $n = 1$. Consider the function

$$g(x) = e^{\frac{a_0}{a_1}x} f(x).$$

Since $g(x_0) = g(x_1) = 0$, by Rolle's theorem there exists a $\xi \in (x_0, x_1)$ for which

$$g'(\xi) = \frac{a_0}{a_1} e^{\frac{a_0}{a_1}\xi} f(\xi) + e^{\frac{a_0}{a_1}\xi} f'\xi) = \frac{e^{\frac{a_0}{a_1}\xi}}{a_1} (a_0 f(\xi) + a_1 f'(\xi)) = 0.$$

Now assume that $n > 1$ and the statement holds for $n-1$. Let $A(x) = (x-c)B(x)$ where c is a real root of polynomial A. By the $n = 1$ case, there exist $y_0 \in (x_0, x_1)$, $y_1 \in (x_1, x_2)$, $\ldots$, $y_{n-1} \in (x_{n-1}, x_n)$ such that $f'(y_j) - cf(y_j) = 0$ for all $j = 0, 1, \ldots, n-1$. Now apply the induction hypothesis for polynomial $B(x)$, function $g = f' - cf$ and points $y_0, \ldots, y_{n-1}$. The hypothesis says that there exists a $\xi \in (y_0, y_{n-1}) \subset (x_0, x_n)$ such that

$$(B(D)g)(\xi) = (B(D)(D - cI)f)(\xi) = (A(D)f)(\xi) = 0.$$

(b) Assume that $u + vi$ is a complex root of polynomial $A(x)$ such that $v \neq 0$. Consider the linear differential equation $a_n g^{(n)} + \ldots + a_1 g' + g = 0$. A solution of this equation is $g_1(x) = e^{ux} \sin vx$ which has infinitely many zeros.

Let k be the smallest index for which $a_k \neq 0$. Choose a small $\varepsilon > 0$ and set $f(x) = g_1(x) + \varepsilon x^k$. If ε is sufficiently small then g has the required number of roots but $a_0 f + a_1 f' + \ldots + a_n f^{(n)} = a_k \varepsilon \neq 0$ everywhere.

13th International Mathematics Competition for University Students

Odessa, July 20-26, 2006

Second Day

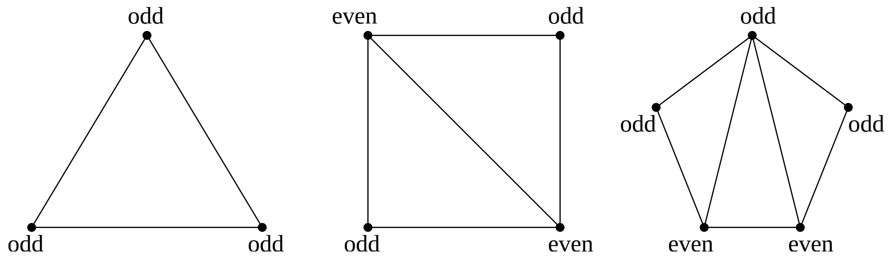
Problem 1. Let V be a convex polygon with n vertices.

(a) Prove that if n is divisible by 3 then V can be triangulated (i.e. dissected into non-overlapping triangles whose vertices are vertices of V) so that each vertex of V is the vertex of an odd number of triangles.

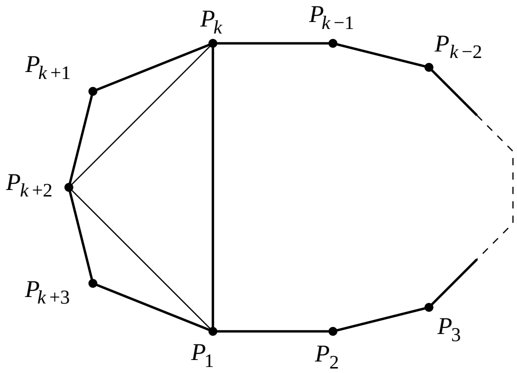
(b) Prove that if n is not divisible by 3 then V can be triangulated so that there are exactly two vertices that are the vertices of an even number of the triangles.

(20 points)

Solution. Apply induction on n. For the initial cases $n = 3, 4, 5$, chose the triangulations shown in the Figure to prove the statement.



Now assume that the statement is true for some $n = k$ and consider the case $n = k + 3$. Denote the vertices of V by $P_1, \ldots, P_{k+3}$. Apply the induction hypothesis on the polygon $P_1P_2 \ldots P_k$; in this triangulation each of vertices $P_1, \ldots, P_k$ belong to an odd number of triangles, except two vertices if n is not divisible by 3. Now add triangles $P_1P_kP_{k+2}$, $P_kP_{k+1}P_{k+2}$ and $P_1P_{k+2}P_{k+3}$. This way we introduce two new triangles at vertices P_1 and P_k so parity is preserved. The vertices P_{k+1}, P_{k+2} and P_{k+3} share an odd number of triangles. Therefore, the number of vertices shared by even number of triangles remains the same as in polygon $P_1P_2 \ldots P_k$.



Problem 2. Find all functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that for any real numbers $a < b$, the image $f\big([a, b]\big)$ is a closed interval of length $b - a$.

(20 points)

Solution. The functions $f(x) = x + c$ and $f(x) = -x + c$ with some constant c obviously satisfy the condition of the problem. We will prove now that these are the only functions with the desired property.

Let f be such a function. Then f clearly satisfies $|f(x) - f(y)| \le |x - y|$ for all x, y; therefore, f is continuous. Given x, y with $x < y$, let $a, b \in [x, y]$ be such that $f(a)$ is the maximum and $f(b)$ is the minimum of f on $[x, y]$. Then $f([x, y]) = [f(b), f(a)]$; hence

$$y - x = f(a) - f(b) \le |a - b| \le y - x$$

This implies $\{a, b\} = \{x, y\}$, and therefore f is a monotone function. Suppose f is increasing. Then $f(x) - f(y) = x - y$ implies $f(x) - x = f(y) - y$, which says that $f(x) = x + c$ for some constant c. Similarly, the case of a decreasing function f leads to $f(x) = -x + c$ for some constant c.

Problem 3. Compare $\tan(\sin x)$ and $\sin(\tan x)$ for all $x \in (0, \frac{\pi}{2})$.
(20 points)
Solution. Let $f(x) = \tan(\sin x) - \sin(\tan x)$. Then

$$f'(x) = \frac{\cos x}{\cos^2(\sin x)} - \frac{\cos(\tan x)}{\cos^2 x} = \frac{\cos^3 x - \cos(\tan x) \cdot \cos^2(\sin x)}{\cos^2 x \cdot \cos^2(\tan x)}$$

Let $0 < x < \arctan \frac{\pi}{2}$. It follows from the concavity of cosine on $(0, \frac{\pi}{2})$ that

$$\sqrt[3]{\cos(\tan x) \cdot \cos^2(\sin x)} < \frac{1}{3}\left[\cos(\tan x) + 2\cos(\sin x)\right] \le \cos\left[\frac{\tan x + 2\sin x}{3}\right] < \cos x\ ,$$

the last inequality follows from $\left[\frac{\tan x + 2\sin x}{3}\right]' = \frac{1}{3}\left[\frac{1}{\cos^2 x} + 2\cos x\right] \ge \sqrt[3]{\frac{1}{\cos^2 x} \cdot \cos x \cdot \cos x} = 1$. This proves that $\cos^3 x - \cos(\tan x) \cdot \cos^2(\sin x) > 0$, so $f'(x) > 0$, so f increases on the interval $[0, \arctan \frac{\pi}{2}]$. To end the proof it is enough to notice that (recall that $4 + \pi^2 < 16$)

$$\tan\left[\sin\left(\arctan \frac{\pi}{2}\right)\right] = \tan \frac{\pi/2}{\sqrt{1 + \pi^2/4}} > \tan \frac{\pi}{4} = 1\ .$$

This implies that if $x \in [\arctan \frac{\pi}{2}, \frac{\pi}{2}]$ then $\tan(\sin x) > 1$ and therefore $f(x) > 0$.

Problem 4. Let v_0 be the zero vector in $\mathbb{R}^n$ and let $v_1, v_2, \dots, v_{n+1} \in \mathbb{R}^n$ be such that the Euclidean norm $|v_i - v_j|$ is rational for every $0 \le i, j \le n + 1$. Prove that $v_1, \dots, v_{n+1}$ are linearly dependent over the rationals.
(20 points)
Solution. By passing to a subspace we can assume that $v_1, \dots, v_n$ are linearly independent over the reals. Then there exist $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ satisfying

$$v_{n+1} = \sum_{j=1}^{n} \lambda_j v_j$$

We shall prove that λ_j is rational for all j. From

$$-2\langle v_i, v_j\rangle = |v_i - v_j|^2 - |v_i|^2 - |v_j|^2$$

we get that $\langle v_i, v_j\rangle$ is rational for all i, j. Define A to be the rational $n \times n$-matrix $A_{ij} = \langle v_i, v_j\rangle$, $w \in \mathbb{Q}^n$ to be the vector $w_i = \langle v_i, v_{n+1}\rangle$, and $\lambda \in \mathbb{R}^n$ to be the vector $(\lambda_i)_i$. Then,

$$\langle v_i, v_{n+1}\rangle = \sum_{j=1}^{n} \lambda_j \langle v_i, v_j\rangle$$

gives $A\lambda = w$. Since $v_1, \dots, v_n$ are linearly independent, A is invertible. The entries of A^{-1} are rationals, therefore $\lambda = A^{-1}w \in \mathbb{Q}^n$, and we are done.

Problem 5. Prove that there exists an infinite number of relatively prime pairs (m, n) of positive integers such that the equation

$$(x+m)^3 = nx$$

has three distinct integer roots.
(20 points)

Solution. Substituting $y = x + m$, we can replace the equation by

$$y^3 - ny + mn = 0.$$

Let two roots be u and v; the third one must be $w = -(u+v)$ since the sum is 0. The roots must also satisfy

$$uv + uw + vw = -(u^2 + uv + v^2) = -n, \qquad \text{i.e.} \quad u^2 + uv + v^2 = n$$

and

$$uvw = -uv(u+v) = mn.$$

So we need some integer pairs (u, v) such that $uv(u+v)$ is divisible by $u^2 + uv + v^2$. Look for such pairs in the form $u = kp$, $v = kq$. Then

$$u^2 + uv + v^2 = k^2(p^2 + pq + q^2),$$

and

$$uv(u+v) = k^3 pq(p+q).$$

Chosing p, q such that they are coprime then setting $k = p^2 + pq + q^2$ we have $\dfrac{uv(u+v)}{u^2+uv+v^2} = p^2 + pq + q^2$.

Substituting back to the original quantites, we obtain the family of cases

$$n = (p^2 + pq + q^2)^3, \qquad m = p^2q + pq^2,$$

and the three roots are

$$x_1 = p^3, \qquad x_2 = q^3, \qquad x_3 = -(p+q)^3.$$

Problem 6. Let A_i, B_i, S_i $(i = 1, 2, 3)$ be invertible real 2×2 matrices such that

(1) not all A_i have a common real eigenvector;

(2) $A_i = S_i^{-1} B_i S_i$ for all $i = 1, 2, 3$;

(3) $A_1A_2A_3 = B_1B_2B_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Prove that there is an invertible real 2×2 matrix S such that $A_i = S^{-1}B_iS$ for all $i = 1, 2, 3$.
(20 points)

Solution. We note that the problem is trivial if $A_j = \lambda I$ for some j, so suppose this is not the case. Consider then first the situation where *some* A_j, say A_3, has two distinct real eigenvalues. We may assume that $A_3 = B_3 = \left(\begin{smallmatrix} \lambda & \\ & \mu \end{smallmatrix}\right)$ by conjugating both sides. Let $A_2 = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $B_2 = \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right)$. Then

$$\begin{aligned} a + d = \operatorname{Tr} A_2 &= \operatorname{Tr} B_2 = a' + d' \\ a\lambda + d\mu = \operatorname{Tr}(A_2A_3) = \operatorname{Tr} A_1^{-1} &= \operatorname{Tr} B_1^{-1} = \operatorname{Tr}(B_2B_3) = a'\lambda + d'\mu. \end{aligned}$$

Hence $a = a'$ and $d = d'$ and so also $bc = b'c'$. Now we cannot have $c = 0$ or $b = 0$, for then $(1,0)^\top$ or $(0,1)^\top$ would be a common eigenvector of all A_j. The matrix $S = \left(\begin{smallmatrix} c' & \\ & c \end{smallmatrix}\right)$ conjugates $A_2 = S^{-1}B_2S$, and as S commutes with $A_3 = B_3$, it follows that $A_j = S^{-1}B_jS$ for all j.

If the distinct eigenvalues of $A_3 = B_3$ are not real, we know from above that $A_j = S^{-1}B_jS$ for some $S \in \mathrm{GL}_2\mathbb{C}$ unless all A_j have a common eigenvector over $\mathbb{C}$. Even if they do, say $A_jv = \lambda_jv$, by taking the conjugate square root it follows that A_j's can be simultaneously diagonalized. If $A_2 = \left(\begin{smallmatrix} a & \\ & d \end{smallmatrix}\right)$ and $B_2 = \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right)$, it follows as above that $a = a'$, $d = d'$ and so $b'c' = 0$. Now B_2 and B_3 (and hence B_1 too) have a common eigenvector over $\mathbb{C}$ so they too can be simultaneously diagonalized. And so $SA_j = B_jS$ for some $S \in \mathrm{GL}_2\mathbb{C}$ in either case. Let $S_0 = \operatorname{Re} S$ and $S_1 = \operatorname{Im} S$. By separating the real and imaginary components, we are done if either S_0 or S_1 is invertible. If not, S_0 may be conjugated to some $T^{-1}S_0T = \left(\begin{smallmatrix} x & 0 \\ y & 0 \end{smallmatrix}\right)$, with $(x, y)^\top \neq (0, 0)^\top$, and it follows that all A_j have a common eigenvector $T(0, 1)^\top$, a contradiction.

We are left with the case when *no* A_j has distinct eigenvalues; then these eigenvalues by necessity are real. By conjugation and division by scalars we may assume that $A_3 = \left(\begin{smallmatrix} 1 & b \\ & 1 \end{smallmatrix}\right)$ and $b \neq 0$. By further conjugation by upper-triangular matrices (which preserves the shape of A_3 up to the value of b) we can also assume that $A_2 = \left(\begin{smallmatrix} 0 & u \\ 1 & v \end{smallmatrix}\right)$. Here $v^2 = \mathrm{Tr}^2 A_2 = 4 \det A_2 = -4u$. Now $A_1 = A_3^{-1}A_2^{-1} = \left(\begin{smallmatrix} -(b+v)/u & 1 \\ 1/u & \end{smallmatrix}\right)$, and hence $(b + v)^2/u^2 = \mathrm{Tr}^2 A_1 = 4 \det A_1 = -4/u$. Comparing these two it follows that $b = -2v$. What we have done is simultaneously reduced all A_j to matrices whose all entries depend on u and v ($= -\det A_2$ and $\mathrm{Tr}\, A_2$, respectively) only, but these themselves are invariant under similarity. So B_j's can be simultaneously reduced to the very same matrices.

IMC2007, Blagoevgrad, Bulgaria

Day 1, August 5, 2007

Problem 1. Let f be a polynomial of degree 2 with integer coefficients. Suppose that $f(k)$ is divisible by 5 for every integer k. Prove that all coefficients of f are divisible by 5.

Solution 1. Let $f(x) = ax^2 + bx + c$. Substituting $x = 0$, $x = 1$ and $x = -1$, we obtain that $5|f(0) = c$, $5|f(1) = (a+b+c)$ and $5|f(-1) = (a-b+c)$. Then $5|f(1)+f(-1)-2f(0) = 2a$ and $5|f(1)-f(-1) = 2b$. Therefore 5 divides $2a$, $2b$ and c and the statement follows.

Solution 2. Consider $f(x)$ as a polynomial over the 5-element field (i.e. modulo 5). The polynomial has 5 roots while its degree is at most 2. Therefore $f \equiv 0 \pmod 5$ and all of its coefficients are divisible by 5.

Problem 2. Let $n \ge 2$ be an integer. What is the minimal and maximal possible rank of an $n \times n$ matrix whose n^2 entries are precisely the numbers $1, 2, \dots, n^2$?

Solution. The minimal rank is 2 and the maximal rank is n. To prove this, we have to show that the rank can be 2 and n but it cannot be 1.

(i) The rank is at least 2. Consider an arbitrary matrix $A = [a_{ij}]$ with entries $1, 2, \dots, n^2$ in some order. Since permuting rows or columns of a matrix does not change its rank, we can assume that $1 = a_{11} < a_{21} < \cdots < a_{n1}$ and $a_{11} < a_{12} < \cdots < a_{1n}$. Hence $a_{n1} \ge n$ and $a_{1n} \ge n$ and at least one of these inequalities is strict. Then $\det \begin{bmatrix} a_{11} & a_{1n} \\ a_{n1} & a_{nn} \end{bmatrix} < 1 \cdot n^2 - n \cdot n = 0$ so $\operatorname{rk}(A) \ge \operatorname{rk} \begin{bmatrix} a_{11} & a_{1n} \\ a_{n1} & a_{nn} \end{bmatrix} \ge 2$.

(ii) The rank can be 2. Let

$$T = \begin{bmatrix} 1 & 2 & \dots & n \\ n+1 & n+2 & \dots & 2n \\ \vdots & \vdots & \ddots & \vdots \\ n^2-n+1 & n^2-n+2 & \dots & n^2 \end{bmatrix}$$

The i^{th} row is $(1, 2, \dots, n) + n(i-1) \cdot (1, 1, \dots, 1)$ so each row is in the two-dimensional subspace generated by the vectors $(1, 2, \dots, n)$ and $(1, 1, \dots, 1)$. We already proved that the rank is at least 2, so $\operatorname{rk}(T) = 2$.

(iii) The rank can be n, i.e. the matrix can be nonsingular. Put odd numbers into the diagonal, only even numbers above the diagonal and arrange the entries under the diagonal arbitrarily. Then the determinant of the matrix is odd, so the rank is complete.

Problem 3. Call a polynomial $P(x_1, \dots, x_k)$ *good* if there exist 2×2 real matrices $A_1, \dots, A_k$ such that

$$P(x_1, \dots, x_k) = \det \left(\sum_{i=1}^{k} x_i A_i \right).$$

Find all values of k for which all homogeneous polynomials with k variables of degree 2 are good.

(A polynomial is homogeneous if each term has the same total degree.)

Solution. The possible values for k are 1 and 2.

If $k = 1$ then $P(x) = \alpha x^2$ and we can choose $A_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \alpha \end{smallmatrix}\right)$.

If $k = 2$ then $P(x, y) = \alpha x^2 + \beta y^2 + \gamma xy$ and we can choose matrices $A_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \alpha \end{smallmatrix}\right)$ and $A_2 = \left(\begin{smallmatrix} 0 & \beta \\ -1 & \gamma \end{smallmatrix}\right)$.

Now let $k \ge 3$. We show that the polynomial $P(x_1, \dots, x_k) = \sum\limits_{i=0}^{k} x_i^2$ is not good. Suppose that $P(x_1, \dots, x_k) = \det \left(\sum\limits_{i=0}^{k} x_i A_i \right)$. Since the first columns of $A_1, \dots, A_k$ are linearly dependent, the first column of some non-trivial linear combination $y_1A_1 + \ldots + y_kA_k$ is zero. Then $\det(y_1A_1 + \ldots + y_kA_k) = 0$ but $P(y_1, \ldots, y_k) \neq 0$, a contradiction.

Problem 4. Let G be a finite group. For arbitrary sets $U, V, W \subset G$, denote by N_{UVW} the number of triples $(x, y, z) \in U \times V \times W$ for which xyz is the unity.

Suppose that G is partitioned into three sets A, B and C (i.e. sets A, B, C are pairwise disjoint and $G = A \cup B \cup C$). Prove that $N_{ABC} = N_{CBA}$.

Solution. We start with three preliminary observations.

Let U, V be two arbitrary subsets of G. For each $x \in U$ and $y \in V$ there is a unique $z \in G$ for which $xyz = e$. Therefore,

$$N_{UVG} = |U \times V| = |U| \cdot |V|. \tag{1}$$

Second, the equation $xyz = e$ is equivalent to $yzx = e$ and $zxy = e$. For arbitrary sets $U, V, W \subset G$, this implies

$$\{(x,y,z) \in U \times V \times W :\ xyz = e\} = \{(x,y,z) \in U \times V \times W :\ yzx = e\} = \{(x,y,z) \in U \times V \times W :\ zxy = e\}$$

and therefore

$$N_{UVW} = N_{VWU} = N_{WUV}. \tag{2}$$

Third, if $U, V \subset G$ and W_1, W_2, W_3 are disjoint sets and $W = W_1 \cup W_2 \cup W_3$ then, for arbitrary $U, V \subset G$,

$$\{(x,y,z) \in U \times V \times W :\ xyz = e\} = \{(x,y,z) \in U \times V \times W_1 :\ xyz = e\} \cup$$
$$\cup \{(x,y,z) \in U \times V \times W_2 :\ xyz = e\} \cup \{(x,y,z) \in U \times V \times W_3 :\ xyz = e\}$$

so

$$N_{UVW} = N_{UVW_1} + N_{UVW_2} + N_{UVW_3}. \tag{3}$$

Applying these observations, the statement follows as

$$N_{ABC} = N_{ABG} - N_{ABA} - N_{ABB} = |A| \cdot |B| - N_{BAA} - N_{BAB} =$$
$$= N_{BAG} - N_{BAA} - N_{BAB} = N_{BAC} = N_{CBA}.$$

Problem 5. Let n be a positive integer and $a_1, \ldots, a_n$ be arbitrary integers. Suppose that a function $f : \mathbb{Z} \to \mathbb{R}$ satisfies $\sum_{i=1}^{n} f(k + a_i\ell) = 0$ whenever k and ℓ are integers and $\ell \neq 0$. Prove that $f = 0$.

Solution. Let us define a subset $\mathcal{I}$ of the polynomial ring $\mathbb{R}[X]$ as follows:

$$\mathcal{I} = \Big\{ P(X) = \sum_{j=0}^{m} b_j X^j \ :\ \sum_{j=0}^{m} b_j f(k + j\ell) = 0 \quad \text{for all } k, \ell \in \mathbb{Z},\ \ell \neq 0 \Big\}.$$

This is a subspace of the real vector space $\mathbb{R}[X]$. Furthermore, $P(X) \in \mathcal{I}$ implies $X \cdot P(X) \in \mathcal{I}$. Hence, $\mathcal{I}$ is an ideal, and it is non-zero, because the polynomial $R(X) = \sum_{i=1}^{n} X^{a_i}$ belongs to $\mathcal{I}$. Thus, $\mathcal{I}$ is generated (as an ideal) by some non-zero polynomial Q.

If Q is constant then the definition of $\mathcal{I}$ implies $f = 0$, so we can assume that Q has a complex zero c. Again, by the definition of $\mathcal{I}$, the polynomial $Q(X^m)$ belongs to $\mathcal{I}$ for every natural number $m \geq 1$; hence $Q(X)$ divides $Q(X^m)$. This shows that all the complex numbers

$$c, c^2, c^3, c^4, \ldots$$

are roots of Q. Since Q can have only finitely many roots, we must have $c^N = 1$ for some $N \geq 1$; in particular, $Q(1) = 0$, which implies $P(1) = 0$ for all $P \in \mathcal{I}$. This contradicts the fact that $R(X) = \sum_{i=1}^{n} X^{a_i} \in \mathcal{I}$, and we are done.

Problem 6. How many nonzero coefficients can a polynomial $P(z)$ have if its coefficients are integers and $|P(z)| \le 2$ for any complex number z of unit length?

Solution. We show that the number of nonzero coefficients can be 0, 1 and 2. These values are possible, for example the polynomials $P_0(z) = 0$, $P_1(z) = 1$ and $P_2(z) = 1 + z$ satisfy the conditions and they have 0, 1 and 2 nonzero terms, respectively.

Now consider an arbitrary polynomial $P(z) = a_0 + a_1 z + \ldots + a_n z^n$ satisfying the conditions and assume that it has at least two nonzero coefficients. Dividing the polynomial by a power of z and optionally replacing $p(z)$ by $-p(z)$, we can achieve $a_0 > 0$ such that conditions are not changed and the number of nonzero terms is preserved. So, without loss of generality, we can assume that $a_0 > 0$.

Let $Q(z) = a_1 z + \ldots + a_{n-1} z^{n-1}$. Our goal is to show that $Q(z) = 0$.

Consider those complex numbers $w_0, w_1, \ldots, w_{n-1}$ on the unit circle for which $a_n w_k^n = |a_n|$; namely, let

$$w_k = \begin{cases} e^{2k\pi i/n} & \text{if } a_n > 0 \\ e^{(2k+1)\pi i/n} & \text{if } a_n < 0 \end{cases} \qquad (k = 0, 1, \ldots, n).$$

Notice that

$$\sum_{k=0}^{n-1} Q(w_k) = \sum_{k=0}^{n-1} Q(w_0 e^{2k\pi i/n}) = \sum_{j=1}^{n-1} a_j w_0^j \sum_{k=0}^{n-1} (e^{2j\pi i/n})^k = 0.$$

Taking the average of polynomial $P(z)$ at the points w_k, we obtain

$$\frac{1}{n}\sum_{k=0}^{n-1} P(w_k) = \frac{1}{n}\sum_{k=0}^{n-1} \big(a_0 + Q(w_k) + a_n w_k^n\big) = a_0 + |a_n|$$

and

$$2 \ge \frac{1}{n}\sum_{k=0}^{n-1} \big|P(w_k)\big| \ge \left|\frac{1}{n}\sum_{k=0}^{n-1} P(w_k)\right| = a_0 + |a_n| \ge 2.$$

This obviously implies $a_0 = |a_n| = 1$ and $\big|P(w_k)\big| = \big|2 + Q(w_k)\big| = 2$ for all k. Therefore, all values of $Q(w_k)$ must lie on the circle $|2 + z| = 2$, while their sum is 0. This is possible only if $Q(w_k) = 0$ for all k. Then polynomial $Q(z)$ has at least n distinct roots while its degree is at most $n - 1$. So $Q(z) = 0$ and $P(z) = a_0 + a_n z^n$ has only two nonzero coefficients.

Remark. From Parseval's formula (i.e. integrating $|P(z)|^2 = P(z)\overline{P(z)}$ on the unit circle) it can be obtained that

$$|a_0|^2 + \ldots + |a_n|^2 = \frac{1}{2\pi}\int_0^{2\pi} \big|P(e^{it})\big|^2 \,\mathrm{d}t \le \frac{1}{2\pi}\int_0^{2\pi} 4 \,\mathrm{d}t = 4. \qquad (4)$$

Hence, there cannot be more than four nonzero coefficients, and if there are more than one nonzero term, then their coefficients are ± 1.

It is also easy to see that equality in (4) cannot hold two or more nonzero coefficients, so it is sufficient to consider only polynomials of the form $1 \pm x^m \pm x^n$. However, we do not know (yet :-)) any simpler argument for these cases than the proof above.

IMC2007, Blagoevgrad, Bulgaria

Day 2, August 6, 2007

Problem 1. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Suppose that for any $c > 0$, the graph of f can be moved to the graph of cf using only a translation or a rotation. Does this imply that $f(x) = ax + b$ for some real numbers a and b?

Solution. No. The function $f(x) = e^x$ also has this property since $ce^x = e^{x+\log c}$.

Problem 2. Let x, y, and z be integers such that $S = x^4 + y^4 + z^4$ is divisible by 29. Show that S is divisible by 29^4.

Solution. We claim that $29 \mid x, y, z$. Then, $x^4 + y^4 + z^4$ is clearly divisible by 29^4.

Assume, to the contrary, that 29 does not divide all of the numbers x, y, z. Without loss of generality, we can suppose that $29 \nmid x$. Since the residue classes modulo 29 form a field, there is some $w \in \mathbb{Z}$ such that $xw \equiv 1 \pmod{29}$. Then, $(xw)^4 + (yw)^4 + (zw)^4$ is also divisible by 29. So we can assume that $x \equiv 1 \pmod{29}$.

Thus, we need to show that $y^4 + z^4 \equiv -1 \pmod{29}$, i.e. $y^4 \equiv -1 - z^4 \pmod{29}$, is impossible. There are only eight fourth powers modulo 29,

$$\begin{aligned}
0 &\equiv 0^4, \\
1 &\equiv 1^4 \equiv 12^4 \equiv 17^4 \equiv 28^4 \pmod{29}, \\
7 &\equiv 8^4 \equiv 9^4 \equiv 20^4 \equiv 21^4 \pmod{29}, \\
16 &\equiv 2^4 \equiv 5^4 \equiv 24^4 \equiv 27^4 \pmod{29}, \\
20 &\equiv 6^4 \equiv 14^4 \equiv 15^4 \equiv 23^4 \pmod{29}, \\
23 &\equiv 3^4 \equiv 7^4 \equiv 22^4 \equiv 26^4 \pmod{29}, \\
24 &\equiv 4^4 \equiv 10^4 \equiv 19^4 \equiv 25^4 \pmod{29}, \\
25 &\equiv 11^4 \equiv 13^4 \equiv 16^4 \equiv 18^4 \pmod{29}.
\end{aligned}$$

The differences $-1 - z^4$ are congruent to 28, 27, 21, 12, 8, 5, 4, and 3. None of these residue classes is listed among the fourth powers.

Problem 3. Let C be a nonempty closed bounded subset of the real line and $f : C \to C$ be a nondecreasing continuous function. Show that there exists a point $p \in C$ such that $f(p) = p$.

(A set is closed if its complement is a union of open intervals. A function g is nondecreasing if $g(x) \le g(y)$ for all $x \le y$.)

Solution. Suppose $f(x) \neq x$ for all $x \in C$. Let $[a, b]$ be the smallest closed interval that contains C. Since C is closed, $a, b \in C$. By our hypothesis $f(a) > a$ and $f(b) < b$. Let $p = \sup\{x \in C : \quad f(x) > x\}$. Since C is closed and f is continuous, $f(p) \ge p$, so $f(p) > p$. For all $x > p$, $x \in C$ we have $f(x) < x$. Therefore $f\big(f(p)\big) < f(p)$ contrary to the fact that f is non-decreasing.

Problem 4. Let $n > 1$ be an odd positive integer and $A = (a_{ij})_{i,j=1\dots n}$ be the $n \times n$ matrix with

$$a_{ij} = \begin{cases} 2 & \text{if } i = j \\ 1 & \text{if } i - j \equiv \pm 2 \pmod{n} \\ 0 & \text{otherwise.} \end{cases}$$

Find $\det A$.

Solution. Notice that $A = B^2$, with $b_{ij} = \begin{cases} 1 & \text{if } i-j \equiv \pm 1 \pmod{n} \\ 0 & \text{otherwise} \end{cases}$. So it is sufficient to find $\det B$.

To find $\det B$, expand the determinant with respect to the first row, and then expad both terms with respect to the first column.

$$\begin{aligned}
\det B &= \begin{vmatrix} 0 & 1 & & & & & 1 \\ 1 & 0 & 1 & & & & \\ & 1 & 0 & 1 & & & \\ & & 1 & \ddots & \ddots & & \\ & & & \ddots & 0 & 1 & \\ & & & & 1 & 0 & 1 \\ 1 & & & & & 1 & 0 \end{vmatrix} = - \begin{vmatrix} 1 & 1 & & & & \\ & 0 & 1 & & & \\ & 1 & \ddots & \ddots & & \\ & & \ddots & 0 & 1 & \\ & & & 1 & 0 & 1 \\ 1 & & & & 1 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 1 & & & \\ & 1 & 0 & 1 & & \\ & & 1 & \ddots & \ddots & \\ & & & \ddots & 0 & 1 \\ & & & & 1 & 0 \\ 1 & & & & & 1 \end{vmatrix} \\
&= -\left(\begin{vmatrix} 0 & 1 & & & \\ 1 & \ddots & \ddots & & \\ & \ddots & 0 & 1 & \\ & & 1 & 0 & 1 \\ & & & 1 & 0 \end{vmatrix} - \begin{vmatrix} 1 & & & & \\ 0 & 1 & & & \\ 1 & \ddots & \ddots & & \\ & \ddots & 0 & 1 & \\ & & 1 & 0 & 1 \end{vmatrix} \right) + \left(\begin{vmatrix} 1 & 0 & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & 0 & 1 \\ & & & 1 & 0 \\ & & & & 1 \end{vmatrix} - \begin{vmatrix} 0 & 1 & & & \\ 1 & 0 & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & 0 & 1 \\ & & & 1 & 0 \end{vmatrix} \right) \\
&= -(0-1) + (1-0) = 2,
\end{aligned}$$

since the second and the third matrices are lower/upper triangular, while in the first and the fourth matrices we have $\text{row}_1 - \text{row}_3 + \text{row}_5 - \dots \pm \text{row}_{n-2} = \bar{0}$.

So $\det B = 2$ and thus $\det A = 4$.

Problem 5. For each positive integer k, find the smallest number n_k for which there exist real $n_k \times n_k$ matrices $A_1, A_2, \dots, A_k$ such that all of the following conditions hold:

(1) $A_1^2 = A_2^2 = \dots = A_k^2 = 0$,

(2) $A_i A_j = A_j A_i$ for all $1 \le i, j \le k$, and

(3) $A_1 A_2 \dots A_k \ne 0$.

Solution. The anwser is $n_k = 2^k$. In that case, the matrices can be constructed as follows: Let V be the n-dimensional real vector space with basis elements $[S]$, where S runs through all $n = 2^k$ subsets of $\{1, 2, \dots, k\}$. Define A_i as an endomorphism of V by

$$A_i[S] = \begin{cases} 0 & \text{if } i \in S \\ [S \cup \{i\}] & \text{if } i \notin S \end{cases}$$

for all $i = 1, 2, \dots, k$ and $S \subset \{1, 2, \dots, k\}$. Then $A_i^2 = 0$ and $A_i A_j = A_j A_i$. Furthermore,

$$A_1 A_2 \dots A_k [\emptyset] = [\{1, 2, \dots, k\}],$$

and hence $A_1 A_2 \dots A_k \ne 0$.

Now let $A_1, A_2, \dots, A_k$ be $n \times n$ matrices satisfying the conditions of the problem; we prove that $n \ge 2^k$. Let v be a real vector satisfying $A_1 A_2 \dots A_k v \ne 0$. Denote by $\mathcal{P}$ the set of all subsets of $\{1, 2, \dots, k\}$. Choose a complete ordering $\prec$ on $\mathcal{P}$ with the property

$$X \prec Y \quad \Rightarrow \quad |X| \le |Y| \quad \text{for all } X, Y \in \mathcal{P}.$$

For every element $X = \{x_1, x_2, \ldots, x_r\} \in \mathcal{P}$, define $A_X = A_{x_1}A_{x_2}\ldots A_{x_r}$ and $v_X = A_X v$. Finally, write $\bar{X} = \{1, 2, \ldots, k\} \setminus X$ for the complement of X.

Now take $X, Y \in \mathcal{P}$ with $X \not\supseteq Y$. Then $A_{\bar{X}}$ annihilates v_Y, because $X \not\supseteq Y$ implies the existence of some $y \in Y \setminus X = Y \cap \bar{X}$, and

$$A_{\bar{X}} v_Y = A_{\bar{X}\setminus\{y\}} A_y A_y v_{Y\setminus\{y\}} = 0,$$

since $A_y^2 = 0$. So, $A_{\bar{X}}$ annihilates the span of all the v_Y with $X \not\supseteq Y$. This implies that v_X does not lie in this span, because $A_{\bar{X}} v_X = v_{\{1,2,\ldots,k\}} \neq 0$. Therefore, the vectors v_X (with $X \in \mathcal{P}$) are linearly independent; hence $n \geq |\mathcal{P}| = 2^k$.

Problem 6. Let $f \neq 0$ be a polynomial with real coefficients. Define the sequence $f_0, f_1, f_2, \ldots$ of polynomials by $f_0 = f$ and $f_{n+1} = f_n + f_n'$ for every $n \geq 0$. Prove that there exists a number N such that for every $n \geq N$, all roots of f_n are real.

Solution. For the proof, we need the following

Lemma 1. For any polynomial g, denote by $d(g)$ the minimum distance of any two of its real zeros ($d(g) = \infty$ if g has at most one real zero). Assume that g and $g + g'$ both are of degree $k \geq 2$ and have k distinct real zeros. Then $d(g + g') \geq d(g)$.

Proof of Lemma 1: Let $x_1 < x_2 < \cdots < x_k$ be the roots of g. Suppose a, b are roots of $g + g'$ satisfying $0 < b - a < d(g)$. Then, a, b cannot be roots of g, and

$$\frac{g'(a)}{g(a)} = \frac{g'(b)}{g(b)} = -1. \tag{1}$$

Since $\frac{g'}{g}$ is strictly decreasing between consecutive zeros of g, we must have $a < x_j < b$ for some j.

For all $i = 1, 2, \ldots, k-1$ we have $x_{i+1} - x_i > b - a$, hence $a - x_i > b - x_{i+1}$. If $i < j$, both sides of this inequality are negative; if $i \geq j$, both sides are positive. In any case, $\frac{1}{a-x_i} < \frac{1}{b-x_{i+1}}$, and hence

$$\frac{g'(a)}{g(a)} = \sum_{i=1}^{k-1} \frac{1}{a - x_i} + \underbrace{\frac{1}{a - x_k}}_{<0} < \sum_{i=1}^{k-1} \frac{1}{b - x_{i+1}} + \underbrace{\frac{1}{b - x_1}}_{>0} = \frac{g'(b)}{g(b)}$$

This contradicts (1).

Now we turn to the proof of the stated problem. Denote by m the degree of f. We will prove by induction on m that f_n has m distinct real zeros for sufficiently large n. The cases $m = 0, 1$ are trivial; so we assume $m \geq 2$. Without loss of generality we can assume that f is monic. By induction, the result holds for f', and by ignoring the first few terms we can assume that f_n' has $m - 1$ distinct real zeros for all n. Let us denote these zeros by $x_1^{(n)} > x_2^{(n)} > \cdots > x_{m-1}^{(n)}$. Then f_n has minima in $x_1^{(n)}, x_3^{(n)}, x_5^{(n)}, \ldots$, and maxima in $x_2^{(n)}, x_4^{(n)}, x_6^{(n)}, \ldots$. Note that in the interval $(x_{i+1}^{(n)}, x_i^{(n)})$, the function $f_{n+1}' = f_n' + f_n''$ must have a zero (this follows by applying Rolle's theorem to the function $e^x f_n'(x)$); the same is true for the interval $(-\infty, x_{m-1}^{(n)})$. Hence, in each of these $m - 1$ intervals, f_{n+1}' has *exactly* one zero. This shows that

$$x_1^{(n)} > x_1^{(n+1)} > x_2^{(n)} > x_2^{(n+1)} > x_3^{(n)} > x_3^{(n+1)} > \ldots \tag{2}$$

Lemma 2. We have $\lim_{n\to\infty} f_n(x_j^{(n)}) = -\infty$ if j is odd, and $\lim\limits_{n\to\infty} f_n(x_j^{(n)}) = +\infty$ if j is even.

Lemma 2 immediately implies the result: For sufficiently large n, the values of all maxima of f_n are positive, and the values of all minima of f_n are negative; this implies that f_n has m distinct zeros.

Proof of Lemma 2: Let $d = \min\{d(f'), 1\}$; then by Lemma 1, $d(f'_n) \geq d$ for all n. Define $\varepsilon = \dfrac{(m-1)d^{m-1}}{m^{m-1}}$; we will show that

$$f_{n+1}(x_j^{(n+1)}) \geq f_n(x_j^{(n)}) + \varepsilon \quad \text{for } j \text{ even.} \tag{3}$$

(The corresponding result for odd j can be shown similarly.) Do to so, write $f = f_n$, $b = x_j^{(n)}$, and choose a satisfying $d \leq b - a \leq 1$ such that f' has no zero inside (a, b). Define ξ by the relation $b - \xi = \dfrac{1}{m}(b - a)$; then $\xi \in (a, b)$. We show that $f(\xi) + f'(\xi) \geq f(b) + \varepsilon$.

Notice, that

$$\begin{aligned}
\frac{f''(\xi)}{f'(\xi)} &= \sum_{i=1}^{m-1} \frac{1}{\xi - x_i^{(n)}} \\
&= \sum_{i<j} \underbrace{\frac{1}{\xi - x_i^{(n)}}}_{<\frac{1}{\xi - a}} + \frac{1}{\xi - b} + \sum_{i>j} \underbrace{\frac{1}{\xi - x_i^{(n)}}}_{<0} \\
&< (m-1)\frac{1}{\xi - a} + \frac{1}{\xi - b} = 0.
\end{aligned}$$

The last equality holds by definition of ξ. Since f' is positive and $\dfrac{f''}{f'}$ is decreasing in (a, b), we have that f'' is negative on (ξ, b). Therefore,

$$f(b) - f(\xi) = \int_\xi^b f'(t)dt \leq \int_\xi^b f'(\xi)dt = (b - \xi)f'(\xi)$$

Hence,

$$\begin{aligned}
f(\xi) + f'(\xi) &\geq f(b) - (b - \xi)f'(\xi) + f'(\xi) \\
&= f(b) + (1 - (\xi - b))f'(\xi) \\
&= f(b) + (1 - \tfrac{1}{m}(b - a))f'(\xi) \\
&\geq f(b) + (1 - \tfrac{1}{m})f'(\xi).
\end{aligned}$$

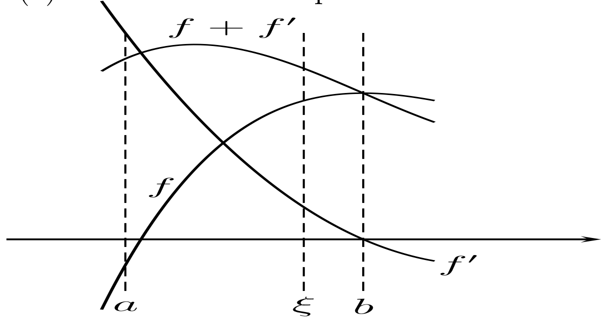
Together with

$$f'(\xi) = |f'(\xi)| = m \prod_{i=1}^{m-1} \underbrace{|\xi - x_i^{(n)}|}_{\geq |\xi - b|} \geq m|\xi - b|^{m-1} \geq \frac{d^{m-1}}{m^{m-2}}$$

we get

$$f(\xi) + f'(\xi) \geq f(b) + \varepsilon.$$

Together with (2) this shows (3). This finishes the proof of Lemma 2.



Problem 6. For a permutation $\sigma = (i_1, i_2, ..., i_n)$ of $(1, 2, ..., n)$ define $D(\sigma) = \sum_{k=1}^{n} |i_k - k|$. Let $Q(n, d)$ be the number of permutations σ of $(1, 2, ..., n)$ with $d = D(\sigma)$. Prove that $Q(n, d)$ is even for $d \geq 2n$.

Solution. Consider the $n \times n$ determinant

$$\Delta(x) = \begin{vmatrix} 1 & x & \dots & x^{n-1} \\ x & 1 & \dots & x^{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ x^{n-1} & x^{n-2} & \dots & 1 \end{vmatrix}$$

where the ij-th entry is $x^{|i-j|}$. From the definition of the determinant we get

$$\Delta(x) = \sum_{(i_1,...,i_n) \in S_n} (-1)^{\mathrm{inv}(i_1,...,i_n)} x^{D(i_1,...,i_n)}$$

where S_n is the set of all permutations of $(1, 2, ..., n)$ and $\mathrm{inv}(i_1, ..., i_n)$ denotes the number of inversions in the sequence $(i_1, ..., i_n)$. So $Q(n, d)$ has the same parity as the coefficient of x^d in $\Delta(x)$.

It remains to evaluate $\Delta(x)$. In order to eliminate the entries below the diagonal, subtract the $(n-1)$-th row, multiplied by x, from the n-th row. Then subtract the $(n-2)$-th row, multiplied by x, from the $(n-1)$-th and so on. Finally, subtract the first row, multiplied by x, from the second row.

$$\Delta(x) = \begin{vmatrix} 1 & x & \dots & x^{n-2} & x^{n-1} \\ x & 1 & \dots & x^{n-3} & x^{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x^{n-2} & x^{n-3} & \dots & 1 & x \\ x^{n-1} & x^{n-2} & \dots & x & 1 \end{vmatrix} = \ldots = \begin{vmatrix} 1 & x & \dots & x^{n-2} & x^{n-1} \\ 0 & 1-x^2 & \dots & x^{n-3}-x^{n-1} & x^{n-2}-x^n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1-x^2 & x-x^3 \\ 0 & 0 & \dots & 0 & 1-x^2 \end{vmatrix} = (1-x^2)^{n-1}.$$

For $d \geq 2n$, the coefficient of x^d is 0 so $Q(n, d)$ is even.

IMC2008, Blagoevgrad, Bulgaria

Day 1, July 27, 2008

Problem 1. Find all continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ such that $f(x) - f(y)$ is rational for all reals x and y such that $x - y$ is rational.

Solution. We prove that $f(x) = ax + b$ where $a \in \mathbb{Q}$ and $b \in \mathbb{R}$. These functions obviously satify the conditions.

Suppose that a function $f(x)$ fulfills the required properties. For an arbitrary rational q, consider the function $g_q(x) = f(x+q) - f(x)$. This is a continuous function which attains only rational values, therefore g_q is constant.

Set $a = f(1) - f(0)$ and $b = f(0)$. Let n be an arbitrary positive integer and let $r = f(1/n) - f(0)$. Since $f(x + 1/n) - f(x) = f(1/n) - f(0) = r$ for all x, we have

$$f(k/n) - f(0) = (f(1/n) - f(0)) + (f(2/n) - f(1/n)) + \ldots + (f(k/n) - f((k-1)/n) = kr$$

and

$$f(-k/n) - f(0) = -(f(0) - f(-1/n)) - (f(-1/n) - f(-2/n)) - \ldots - (f(-(k-1)/n) - f(-k/n) = -kr$$

for $k \geq 1$. In the case $k = n$ we get $a = f(1) - f(0) = nr$, so $r = a/n$. Hence, $f(k/n) - f(0) = kr = ak/n$ and then $f(k/n) = a \cdot k/n + b$ for all integers k and $n > 0$.

So, we have $f(x) = ax + b$ for all rational x. Since the function f is continous and the rational numbers form a dense subset of $\mathbb{R}$, the same holds for all real x.

Problem 2. Denote by V the real vector space of all real polynomials in one variable, and let $P\colon V \to \mathbb{R}$ be a linear map. Suppose that for all $f, g \in V$ with $P(fg) = 0$ we have $P(f) = 0$ or $P(g) = 0$. Prove that there exist real numbers x_0, c such that $P(f) = c\,f(x_0)$ for all $f \in V$.

Solution. We can assume that $P \neq 0$.

Let $f \in V$ be such that $P(f) \neq 0$. Then $P(f^2) \neq 0$, and therefore $P(f^2) = aP(f)$ for some non-zero real a. Then $0 = P(f^2 - af) = P(f(f-a))$ implies $P(f - a) = 0$, so we get $P(a) \neq 0$. By rescaling, we can assume that $P(1) = 1$. Now $P(X + b) = 0$ for $b = -P(X)$. Replacing P by $\hat{P}$ given as

$$\hat{P}(f(X)) = P(f(X + b))$$

we can assume that $P(X) = 0$.

Now we are going to prove that $P(X^k) = 0$ for all $k \geq 1$. Suppose this is true for all $k < n$. We know that $P(X^n + e) = 0$ for $e = -P(X^n)$. From the induction hypothesis we get

$$P\big((X + e)(X + 1)^{n-1}\big) = P(X^n + e) = 0,$$

and therefore $P(X + e) = 0$ (since $P(X + 1) = 1 \neq 0$). Hence $e = 0$ and $P(X^n) = 0$, which completes the inductive step. From $P(1) = 1$ and $P(X^k) = 0$ for $k \geq 1$ we immediately get $P(f) = f(0)$ for all $f \in V$.

Problem 3. Let p be a polynomial with integer coefficients and let $a_1 < a_2 < \ldots < a_k$ be integers.

a) Prove that there exists $a \in \mathbb{Z}$ such that $p(a_i)$ divides $p(a)$ for all $i = 1, 2, \ldots, k$.

b) Does there exist an $a \in \mathbb{Z}$ such that the product $p(a_1) \cdot p(a_2) \cdot \ldots \cdot p(a_k)$ divides $p(a)$?

Solution. The theorem is obvious if $p(a_i) = 0$ for some i, so assume that all $p(a_i)$ are nonzero and pairwise different.

There exist numbers s, t such that $s|p(a_1)$, $t|p(a_2)$, $st = \operatorname{lcm}(p(a_1), p(a_2))$ and $\gcd(s,t) = 1$.

As s, t are relatively prime numbers, there exist $m, n \in \mathbb{Z}$ such that $a_1 + sn = a_2 + tm =: b_2$. Obviously $s|p(a_1 + sn) - p(a_1)$ and $t|p(a_2 + tm) - p(a_2)$, so $st|p(b_2)$.

Similarly one obtains b_3 such that $p(a_3)|p(b_3)$ and $p(b_2)|p(b_3)$ thus also $p(a_1)|p(b_3)$ and $p(a_2)|p(b_3)$.

Reasoning inductively we obtain the existence of $a = b_k$ as required.

The polynomial $p(x) = 2x^2 + 2$ shows that the second part of the problem is not true, as $p(0) = 2$, $p(1) = 4$ but no value of $p(a)$ is divisible by 8 for integer a.

Remark. One can assume that the $p(a_i)$ are nonzero and ask for a such that $p(a)$ is a nonzero multiple of all $p(a_i)$. In the solution above, it can happen that $p(a) = 0$. But every number $p(a + np(a_1)p(a_2)\ldots p(a_k))$ is also divisible by every $p(a_i)$, since the polynomial is nonzero, there exists n such that $p(a + np(a_1)p(a_2)\ldots p(a_k))$ satisfies the modified thesis.

Problem 4. We say a triple (a_1, a_2, a_3) of nonnegative reals is *better* than another triple (b_1, b_2, b_3) if **two out of the three** following inequalities $a_1 > b_1$, $a_2 > b_2$, $a_3 > b_3$ are satisfied. We call a triple (x, y, z) *special* if x, y, z are nonnegative and $x + y + z = 1$. Find all natural numbers n for which there is a set S of n *special* triples such that for any given *special* triple we can find at least one *better* triple in S.

Solution. The answer is $n \geqslant 4$.

Consider the following set of special triples:

$$\left(0, \frac{8}{15}, \frac{7}{15}\right), \quad \left(\frac{2}{5}, 0, \frac{3}{5}\right), \quad \left(\frac{3}{5}, \frac{2}{5}, 0\right), \quad \left(\frac{2}{15}, \frac{11}{15}, \frac{2}{15}\right).$$

We will prove that any special triple (x, y, z) is worse than one of these (triple a is worse than triple b if triple b is better than triple a). We suppose that some special triple (x, y, z) is actually not worse than the first three of the triples from the given set, derive some conditions on x, y, z and prove that, under these conditions, (x, y, z) is worse than the fourth triple from the set.

Triple (x, y, z) is not worse than $\left(0, \frac{8}{15}, \frac{7}{15}\right)$ means that $y \geqslant \frac{8}{15}$ or $z \geqslant \frac{7}{15}$. Triple (x, y, z) is not worse than $\left(\frac{2}{5}, 0, \frac{3}{5}\right)$ — $x \geqslant \frac{2}{5}$ or $z \geqslant \frac{3}{5}$. Triple (x, y, z) is not worse than $\left(\frac{3}{5}, \frac{2}{5}, 0\right)$ — $x \geqslant \frac{3}{5}$ or $y \geqslant \frac{2}{5}$. Since $x + y + z = 1$, then it is impossible that all inequalities $x \geqslant \frac{2}{5}$, $y \geqslant \frac{2}{5}$ and $z \geqslant \frac{7}{15}$ are true. Suppose that $x < \frac{2}{5}$, then $y \geqslant \frac{2}{5}$ and $z \geqslant \frac{3}{5}$. Using $x + y + z = 1$ and $x \geqslant 0$ we get $x = 0, y = \frac{2}{5}, z = \frac{3}{5}$. We obtain the triple $\left(0, \frac{2}{5}, \frac{3}{5}\right)$ which is worse than $\left(\frac{2}{15}, \frac{11}{15}, \frac{2}{15}\right)$. Suppose that $y < \frac{2}{5}$, then $x \geqslant \frac{3}{5}$ and $z \geqslant \frac{7}{15}$ and this is a contradiction to the admissibility of (x, y, z). Suppose that $z < \frac{7}{15}$, then $x \geqslant \frac{2}{5}$ and $y \geqslant \frac{8}{15}$. We get (by admissibility, again) that $z \leqslant \frac{1}{15}$ and $y \leqslant \frac{3}{5}$. The last inequalities imply that $\left(\frac{2}{15}, \frac{11}{15}, \frac{2}{15}\right)$ is better than (x, y, z).

We will prove that for any given set of three special triples one can find a special triple which is not worse than any triple from the set. Suppose we have a set S of three special triples

$$(x_1, y_1, z_1), \quad (x_2, y_2, z_2), \quad (x_3, y_3, z_3).$$

Denote $a(S) = min(x_1, x_2, x_3), b(S) = min(y_1, y_2, y_3), c(S) = min(z_1, z_2, z_3)$. It is easy to check that S_1:

$$\left(\frac{x_1 - a}{1-a-b-c}, \frac{y_1 - b}{1-a-b-c}, \frac{z_1 - c}{1-a-b-c}\right)$$
$$\left(\frac{x_2 - a}{1-a-b-c}, \frac{y_2 - b}{1-a-b-c}, \frac{z_2 - c}{1-a-b-c}\right)$$
$$\left(\frac{x_3 - a}{1-a-b-c}, \frac{y_3 - b}{1-a-b-c}, \frac{z_3 - c}{1-a-b-c}\right)$$

is a set of three special triples also (we may suppose that $a + b + c < 1$, because otherwise all three triples are equal and our statement is trivial).

If there is a special triple (x, y, z) which is not worse than any triple from S_1, then the triple

$$((1-a-b-c)x + a, (1-a-b-c)y + b, (1-a-b-c)z + c)$$

is special and not worse than any triple from S. We also have $a(S_1) = b(S_1) = c(S_1) = 0$, so we may suppose that the same holds for our starting set S.

Suppose that one element of S has two entries equal to 0.

Note that one of the two remaining triples from S is not worse than the other. This triple is also not worse than all triples from S because any special triple is not worse than itself and the triple with two zeroes.

So we have $a = b = c = 0$ but we may suppose that all triples from S contain at most one zero. By transposing triples and elements in triples (elements in all triples must be transposed simultaneously) we may achieve the following situation $x_1 = y_2 = z_3 = 0$ and $x_2 \geqslant x_3$. If $z_2 \geqslant z_1$, then the second triple $(x_2, 0, z_2)$ is not worse than the other two triples from S. So we may assume that $z_1 \geqslant z_2$. If $y_1 \geqslant y_3$, then the first triple is not worse than the second and the third and we assume $y_3 \geqslant y_1$. Consider the three pairs of numbers x_2, y_1; z_1, x_3; y_3, z_2. The sum of all these numbers is three and consequently the sum of the numbers in one of the pairs is less than or equal to one. If it is the first pair then the triple $(x_2, 1 - x_2, 0)$ is not worse than all triples from S, for the second we may take $(1 - z_1, 0, z_1)$ and for the third — $(0, y_3, 1 - y_3)$. So we found a desirable special triple for any given S.

Problem 5. Does there exist a finite group G with a normal subgroup H such that $|\operatorname{Aut} H| > |\operatorname{Aut} G|$?

Solution. Yes. Let H be the commutative group $H = \mathbb{F}_2^3$, where $\mathbb{F}_2 \cong \mathbb{Z}/2\mathbb{Z}$ is the field with two elements. The group of automorphisms of H is the general linear group $\mathrm{GL}_3\mathbb{F}_2$; it has

$$(8-1) \cdot (8-2) \cdot (8-4) = 7 \cdot 6 \cdot 4 = 168$$

elements. One of them is the shift operator $\phi : (x_1, x_2, x_3) \mapsto (x_2, x_3, x_1)$.

Now let $T = \{a^0, a^1, a^2\}$ be a group of order 3 (written multiplicatively); it acts on H by $\tau(a) = \phi$. Let G be the semidirect product $G = H \rtimes_\tau T$. In other words, G is the group of 24 elements

$$G = \{ba^i : \; b \in H, \, i \in (\mathbb{Z}/3\mathbb{Z})\}, \quad ab = \phi(b)a.$$

G has one element e of order 1 and seven elements b, $b \in H$, $b \neq e$ of order 2.

If $g = ba$, we find that $g^2 = baba = b\phi(b)a^2 \neq e$, and that

$$g^3 = b\phi(b)a^2ba = b\phi(b)a\phi(b)a^2 = b\phi(b)\phi^2(b)a^3 = \psi(b),$$

where the homomorphism $\psi : H \to H$ is defined as $\psi : (x_1, x_2, x_3) \mapsto (x_1 + x_2 + x_3)(1, 1, 1)$. It is clear that $g^3 = \psi(b) = e$ for 4 elements $b \in H$, while $g^6 = \psi^2(b) = e$ for all $b \in H$.

We see that G has 8 elements of order 3, namely ba and ba^2 with $b \in \operatorname{Ker}\psi$, and 8 elements of order 6, namely ba and ba^2 with $b \notin \operatorname{Ker}\psi$. That accounts for orders of all elements of G.

Let $b_0 \in H \setminus \operatorname{Ker}\psi$ be arbitrary; it is easy to see that G is generated by b_0 and a. As every automorphism of G is fully determined by its action on b_0 and a, it follows that G has no more than

$$7 \cdot 8 = 56$$

automorphisms.

Remark. G and H can be equivalently presented as subgroups of S_6, namely as $H = \langle(12), (34), (56)\rangle$ and $G = \langle(135)(246), (12)\rangle$.

is a Cauchy sequence in $\mathcal{H}$. (This is the crucial observation.) Indeed, for $m > n$, the norm $\|y_m - y_n\|$ may be computed by the above remark as

$$\|y_m - y_n\|^2 = \frac{d^2}{2}\left\|\left(\frac{1}{m}-\frac{1}{n},\ldots,\frac{1}{m}-\frac{1}{n},\frac{1}{m},\ldots,\frac{1}{m}\right)^{\top}\right\|^2_{\mathbb{R}^m} = \frac{d^2}{2}\left(\frac{n(m-n)^2}{m^2n^2}+\frac{m-n}{m^2}\right)$$
$$= \frac{d^2}{2}\frac{(m-n)(m-n+n)}{m^2n} = \frac{d^2}{2}\frac{m-n}{mn} = \frac{d^2}{2}\left(\frac{1}{n}-\frac{1}{m}\right) \to 0, \quad m,n\to\infty.$$

By completeness of $\mathcal{H}$, it follows that there exists a limit

$$y = \lim_{n\to\infty} y_n \in \mathcal{H}.$$

We claim that y sastisfies all conditions of the problem. For $m > n > p$, with n, p fixed, we compute

$$\|x_n - y_m\|^2 = \frac{d^2}{2}\left\|\left(-\frac{1}{m},\ldots,-\frac{1}{m},1-\frac{1}{m},-\frac{1}{m},\ldots,-\frac{1}{m}\right)^{\top}\right\|^2_{\mathbb{R}^m}$$
$$= \frac{d^2}{2}\left[\frac{m-1}{m^2}+\frac{(m-1)^2}{m^2}\right] = \frac{d^2}{2}\frac{m-1}{m} \to \frac{d^2}{2}, \quad m\to\infty,$$

showing that $\|x_n - y\| = d/\sqrt{2}$, as well as

$$\langle x_n - y_m, x_p - y_m\rangle = \frac{d^2}{2}\left\langle\left(-\frac{1}{m},\ldots,-\frac{1}{m},\ldots,1-\frac{1}{m},\ldots,-\frac{1}{m}\right)^{\top},\right.$$
$$\left.\left(-\frac{1}{m},\ldots,1-\frac{1}{m},\ldots,-\frac{1}{m},\ldots,-\frac{1}{m}\right)^{\top}\right\rangle_{\mathbb{R}^m}$$
$$= \frac{d^2}{2}\left[\frac{m-2}{m^2}-\frac{2}{m}\left(1-\frac{1}{m}\right)\right] = -\frac{d^2}{2m} \to 0, \quad m\to\infty,$$

showing that $\langle x_n - y, x_p - y\rangle = 0$, so that

$$\left\{\frac{\sqrt{2}}{d}(x_n - y):\ n\in\mathbb{N}\right\}$$

is indeed an orthonormal system of vectors.

This completes the proof in the case when $T = S$, which we can always take if S is countable. If it is not, let x', x'' be any two distinct points in $S\setminus T$. Then applying the above procedure to the set

$$T' = \{x', x'', x_1, x_2, \ldots, x_n, \ldots\}$$

it follows that

$$\lim_{n\to\infty}\frac{x'+x''+x_1+x_2+\cdots+x_n}{n+2} = \lim_{n\to\infty}\frac{x_1+x_2+\cdots+x_n}{n} = y$$

satisfies that

$$\left\{\frac{\sqrt{2}}{d}(x'-y), \frac{\sqrt{2}}{d}(x''-y)\right\} \cup \left\{\frac{\sqrt{2}}{d}(x_n - y):\ n\in\mathbb{N}\right\}$$

is still an orthonormal system.

This it true for any distinct $x', x'' \in S\setminus T$; it follows that the entire system

$$\left\{\frac{\sqrt{2}}{d}(x - y):\ x\in S\right\}$$

is an orthonormal system of vectors in $\mathcal{H}$, as required.

IMC2008, Blagoevgrad, Bulgaria

Day 2, July 28, 2008

Problem 1. Let n, k be positive integers and suppose that the polynomial $x^{2k} - x^k + 1$ divides $x^{2n} + x^n + 1$. Prove that $x^{2k} + x^k + 1$ divides $x^{2n} + x^n + 1$.

Solution. Let $f(x) = x^{2n} + x^n + 1$, $g(x) = x^{2k} - x^k + 1$, $h(x) = x^{2k} + x^k + 1$. The complex number $x_1 = \cos(\frac{\pi}{3k}) + i\sin(\frac{\pi}{3k})$ is a root of $g(x)$.

Let $\alpha = \frac{\pi n}{3k}$. Since $g(x)$ divides $f(x)$, $f(x_1) = g(x_1) = 0$. So, $0 = x_1^{2n} + x_1^n + 1 = (\cos(2\alpha) + i\sin(2\alpha)) + (\cos\alpha + i\sin\alpha) + 1 = 0$, and $(2\cos\alpha + 1)(\cos\alpha + i\sin\alpha) = 0$. Hence $2\cos\alpha + 1 = 0$, i.e. $\alpha = \pm\frac{2\pi}{3} + 2\pi c$, where $c \in \mathbb{Z}$.

Let x_2 be a root of the polynomial $h(x)$. Since $h(x) = \frac{x^{3k}-1}{x^k-1}$, the roots of the polynomial $h(x)$ are distinct and they are $x_2 = \cos\frac{2\pi s}{3k} + i\sin\frac{2\pi s}{3k}$, where $s = 3a \pm 1, a \in \mathbb{Z}$. It is enough to prove that $f(x_2) = 0$. We have $f(x_2) = x_2^{2n} + x_2^n + 1 = (\cos(4s\alpha) + \sin(4s\alpha)) + (\cos(2s\alpha) + \sin(2s\alpha)) + 1 = (2\cos(2s\alpha) + 1)(\cos(2s\alpha) + i\sin(2s\alpha)) = 0$ (since $2\cos(2s\alpha) + 1 = 2\cos(2s(\pm\frac{2\pi}{3} + 2\pi c)) + 1 = 2\cos(\frac{4\pi s}{3}) + 1 = 2\cos(\frac{4\pi}{3}(3a \pm 1)) + 1 = 0$).

Problem 2. Two different ellipses are given. One focus of the first ellipse coincides with one focus of the second ellipse. Prove that the ellipses have at most two points in common.

Solution. It is well known that an ellipse might be defined by a focus (a point) and a directrix (a straight line), as a locus of points such that the distance to the focus divided by the distance to directrix is equal to a given number $e < 1$. So, if a point X belongs to both ellipses with the same focus F and directrices l_1, l_2, then $e_1 \cdot l_1X = FX = e_2 \cdot l_2X$ (here we denote by l_1X, l_2X distances between the corresponding line and the point X). The equation $e_1 \cdot l_1X = e_2 \cdot l_2X$ defines two lines, whose equations are linear combinations with coefficients $e_1, \pm e_2$ of the normalized equations of lines l_1, l_2 but of those two only one is relevant, since X and F should lie on the same side of each directrix. So, we have that all possible points lie on one line. The intersection of a line and an ellipse consists of at most two points.

Problem 3. Let n be a positive integer. Prove that 2^{n-1} divides

$$\sum_{0\le k<n/2}\binom{n}{2k+1}5^k.$$

Solution. As is known, the Fibonacci numbers F_n can be expressed as $F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$. Expanding this expression, we obtain that $F_n = \frac{1}{2^{n-1}}\left(\binom{n}{1} + \binom{n}{3}5 + \ldots + \binom{n}{l}5^{\frac{l-1}{2}}\right)$, where l is the greatest odd number such that $l \le n$ and $s = \frac{l-1}{2} \le \frac{n}{2}$.

So, $F_n = \frac{1}{2^{n-1}}\sum_{k=0}^{s}\binom{n}{2k+1}5^k$, which implies that 2^{n-1} divides $\sum_{0\le k<n/2}\binom{n}{2k+1}5^k$.

Problem 4. Let $\mathbb{Z}[x]$ be the ring of polynomials with integer coefficients, and let $f(x), g(x) \in \mathbb{Z}[x]$ be nonconstant polynomials such that $g(x)$ divides $f(x)$ in $\mathbb{Z}[x]$. Prove that if the polynomial $f(x) - 2008$ has at least 81 distinct integer roots, then the degree of $g(x)$ is greater than 5.

Solution. Let $f(x) = g(x)h(x)$ where $h(x)$ is a polynomial with integer coefficients.

Let $a_1, \ldots, a_{81}$ be distinct integer roots of the polynomial $f(x) - 2008$. Then $f(a_i) = g(a_i)h(a_i) = 2008$ for $i = 1, \ldots, 81$, Hence, $g(a_1), \ldots, g(a_{81})$ are integer divisors of 2008.

Since $2008 = 2^3 \cdot 251$ (2, 251 are primes) then 2008 has exactly 16 distinct integer divisors (including the negative divisors as well). By the pigeonhole principle, there are at least 6 equal numbers among $g(a_1), \ldots, g(a_{81})$ (because $81 > 16 \cdot 5$). For example, $g(a_1) = g(a_2) = \ldots = g(a_6) = c$. So $g(x) - c$ is a nonconstant polynomial which has at least 6 distinct roots (namely $a_1, \dots, a_6$). Then the degree of the polynomial $g(x) - c$ is at least 6.

Problem 5. Let n be a positive integer, and consider the matrix $A = (a_{ij})_{1 \le i,j \le n}$, where

$$a_{ij} = \begin{cases} 1 & \text{if } i+j \text{ is a prime number,} \\ 0 & \text{otherwise.} \end{cases}$$

Prove that $|\det A| = k^2$ for some integer k.

Solution. Call a square matrix *of type (B)*, if it is of the form

$$\begin{pmatrix} 0 & b_{12} & 0 & \dots & b_{1,2k-2} & 0 \\ b_{21} & 0 & b_{23} & \dots & 0 & b_{2,2k-1} \\ 0 & b_{32} & 0 & \dots & b_{3,2k-2} & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{2k-2,1} & 0 & b_{2k-2,3} & \dots & 0 & b_{2k-2,2k-1} \\ 0 & b_{2k-1,2} & 0 & \dots & b_{2k-1,2k-2} & 0 \end{pmatrix}.$$

Note that every matrix of this form has determinant zero, because it has k columns spanning a vector space of dimension at most $k-1$.

Call a square matrix *of type (C)*, if it is of the form

$$C' = \begin{pmatrix} 0 & c_{11} & 0 & c_{12} & \dots & 0 & c_{1,k} \\ c_{11} & 0 & c_{12} & 0 & \dots & c_{1,k} & 0 \\ 0 & c_{21} & 0 & c_{22} & \dots & 0 & c_{2,k} \\ c_{21} & 0 & c_{22} & 0 & \dots & c_{2,k} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & c_{k,1} & 0 & c_{k,2} & \dots & 0 & c_{k,k} \\ c_{k,1} & 0 & c_{k,2} & 0 & \dots & c_{k,k} & 0 \end{pmatrix}$$

By permutations of rows and columns, we see that

$$|\det C'| = \left| \det \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix} \right| = |\det C|^2,$$

where C denotes the $k \times k$-matrix with coefficients $c_{i,j}$. Therefore, the determinant of any matrix of type (C) is a perfect square (up to a sign).

Now let X' be the matrix obtained from A by replacing the first row by $\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \end{pmatrix}$, and let Y be the matrix obtained from A by replacing the entry a_{11} by 0. By multi-linearity of the determinant, $\det(A) = \det(X') + \det(Y)$. Note that X' can be written as

$$X' = \begin{pmatrix} 1 & 0 \\ v & X \end{pmatrix}$$

for some $(n-1) \times (n-1)$-matrix X and some column vector v. Then $\det(A) = \det(X) + \det(Y)$. Now consider two cases. If n is odd, then X is of type (C), and Y is of type (B). Therefore, $|\det(A)| = |\det(X)|$ is a perfect square. If n is even, then X is of type (B), and Y is of type (C); hence $|\det(A)| = |\det(Y)|$ is a perfect square.

The set of primes can be replaced by any subset of $\{2\} \cup \{3, 5, 7, 9, 11, \dots\}$.

Problem 6. Let $\mathcal{H}$ be an infinite-dimensional real Hilbert space, let $d > 0$, and suppose that S is a set of points (not necessarily countable) in $\mathcal{H}$ such that the distance between any two distinct points in S is equal to d. Show that there is a point $y \in \mathcal{H}$ such that

$$\left\{ \frac{\sqrt{2}}{d}(x - y) : \ x \in S \right\}$$

is an orthonormal system of vectors in $\mathcal{H}$.

Solution. It is clear that, if $\mathcal{B}$ is an orthonormal system in a Hilbert space $\mathcal{H}$, then $\{(d/\sqrt{2})e : \ e \in \mathcal{B}\}$ is a set of points in $\mathcal{H}$, any two of which are at distance d apart. We need to show that every set S of equidistant points is a translate of such a set.

We begin by noting that, if $x_1, x_2, x_3, x_4 \in S$ are four distinct points, then

$$\begin{aligned} \langle x_2 - x_1, x_2 - x_1 \rangle &= d^2, \\ \langle x_2 - x_1, x_3 - x_1 \rangle &= \frac{1}{2}\left(\|x_2 - x_1\|^2 + \|x_3 - x_1\|^2 - \|x_2 - x_3\|^2 \right) = \frac{1}{2}d^2, \\ \langle x_2 - x_1, x_4 - x_3 \rangle &= \langle x_2 - x_1, x_4 - x_1 \rangle - \langle x_2 - x_1, x_3 - x_1 \rangle = \frac{1}{2}d^2 - \frac{1}{2}d^2 = 0. \end{aligned}$$

This shows that scalar products among vectors which are finite linear combinations of the form

$$\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n,$$

where $x_1, x_2, \dots, x_n$ are distinct points in S and $\lambda_1, \lambda_2, \dots, \lambda_n$ are integers with $\lambda_1 + \lambda_2 + \cdots + \lambda_n = 0$, are universal across all such sets S in all Hilbert spaces $\mathcal{H}$; in particular, we may conveniently evaluate them using examples of our choosing, such as the canonical example above in $\mathbb{R}^n$. In fact this property trivially follows also when coefficients λ_i are rational, and hence by continuity any real numbers with sum 0.

If $S = \{x_1, x_2, \dots, x_n\}$ is a finite set, we form

$$x = \frac{1}{n}\left(x_1 + x_2 + \cdots + x_n \right),$$

pick a non-zero vector $z \in [\mathrm{Span}(x_1 - x, x_2 - x, \dots, x_n - x)]^\perp$ and seek y in the form $y = x + \lambda z$ for a suitable $\lambda \in \mathbb{R}$. We find that

$$\langle x_1 - y, x_2 - y \rangle = \langle x_1 - x - \lambda z, x_2 - x - \lambda z \rangle = \langle x_1 - x, x_2 - x \rangle + \lambda^2 \|z\|^2.$$

$\langle x_1 - x, x_2 - x \rangle$ may be computed by our remark above as

$$\begin{aligned} \langle x_1 - x, x_2 - x \rangle &= \frac{d^2}{2} \left\langle \left(\frac{1}{n} - 1, \frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n} \right)^\top, \left(\frac{1}{n}, \frac{1}{n} - 1, \frac{1}{n}, \dots, \frac{1}{n} \right)^\top \right\rangle_{\mathbb{R}^n} \\ &= \frac{d^2}{2} \left(\frac{2}{n}\left(\frac{1}{n} - 1 \right) + \frac{n-2}{n^2} \right) = -\frac{d^2}{2n}. \end{aligned}$$

So the choice $\lambda = \dfrac{d}{\sqrt{2n}\|z\|}$ will make all vectors $\dfrac{\sqrt{2}}{d}(x_i - y)$ orthogonal to each other; it is easily checked as above that they will also be of length one.

Let now S be an infinite set. Pick an infinite sequence $T = \{x_1, x_2, \dots, x_n, \dots\}$ of distinct points in S. We claim that the sequence

$$y_n = \frac{1}{n}(x_1 + x_2 + \cdots + x_n)$$

International Mathematics Competition for University Students
July 25–30 2009, Budapest, Hungary

Day 1

Problem 1.
Suppose that f and g are real-valued functions on the real line and $f(r) \le g(r)$ for every rational r. Does this imply that $f(x) \le g(x)$ for every real x if

a) f and g are non-decreasing?

b) f and g are continuous?

Solution. a) No. Counter-example: f and g can be chosen as the characteristic functions of $[\sqrt{3}, \infty)$ and $(\sqrt{3}, \infty)$, respectively.

b) Yes. By the assumptions $g - f$ is continuous on the whole real line and nonnegative on the rationals. Since any real number can be obtained as a limit of rational numbers we get that $g - f$ is nonnegative on the whole real line.

Problem 2.
Let A, B and C be real square matrices of the same size, and suppose that A is invertible. Prove that if $(A-B)C = BA^{-1}$, then $C(A-B) = A^{-1}B$.

Solution. A straightforward calculation shows that $(A-B)C = BA^{-1}$ is equivalent to $AC - BC - BA^{-1} + AA^{-1} = I$, where I denotes the identity matrix. This is equivalent to $(A-B)(C+A^{-1}) = I$. Hence, $(A-B)^{-1} = C + A^{-1}$, meaning that $(C + A^{-1})(A-B) = I$ also holds. Expansion yields the desired result.

Problem 3.
In a town every two residents who are not friends have a friend in common, and no one is a friend of everyone else. Let us number the residents from 1 to n and let a_i be the number of friends of the i-th resident. Suppose that $\sum_{i=1}^{n} a_i^2 = n^2 - n$. Let k be the smallest number of residents (at least three) who can be seated at a round table in such a way that any two neighbors are friends. Determine all possible values of k.

Solution. Let us define the simple, undirected graph G so that the vertices of G are the town's residents and the edges of G are the friendships between the residents. Let $V(G) = \{v_1, v_2, \dots, v_n\}$ denote the vertices of G; a_i is degree of v_i for every i. Let $E(G)$ denote the edges of G. In this terminology, the problem asks us to describe the length k of the shortest cycle in G.

Let us count the walks of length 2 in G, that is, the ordered triples (v_i, v_j, v_l) of vertices with $v_iv_j, v_jv_l \in E(G)$ ($i = l$ being allowed). For a given j the number is obviously a_j^2, therefore the total number is $\sum_{i=1}^{n} a_i^2 = n^2 - n$.

Now we show that there is an injection f from the set of ordered pairs of distinct vertices to the set of these walks. For $v_iv_j \notin E(G)$, let $f(v_i, v_j) = (v_i, v_l, v_j)$ with arbitrary l such that $v_iv_l, v_lv_j \in E(G)$. For $v_iv_j \in E(G)$, let $f(v_i, v_j) = (v_i, v_j, v_i)$. f is an injection since for $i \ne l$, (v_i, v_j, v_l) can only be the image of (v_i, v_l), and for $i = l$, it can only be the image of (v_i, v_j).

Since the number of ordered pairs of distinct vertices is $n^2 - n$, $\sum_{i=1}^{n} a_i^2 \ge n^2 - n$. Equality holds iff f is surjective, that is, iff there is exactly one l with $v_iv_l, v_lv_j \in E(G)$ for every i, j with $v_iv_j \notin E(G)$ and there is no such l for any i, j with $v_iv_j \in E(G)$. In other words, iff G contains neither C_3 nor C_4 (cycles of length 3 or 4), that is, G is either a forest (a cycle-free graph) or the length of its shortest cycle is at least 5.

It is easy to check that if every two vertices of a forest are connected by a path of length at most 2, then the forest is a star (one vertex is connected to all others by an edge). But G has n vertices, and none of them has degree $n - 1$. Hence G is not forest, so it has cycles. On the other hand, if the length of a cycle C of G is at least 6 then it has two vertices such that both arcs of C connecting them are longer than 2. Hence there is a path connecting them that is shorter than both arcs. Replacing one of the arcs by this path, we have a closed walk shorter than C. Therefore length of the shortest cycle is 5.

Finally, we must note that there is at least one G with the prescribed properties – e.g. the cycle C_5 itself satisfies the conditions. Thus 5 is the sole possible value of k.

Problem 4.

Let $p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ be a complex polynomial. Suppose that $1 = c_0 \geq c_1 \geq \cdots \geq c_n \geq 0$ is a sequence of real numbers which is convex (i.e. $2c_k \leq c_{k-1} + c_{k+1}$ for every $k = 1, 2, \ldots, n-1$), and consider the polynomial

$$q(z) = c_0 a_0 + c_1 a_1 z + c_2 a_2 z^2 + \cdots + c_n a_n z^n.$$

Prove that

$$\max_{|z|\leq 1} \big|q(z)\big| \leq \max_{|z|\leq 1} \big|p(z)\big|.$$

Solution. The polynomials p and q are regular on the complex plane, so by the Maximum Principle, $\max_{|z|\leq 1} |q(z)| = \max_{|z|=1} |q(z)|$, and similarly for p. Let us denote $M_f = \max_{|z|=1} |f(z)|$ for any regular function f. Thus it suffices to prove that $M_q \leq M_p$.

First, note that we can assume $c_n = 0$. Indeed, for $c_n = 1$, we get $p = q$ and the statement is trivial; otherwise, $q(z) = c_n p(z) + (1 - c_n) r(z)$, where $r(z) = \sum_{j=0}^{n} \frac{c_j - c_n}{1 - c_n} a_j z^j$. The sequence $c_j' = \frac{c_j - c_n}{1 - c_n}$ also satisfies the prescribed conditions (it is a positive linear transform of the sequence c_n with $c_0' = 1$), but $c_n' = 0$ too, so we get $M_r \leq M_p$. This is enough: $M_q = |q(z_0)| \leq c_n |p(z_0)| + (1 - c_n)|r(z_0)| \leq c_n M_p + (1 - c_n) M_r \leq M_p$.

Using the Cauchy formulas, we can express the coefficients a_j of p from its values taken over the positively oriented circle $S = \{|z| = 1\}$:

$$a_j = \frac{1}{2\pi i} \int_S \frac{p(z)}{z^{j+1}} \mathrm{d}z = \frac{1}{2\pi} \int_S \frac{p(z)}{z^j} |dz|$$

for $0 \leq j \leq n$, otherwise

$$\int_S \frac{p(z)}{z^j} |dz| = 0.$$

Let us use these identities to get a new formula for q, using only the values of p over S:

$$2\pi \cdot q(w) = \sum_{j=0}^{n} c_j \left(\int_S p(z) z^{-j} |dz| \right) w^j.$$

We can exchange the order of the summation and the integration (sufficient conditions to do this obviously apply):

$$2\pi \cdot q(w) = \int_S \left(\sum_{j=0}^{n} c_j (w/z)^j \right) p(z) |dz|.$$

It would be nice if the integration kernel (the sum between the brackets) was real. But this is easily arranged – for $-n \leq j \leq -1$, we can add the conjugate expressions, because by the above remarks, they are zero anyway:

$$2\pi \cdot q(w) = \sum_{j=0}^{n} c_j \left(\int_S p(z) z^{-j} |dz| \right) w^j = \sum_{j=-n}^{n} c_{|j|} \left(\int_S p(z) z^{-j} |dz| \right) w^j,$$

$$2\pi \cdot q(w) = \int_S \left(\sum_{j=-n}^{n} c_{|j|} (w/z)^j \right) p(z) |dz| = \int_S K(w/z) p(z) |dz|,$$

where

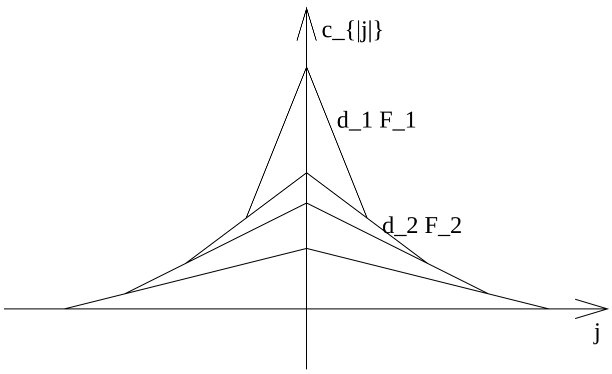
$$K(u) = \sum_{j=-n}^{n} c_{|j|} u^j = c_0 + 2 \sum_{j=1}^{n} c_j \Re(u^j)$$

for $u \in S$.

Let us examine $K(u)$. It is a real-valued function. Again from the Cauchy formulas, $\int_S K(u)|\mathrm{d}u| = 2\pi c_0 = 2\pi$. If $\int_S |K(u)||\mathrm{d}u| = 2\pi$ still holds (taking the absolute value does not increase the integral), then for every w:

$$2\pi |q(w)| = \left| \int_S K(w/z) p(z) |dz| \right| \leq \int_S |K(w/z)| \cdot |p(z)| |dz| \leq M_p \int_S |K(u)| |du| = 2\pi M_p;$$

this would conclude the proof. So it suffices to prove that $\int_S |K(u)||\mathrm{d}u| = \int_S K(u)|\mathrm{d}u|$, which is to say, K is non-negative.



Now let us decompose K into a sum using the given conditions for the numbers c_j (including $c_n = 0$). Let $d_k = c_{k-1} - 2c_k + c_{k+1}$ for $k = 1, \dots, n$ (setting $c_{n+1} = 0$); we know that $d_k \geq 0$. Let $F_k(u) = \sum_{j=-k+1}^{k-1} (k - |j|)u^j$. Then $K(u) = \sum_{k=1}^{n} d_k F_k(u)$ by easy induction (or see Figure for a graphical illustration). So it suffices to prove that $F_k(u)$ is real and $F_k(u) \geq 0$ for $u \in S$. This is reasonably well-known (as $\frac{F_k}{k}$ is the Fejér kernel), and also very easy:

$$F_k(u) = (1 + u + u^2 + \dots + u^{k-1})(1 + u^{-1} + u^{-2} + \dots + u^{-(k-1)}) =$$
$$= (1 + u + u^2 + \dots + u^{k-1})\overline{(1 + u + u^2 + \dots + u^{k-1})} = |1 + u + u^2 + \dots + u^{k-1}|^2 \geq 0$$

This completes the proof.

Problem 5.
Let n be a positive integer. An *n-simplex* in $\mathbb{R}^n$ is given by $n+1$ points $P_0, P_1, \dots, P_n$, called its *vertices*, which do not all belong to the same hyperplane. For every n-simplex S we denote by $v(S)$ the volume of S, and we write $C(S)$ for the center of the unique sphere containing all the vertices of S.

Suppose that P is a point inside an n-simplex S. Let S_i be the n-simplex obtained from S by replacing its i-th vertex by P. Prove that

$$v(S_0)C(S_0) + v(S_1)C(S_1) + \dots + v(S_n)C(S_n) = v(S)C(S).$$

Solution 1. We will prove this by induction on n, starting with $n = 1$. In that case we are given an interval $[a, b]$ with a point $p \in (a, b)$, and we have to verify

$$(b - p)\frac{b + p}{2} + (p - a)\frac{p + a}{2} = (b - a)\frac{b + a}{2},$$

which is true.

Now let assume the result is true for $n - 1$ and prove it for n. We have to show that the point

$$X = \sum_{j=0}^{n} \frac{v(S_j)}{v(S)} O(S_j)$$

has the same distance to all the points $P_0, P_1, \dots, P_n$. Let $i \in \{0, 1, 2, \dots, n\}$ and define the sets $M_i = \{P_0, P_1, \dots, P_{i-1}, P_{i+1}, \dots, P_n\}$. The set of all points having the same distance to all points in M_i is a line h_i orthogonal to the hyperplane E_i determined by the points in M_i. We are going to show that X lies on every h_i. To do so, fix some index i and notice that

$$X = \frac{v(S_i)}{v(S)} O(S_i) + \frac{v(S) - v(S_i)}{v(S)} \cdot \underbrace{\sum_{j \neq i} \frac{v(S_j)}{v(S) - v(S_i)} O(S_j)}_{Y}$$

and $O(S_i)$ lies on h_i, so that it is enough to show that Y lies on h_i.

A map $f : \mathbb{R}_{>0} \to \mathbb{R}^n$ will be called *affine* if there are points $A, B \in \mathbb{R}^n$ such that $f(\lambda) = \lambda A + (1 - \lambda)B$. Consider the ray g starting in P_i and passing through P. For $\lambda > 0$ let $P_\lambda = (1 - \lambda)P + \lambda P_i$, so that P_λ is an affine function describing the points of g. For every such λ let S_j^λ be the n-simplex obtained from S by replacing the j-th vertex by P_λ. The point $O(S_j^\lambda)$ is the intersection of the fixed line h_j with the hyperplane orthogonal to

g and passing through the midpoint of the segment $\overline{P_iP_\lambda}$ which is given by an affine function. This implies that also $O(S_j^\lambda)$ is an affine function. We write $\varphi_j = \frac{v(S_j)}{v(S)-s(S_i)}$, and then

$$Y_\lambda = \sum_{j\neq i} \varphi_j O(S_j^\lambda)$$

is an affine function. We want to show that $Y_\lambda \in h_i$ for all λ (then specializing to $\lambda = 1$ gives the desired result). It is enough to do this for two different values of λ.

Let g intersect the sphere containing the vertices of S in a point Z; then $Z = P_\lambda$ for a suitable $\lambda > 0$, and we have $O(S_j^\lambda) = O(S)$ for all j, so that $Y_\lambda = O(S) \in h_i$. Now let g intersect the hyperplane E_i in a point Q; then $Q = P_\lambda$ for some $\lambda > 0$, and Q is different from Z. Define T to be the $(n-1)$-simplex with vertex set M_i, and let T_j be the $(n-1)$-simplex obtained from T by replacing the vertex P_j by Q. If we write v' for the volume of $(n-1)$-simplices in the hyperplane E_i, then

$$\begin{aligned} \frac{v'(T_j)}{v'(T)} &= \frac{v(S_j^\lambda)}{v(S)} = \frac{v(S_j^\lambda)}{\sum_{k\neq i} v(S_k^\lambda)} \\ &= \frac{\lambda v(S_j)}{\sum_{k\neq i} \lambda v(S_k)} = \frac{v(S_j)}{v(S)-v(S_i)} = \varphi_j. \end{aligned}$$

If p denotes the orthogonal projection onto E_i then $p(O(S_j^\lambda)) = O(T_j)$, so that $p(Y_\lambda) = \sum_{j\neq i} \varphi_j O(T_j)$ equals $O(T)$ by induction hypothesis, which implies $Y_\lambda \in p^{-1}(O(T)) = h_i$, and we are done.

Solution 2. For $n = 1$, the statement is checked easily.

Assume $n \geq 2$. Denote $O(S_j) - O(S)$ by q_j and $P_j - P$ by p_j. For all distinct j and k in the range $0, ..., n$ the point $O(S_j)$ lies on a hyperplane orthogonal to p_k and P_j lies on a hyperplane orthogonal to q_k. So we have

$$\begin{cases} \langle p_i, q_j - q_k \rangle = 0 \\ \langle q_i, p_j - p_k \rangle = 0 \end{cases}$$

for all $j \neq i \neq k$. This means that the value $\langle p_i, q_j \rangle$ is independent of j as long as $j \neq i$, denote this value by λ_i. Similarly, $\langle q_i, p_j \rangle = \mu_i$ for some μ_i. Since $n \geq 2$, these equalities imply that all the λ_i and μ_i values are equal, in particular, $\langle p_i, q_j \rangle = \langle p_j, q_i \rangle$ for any i and j.

We claim that for such p_i and q_i, the volumes

$$V_j = |\det(p_0, ..., p_{j-1}, p_{j+1}, ..., p_n)|$$

and

$$W_j = |\det(q_0, ..., q_{j-1}, q_{j+1}, ..., q_n)|$$

are proportional. Indeed, first assume that $p_0, ..., p_{n-1}$ and $q_0, ..., q_{n-1}$ are bases of $\mathbb{R}^n$, then we have

$$\begin{aligned} V_j &= \frac{1}{|\det(q_0, ..., q_{n-1})|} \left| \det\left((\langle p_k, q_l \rangle) \right)_{\substack{k\neq j \\ l<n}} \right| = \\ &= \frac{1}{|\det(q_0, ..., q_{n-1})|} \left| \det((\langle p_k, q_l \rangle))_{\substack{l\neq j \\ k<n}} \right| = \left| \frac{\det(p_0, ..., p_{n-1})}{\det(q_0, ..., q_{n-1})} \right| W_j. \end{aligned}$$

If our assumption did not hold after any reindexing of the vectors p_i and q_i, then both p_i and q_i span a subspace of dimension at most $n-1$ and all the volumes are 0.

Finally, it is clear that $\sum q_j W_j / \det(q_0, ..., q_n) = 0$: the weight of p_j is the height of 0 over the hyperplane spanned by the rest of the vectors q_k relative to the height of p_j over the same hyperplane, so the sum is parallel to all the faces of the simplex spanned by $q_0, ..., q_n$. By the argument above, we can change the weights to the proportional set of weights $V_j / \det(p_0, ..., p_n)$ and the sum will still be 0. That is,

$$\begin{aligned} 0 &= \sum q_j \frac{V_j}{\det(p_0, ..., p_n)} = \sum (O(S_j) - O(S)) \frac{v(S_j)}{v(S)} = \\ &= \frac{1}{v(S)} \left(\sum O(S_j) v(S_j) - O(S) \sum v(S_j) \right) = \frac{1}{v(S)} \left(\sum O(S_j) v(S_j) - O(S) v(S) \right), \end{aligned}$$

q.e.d.

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Day 2

Problem 1.
Let ℓ be a line and P a point in $\mathbb{R}^3$. Let S be the set of points X such that the distance from X to ℓ is greater than or equal to two times the distance between X and P. If the distance from P to ℓ is $d > 0$, find the volume of S.

Solution. We can choose a coordinate system of the space such that the line ℓ is the z-axis and the point P is $(d, 0, 0)$. The distance from the point (x, y, z) to ℓ is $\sqrt{x^2 + y^2}$, while the distance from P to X is $|PX| = \sqrt{(x-d)^2 + y^2 + z^2}$. Square everything to get rid of the square roots. The condition can be reformulated as follows: the square of the distance from ℓ to X is at least $4|PX|^2$.

$$x^2 + y^2 \geq 4((x-d)^2 + y^2 + z^2)$$

$$0 \geq 3x^2 - 8dx + 4d^2 + 3y^2 + 4z^2$$

$$\left(\frac{16}{3} - 4\right) d^2 \geq 3\left(x - \frac{4}{3}d\right)^2 + 3y^2 + 4z^2$$

A translation by $\frac{4}{3}d$ in the x-direction does not change the volume, so we get

$$\frac{4}{3}d^2 \geq 3x_1^2 + 3y^2 + 4z^2$$

$$1 \geq \left(\frac{3x_1}{2d}\right)^2 + \left(\frac{3y}{2d}\right)^2 + \left(\frac{\sqrt{3}z}{d}\right)^2,$$

where $x_1 = x - \frac{4}{3}d$. This equation defines a solid ellipsoid in canonical form. To compute its volume, perform a linear transformation: we divide x_1 and y by $\frac{2d}{3}$ and z by $\frac{d}{\sqrt{3}}$. This changes the volume by the factor $\left(\frac{2d}{3}\right)^2 \frac{d}{\sqrt{3}} = \frac{4d^3}{9\sqrt{3}}$ and turns the ellipsoid into the unit ball of volume $\frac{4}{3}\pi$. So before the transformation the volume was $\frac{4d^3}{9\sqrt{3}} \cdot \frac{4}{3}\pi = \frac{16\pi d^3}{27\sqrt{3}}$.

Problem 2.
Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is a two times differentiable function satisfying $f(0) = 1$, $f'(0) = 0$, and for all $x \in [0, \infty)$,

$$f''(x) - 5f'(x) + 6f(x) \geq 0.$$

Prove that for all $x \in [0, \infty)$,

$$f(x) \geq 3e^{2x} - 2e^{3x}.$$

Solution. We have $f''(x) - 2f'(x) - 3(f'(x) - 2f(x)) \geq 0$, $x \in [0, \infty)$.

Let $g(x) = f'(x) - 2f(x)$, $x \in [0, \infty)$. It follows that

$$g'(x) - 3g(x) \geq 0,\ x \in [0, \infty),$$

hence

$$(g(x)e^{-3x})' \geq 0, \quad x \in [0, \infty),$$

therefore

$$g(x)e^{-3x} \geq g(0) = -2,\ x \in [0, \infty) \text{ or equivalently}$$
$$f'(x) - 2f(x) \geq -2e^{3x},\ x \in [0, \infty).$$

Analogously we get

$$(f(x)e^{-2x})' \geq -2e^{x},\ x \in [0, \infty) \text{ or equivalently}$$
$$(f(x)e^{-2x} + 2e^{x})' \geq 0,\ x \in [0, \infty).$$

It follows that

$$f(x)e^{-2x} + 2e^{x} \geq f(0) + 2 = 3,\ x \in [0, \infty) \text{ or equivalently}$$
$$f(x) \geq 3e^{2x} - 2e^{3x},\ x \in [0, \infty).$$

Problem 3.
Let $A, B \in M_n(\mathbb{C})$ be two $n \times n$ matrices such that

$$A^2B + BA^2 = 2ABA.$$

Prove that there exists a positive integer k such that $(AB - BA)^k = 0$.

Solution 1. Let us fix the matrix $A \in M_n(\mathbb{C})$. For every matrix $X \in M_n(\mathbb{C})$, let $\Delta X := AX - XA$. We need to prove that the matrix ΔB is nilpotent.

Observe that the condition $A^2B + BA^2 = 2ABA$ is equivalent to

$$\Delta^2 B = \Delta(\Delta B) = 0. \tag{1}$$

Δ is linear; moreover, it is a derivation, i.e. it satisfies the Leibniz rule:

$$\Delta(XY) = (\Delta X)Y + X(\Delta Y), \quad \forall X, Y \in M_n(\mathbb{C}).$$

Using induction, one can easily generalize the above formula to k factors:

$$\Delta(X_1 \cdots X_k) = (\Delta X_1)X_2 \cdots X_k + \cdots + X_1 \cdots X_{j-1}(\Delta X_j)X_{j+1} \cdots X_k + X_1 \cdots X_{n-1}\Delta X_k, \tag{2}$$

for any matrices $X_1, X_2, \ldots, X_k \in M_n(\mathbb{C})$. Using the identities (1) and (2) we obtain the equation for $\Delta^k(B^k)$:

$$\Delta^k(B^k) = k!(\Delta B)^k, \quad \forall k \in \mathbb{N}. \tag{3}$$

By the last equation it is enough to show that $\Delta^n(B^n) = 0$.

To prove this, first we observe that equation (3) together with the fact that $\Delta^2 B = 0$ implies that $\Delta^{k+1}B^k = 0$, for every $k \in \mathbb{N}$. Hence, we have

$$\Delta^k(B^j) = 0, \quad \forall k, j \in \mathbb{N},\ j < k. \tag{4}$$

By the Cayley–Hamilton Theorem, there are scalars $\alpha_0, \alpha_1, \ldots, \alpha_{n-1} \in \mathbb{C}$ such that

$$B^n = \alpha_0 I + \alpha_1 B + \cdots + \alpha_{n-1}B^{n-1},$$

which together with (4) implies that $\Delta^n B^n = 0$.

Solution 2. Set $X = AB - BA$. The matrix X commutes with A because

$$AX - XA = (A^2B - ABA) - (ABA - BA^2) = A^2B + BA^2 - 2ABA = 0.$$

Hence for any $m \geq 0$ we have

$$X^{m+1} = X^m(AB - BA) = AX^mB - X^mBA.$$

Take the trace of both sides:

$$\operatorname{tr} X^{m+1} = \operatorname{tr} A(X^mB) - \operatorname{tr}(X^mB)A = 0$$

(since for any matrices U and V, we have $\operatorname{tr} UV = \operatorname{tr} VU$). As $\operatorname{tr} X^{m+1}$ is the sum of the $m+1$-st powers of the eigenvalues of X, the values of $\operatorname{tr} X$, …, $\operatorname{tr} X^n$ determine the eigenvalues of X uniquely, therefore all of these eigenvalues have to be 0. This implies that X is nilpotent.

Problem 4.
Let p be a prime number and $\mathbb{F}_p$ be the field of residues modulo p. Let W be the smallest set of polynomials with coefficients in $\mathbb{F}_p$ such that

- the polynomials $x + 1$ and $x^{p-2} + x^{p-3} + \cdots + x^2 + 2x + 1$ are in W, and
- for any polynomials $h_1(x)$ and $h_2(x)$ in W the polynomial $r(x)$, which is the remainder of $h_1(h_2(x))$ modulo $x^p - x$, is also in W.

How many polynomials are there in W?

Solution. Note that both of our polynomials are bijective functions on $\mathbb{F}_p$: $f_1(x) = x + 1$ is the cycle $0 \to 1 \to 2 \to \cdots \to (p-1) \to 0$ and $f_2(x) = x^{p-2} + x^{p-3} + \cdots + x^2 + 2x + 1$ is the transposition $0 \leftrightarrow 1$ (this follows from the formula $f_2(x) = \frac{x^{p-1}-1}{x-1} + x$ and Fermat's little theorem). So any composition formed from them is also a bijection, and reduction modulo $x^p - x$ does not change the evaluation in $\mathbb{F}_p$. Also note that the transposition and the cycle generate the symmetric group ($f_1^k \circ f_2 \circ f_1^{p-k}$ is the transposition $k \leftrightarrow (k+1)$, and transpositions of consecutive elements clearly generate S_p), so we get all $p!$ permutations of the elements of $\mathbb{F}_p$.

The set W only contains polynomials of degree at most $p-1$. This means that two distinct elements of W cannot represent the same permutation. So W must contain those polynomials of degree at most $p-1$ which permute the elements of $\mathbb{F}_p$. By minimality, W has exactly these $p!$ elements.

Problem 5.
Let $\mathbb{M}$ be the vector space of $m \times p$ real matrices. For a vector subspace $S \subseteq \mathbb{M}$, denote by $\delta(S)$ the dimension of the vector space generated by all columns of all matrices in S.

Say that a vector subspace $T \subseteq \mathbb{M}$ is a *covering matrix space* if

$$\bigcup_{A \in T,\, A \neq 0} \ker A = \mathbb{R}^p.$$

Such a T is *minimal* if it does not contain a proper vector subspace $S \subset T$ which is also a covering matrix space.

(a) (8 points) Let T be a minimal covering matrix space and let $n = \dim T$. Prove that

$$\delta(T) \le \binom{n}{2}.$$

(b) (2 points) Prove that for every positive integer n we can find m and p, and a minimal covering matrix space T as above such that $\dim T = n$ and $\delta(T) = \binom{n}{2}$.

Solution 1. (a) We will prove the claim by constructing a suitable decomposition $T = Z_0 \oplus Z_1 \oplus \cdots$ and a corresponding decomposition of the space spanned by all columns of T as $W_0 \oplus W_1 \oplus \cdots$, such that $\dim W_0 \leqslant n-1$, $\dim W_1 \leqslant n-2$, etc., from which the bound follows.

We first claim that, in every covering matrix space S, we can find an $A \in S$ with $\operatorname{rk} A \leqslant \dim S - 1$. Indeed, let $S_0 \subseteq S$ be some minimal covering matrix space. Let $s = \dim S_0$ and fix some subspace $S' \subset S_0$ of dimension $s-1$. S' is not covering by minimality of S_0, so that we can find an $u \in \mathbb{R}^p$ with $u \notin \cup_{B \in S',\, B \neq 0} \operatorname{Ker} B$. Let $V = S'(u)$; by the rank-nullity theorem, $\dim V = s-1$. On the other hand, as S_0 is covering, we have that $Au = 0$ for some $A \in S_0 \setminus S'$. We claim that $\operatorname{Im} A \subset V$ (and therefore $\operatorname{rk}(A) \leqslant s-1$).

For suppose that $Av \notin V$ for some $v \in \mathbb{R}^p$. For every $\alpha \in \mathbb{R}$, consider the map $f_\alpha : S_0 \to \mathbb{R}^m$ defined by $f_\alpha : (\tau + \beta A) \mapsto \tau(u + \alpha v) + \beta A v$, $\tau \in S'$, $\beta \in \mathbb{R}$. Note that f_0 is of rank $s = \dim S_0$ by our assumption, so that some $s \times s$ minor of the matrix of f_0 is non-zero. The corresponding minor of f_α is thus a nonzero polynomial of α, so that it follows that $\operatorname{rk} f_\alpha = s$ for all but finitely many α. For such an $\alpha \neq 0$, we have that $\operatorname{Ker} f_\alpha = \{0\}$ and thus

$$0 \neq \tau(u + \alpha v) + \beta A v = (\tau + \alpha^{-1}\beta A)(u + \alpha v)$$

for all $\tau \in S'$, $\beta \in \mathbb{R}$ not both zero, so that $B(u + \alpha v) \neq 0$ for all nonzero $B \in S_0$, a contradiction.

Let now T be a minimal covering matrix space, and write $\dim T = n$. We have shown that we can find an $A \in T$ such that $W_0 = \operatorname{Im} A$ satisfies $w_0 = \dim W_0 \leqslant n-1$. Denote $Z_0 = \{B \in T \,:\, \operatorname{Im} B \subset W_0\}$; we know that $t_0 = \dim Z_0 \geqslant 1$. If $T = Z_0$, then $\delta(T) \leqslant n-1$ and we are done. Else, write $T = Z_0 \oplus T_1$, also write $\mathbb{R}^m = W_0 \oplus V_1$ and let $\pi_1 : \mathbb{R}^m \to \mathbb{R}^m$ be the projection onto the V_1-component. We claim that

$$T_1^\sharp = \{\pi_1 \tau_1 \,:\, \tau_1 \in T_1\}$$

is also a covering matrix space. Note here that $\pi_1^\sharp : T_1 \to T_1^\sharp$, $\tau_1 \mapsto (\pi_1 \tau_1)$ is an isomorphism. In particular we note that $\delta(T) = w_0 + \delta(T_1^\sharp)$.

Suppose that $T_1^\sharp$ is not a covering matrix space, so we can find a $v_1 \in \mathbb{R}^p$ with $v_1 \notin \cup_{\tau_1 \in T_1, \tau_1 \neq 0} \operatorname{Ker}(\pi_1 \tau_1)$. On the other hand, by minimality of T we can find a $u_1 \in \mathbb{R}^p$ with $u_1 \notin \cup_{\tau_0 \in Z_0,\, \tau_0 \neq 0} \operatorname{Ker} \tau_0$. The maps $g_\alpha : Z_0 \to V$,

$\tau_0 \mapsto \tau_0(u_1 + \alpha v_1)$ and $h_\beta : T_1 \to V_1$, $\tau_1 \mapsto \pi_1(\tau_1(v_1 + \beta u_1))$ have $\operatorname{rk} g_0 = t_0$ and $\operatorname{rk} h_0 = n - t_0$ and thus both $\operatorname{rk} g_\alpha = t_0$ and $\operatorname{rk} h_{\alpha^{-1}} = n - t_0$ for all but finitely many $\alpha \neq 0$ by the same argument as above. Pick such an α and suppose that

$$(\tau_0 + \tau_1)(u_1 + \alpha v_1) = 0$$

for some $\tau_0 \in Z_0$, $\tau_1 \in T_1$. Applying π_1 to both sides we see that we can only have $\tau_1 = 0$, and then $\tau_0 = 0$ as well, a contradiction given that T is a covering matrix space.

In fact, the exact same proof shows that, in general, if T is a minimal covering matrix space, $\mathbb{R}^m = V_0 \oplus V_1$, $T_0 = \{\tau \in T \ : \ \operatorname{Im} \tau \subset V_0\}$, $T = T_0 \oplus T_1$, $\pi_1 : \mathbb{R}^m \to \mathbb{R}^m$ is the projection onto the V_1-component, and $T_1^\sharp = \{\pi_1 \tau_1 \ : \ \tau_1 \in T_1\}$, then $T_1^\sharp$ is a covering matrix space.

We can now repeat the process. We choose a $\pi_1 A_1 \in T_1^\sharp$ such that $W_1 = (\pi_1 A_1)(\mathbb{R}^p)$ has $w_1 = \dim W_1 \leqslant n - t_0 - 1 \leqslant n - 2$. We write $Z_1 = \{\tau_1 \in T_1 \ : \ \operatorname{Im}(\pi_1 \tau_1) \subset W_1\}$, $T_1 = Z_1 \oplus T_2$ (and so $T = (Z_0 \oplus Z_1) \oplus T_2$), $t_1 = \dim Z_1 \geqslant 1$, $V_1 = W_1 \oplus V_2$ (and so $\mathbb{R}^m = (W_0 \oplus W_1) \oplus V_2$), $\pi_2 : \mathbb{R}^m \to \mathbb{R}^m$ is the projection onto the V_2-component, and $T_2^\sharp = \{\pi_2 \tau_2 \ : \ \tau_2 \in T_2\}$, so that $T_2^\sharp$ is also a covering matrix space, etc.

We conclude that

$$\delta(T) = w_0 + \delta(T_1) = w_0 + w_1 + \delta(T_2) = \cdots$$
$$\leqslant (n-1) + (n-2) + \cdots \leqslant \binom{n}{2}.$$

(b) We consider $\binom{n}{2} \times n$ matrices whose rows are indexed by $\binom{n}{2}$ pairs (i,j) of integers $1 \leqslant i < j \leqslant n$. For every $u = (u_1, u_2, \ldots, u_n) \in \mathbb{R}^n$, consider the matrix $A(u)$ whose entries $A(u)_{(i,j),k}$ with $1 \leqslant i < j \leqslant n$ and $1 \leqslant k \leqslant n$ are given by

$$(A(u))_{(i,j),k} = \begin{cases} u_i, \ k = j, \\ -u_j, \ k = i, \\ 0, \ \text{otherwise}. \end{cases}$$

It is immediate that $\operatorname{Ker} A(u) = \mathbb{R} \cdot u$ for every $u \neq 0$, so that $S = \{A(u) \ : \ u \in \mathbb{R}^n\}$ is a covering matrix space, and in fact a minimal one.

On the other hand, for any $1 \leqslant i < j \leqslant n$, we have that $A(e_i)_{(i,j),j}$ is the $(i,j)^{\text{th}}$ vector in the standard basis of $\mathbb{R}^{\binom{n}{2}}$, where e_i denotes the i^{th} vector in the standard basis of $\mathbb{R}^n$. This means that $\delta(S) = \binom{n}{2}$, as required.

Solution 2. (for part a)

Let us denote $X = \mathbb{R}^p$, $Y = \mathbb{R}^m$. For each $x \in X$, denote by $\mu_x : T \to Y$ the evaluation map $\tau \mapsto \tau(x)$. As T is a covering matrix space, $\ker \mu_x > 0$ for every $x \in X$. Let $U = \{x \in X : \dim \ker \mu_x = 1\}$.

Let T_1 be the span of the family of subspaces $\{\ker \mu_x \ : \ x \in U\}$. We claim that $T_1 = T$. For suppose the contrary, and let $T' \subset T$ be a subspace of T of dimension $n-1$ such that $T_1 \subseteq T'$. This implies that T' is a covering matrix space. Indeed, for $x \in U$, $(\ker \mu_x) \cap T' = \ker \mu_x \neq 0$, while for $x \notin U$ we have $\dim \mu_x \geq 2$, so that $(\ker \mu_x) \cap T' \neq 0$ by computing dimensions. However, this is a contradiction as T is minimal.

Now we may choose $x_1, x_2, \ldots, x_n \in U$ and $\tau_1, \tau_2, \ldots, \tau_n \in T$ in such a way that $\ker \mu_{x_i} = \mathbb{R}\tau_i$ and τ_i form a basis of T. Let us complete $x_1, \ldots, x_n$ to a sequence $x_1, \ldots, x_d$ which spans X. Put $y_{ij} = \tau_i(x_j)$. It is clear that y_{ij} span the vector space generated by the columns of all matrices in T. We claim that the subset $\{y_{ij} : i > j\}$ is enough to span this space, which clearly implies that $\delta(T) \leqslant \binom{n}{2}$.

We have $y_{ii} = 0$. So it is enough to show that every y_{ij} with $i < j$ can be expressed as a linear combination of y_{ki}, $k = 1, \ldots, n$. This follows from the following lemma:

Lemma. For every $x_0 \in U$, $0 \neq \tau_0 \in \ker \mu_{x_0}$ and $x \in X$, there exists a $\tau \in T$ such that $\tau_0(x) = \tau(x_0)$.

Proof. The operator μ_{x_0} has rank $n-1$, which implies that for small ε the operator $\mu_{x_0 + \varepsilon x}$ also has rank $n-1$. Therefore one can produce a rational function $\tau(\varepsilon)$ with values in T such that $m_{x_0 + \varepsilon x}(\tau(\varepsilon)) = 0$. Taking the derivative at $\varepsilon = 0$ gives $\mu_{x_0}(\tau_0) + \mu_x(\tau'(0)) = 0$. Therefore $\tau = -\tau'(0)$ satisfies the desired property.

Remark. Lemma in solution 2 is the same as the claim $\operatorname{Im} A \subset V$ at the beginning of solution 1, but the proof given here is different. It can be shown that all minimal covering spaces T with $\dim T = \binom{n}{2}$ are essentially the ones described in our example.

IMC2010, Blagoevgrad, Bulgaria

Day 1, July 26, 2010

Problem 1. Let $0 < a < b$. Prove that

$$\int_a^b (x^2+1)e^{-x^2}dx \ge e^{-a^2} - e^{-b^2}.$$

Solution 1. Let $f(x) = \int_0^x (t^2+1)e^{-t^2}dt$ and let $g(x) = -e^{-x^2}$; both functions are increasing.

By Cauchy's Mean Value Theorem, there exists a real number $x \in (a,b)$ such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(x)}{g'(x)} = \frac{(x^2+1)e^{-x^2}}{2xe^{-x^2}} = \frac{1}{2}\left(x+\frac{1}{x}\right) \ge \sqrt{x\cdot\frac{1}{x}} = 1.$$

Then

$$\int_a^b (x^2+1)e^{-x^2}dx = f(b)-f(a) \ge g(b)-g(a) = e^{-a^2} - e^{-b^2}.$$

Solution 2.

$$\int_a^b (x^2+1)e^{-x^2}dx \ge \int_a^b 2xe^{-x^2}dx = \left[-e^{-x^2}\right]_a^b = e^{-a^2} - e^{-b^2}.$$

Problem 2. Compute the sum of the series

$$\sum_{k=0}^{\infty} \frac{1}{(4k+1)(4k+2)(4k+3)(4k+4)} = \frac{1}{1\cdot 2\cdot 3\cdot 4} + \frac{1}{5\cdot 6\cdot 7\cdot 8} + \cdots .$$

Solution 1. Let

$$F(x) = \sum_{k=0}^{\infty} \frac{x^{4k+4}}{(4k+1)(4k+2)(4k+3)(4k+4)}.$$

This power series converges for $|x| \le 1$ and our goal is to compute $F(1)$.

Differentiating 4 times, we get

$$F^{(IV)}(x) = \sum_{k=0}^{\infty} x^{4k} = \frac{1}{1-x^4}.$$

Since $F(0) = F'(0) = F''(0) = F'''(0) = 0$ and F is continuous at $1-0$ by Abel's continuity theorem, integrating 4 times we get

$$F'''(y)=F'''(0)+\int_0^y F^{(IV)}(x)\,\mathrm{d}x=\int_0^y\frac{\mathrm{d}x}{1-x^4}=\frac{1}{2}\arctan y+\frac{1}{4}\log(1+y)-\frac{1}{4}\log(1-y)\,,$$

$$F''(z)=F''(0)+\int_0^z F'''(y)\,\mathrm{d}x=\int_0^z\left(\frac{1}{2}\arctan y+\frac{1}{4}\log(1+y)-\frac{1}{4}\log(1-y)\right)\,\mathrm{d}y=$$

$$=\frac{1}{2}\left(z\arctan z-\int_0^z\frac{y}{1+y^2}\,\mathrm{d}y\right)+\frac{1}{4}\left((1+z)\log(1+z)-\int_0^z\,\mathrm{d}y\right)+\frac{1}{4}\left((1-z)\log(1-z)+\int_0^z\,\mathrm{d}y\right)=$$

$$=\frac{1}{2}z\arctan z-\frac{1}{4}\log(1+z^2)+\frac{1}{4}(1+z)\log(1+z)+\frac{1}{4}(1-z)\log(1-z)\,,$$

$$F'(t)=\int_0^t\left(\frac{1}{2}z\arctan z-\frac{1}{4}\log(1+z^2)+\frac{1}{4}(1+z)\log(1+z)+\frac{1}{4}(1-z)\log(1-z)\right)\,\mathrm{d}t=$$

$$=\frac{1}{4}\Big((1+t^2)\arctan t-t\Big)-\frac{1}{4}\Big(t\log(1+t^2)-2t+2\arctan t\Big)+$$

$$+\frac{1}{8}\left((1+t)^2\log(1+t)-t-\frac{1}{2}t^2\right)-\frac{1}{8}\left((1-t)^2\log(1-t)+t-\frac{1}{2}t^2\right)=$$

$$=\frac{1}{4}(-1+t^2)\arctan t-\frac{1}{4}t\log(1+t^2)+\frac{1}{8}(1+t)^2\log(1+t)-\frac{1}{8}(1-t)^2\log(1-t)\,,$$

$$F(1)=\int_0^1\left(\frac{1}{4}(-1+t^2)\arctan t-\frac{1}{4}t\log(1+t^2)+\frac{1}{8}(1+t)^2\log(1+t)-\frac{1}{8}(1-t)^2\log(1-t)\right)\,\mathrm{d}t=$$

$$=\left[\frac{-3t+t^3}{12}\arctan t+\frac{1-3t^2}{24}\log(1+t^2)+\frac{(1+t)^3}{24}\log(1+t)+\frac{(1-t)^3}{24}\log(1-t)\right]_0^1=\frac{\ln 2}{4}-\frac{\pi}{24}.$$

Remark. The computation can be shorter if we change the order of integrations.

$$F(1)=\int_{t=0}^1\int_{z=0}^t\int_{y=0}^z\int_{x=0}^y\frac{1}{1-x^4}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}t=\int_{x=0}^1\frac{1}{1-x^4}\int_{y=x}^1\int_{z=y}^1\int_{t=z}^1\,\mathrm{d}t\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}z=$$

$$=\int_{x=0}^1\frac{1}{1-x^4}\left(\frac{1}{6}\int_{y=x}^1\int_{z=x}^1\int_{t=x}^1\,\mathrm{d}t\,\mathrm{d}z\,\mathrm{d}y\right)\,\mathrm{d}x=\int_0^1\frac{1}{1-x^4}\cdot\frac{(1-x)^3}{6}\,\mathrm{d}x=$$

$$=\left[-\frac{1}{6}\arctan x-\frac{1}{12}\log(1+x^2)+\frac{1}{3}\log(1+x)\right]_0^1=\frac{\ln 2}{4}-\frac{\pi}{24}.$$

Solution 2. Let

$$A_m=\sum_{k=0}^m\frac{1}{(4k+1)(4k+2)(4k+3)(4k+4)}=\sum_{k=0}^m\left(\frac{1}{6}\cdot\frac{1}{4k+1}-\frac{1}{2}\cdot\frac{1}{4k+2}+\frac{1}{2}\cdot\frac{1}{4k+3}-\frac{1}{6}\cdot\frac{1}{4k+4}\right),$$

$$B_m=\sum_{k=0}^m\left(\frac{1}{4k+1}-\frac{1}{4k+3}\right),$$

$$C_m=\sum_{k=0}^m\left(\frac{1}{4k+1}-\frac{1}{4k+2}+\frac{1}{4k+3}-\frac{1}{4k+4}\right)\quad\text{and}$$

$$D_m=\sum_{k=0}^m\left(\frac{1}{4k+2}-\frac{1}{4k+4}\right).$$

It is easy check that

$$A_m=\frac{1}{3}C_m-\frac{1}{6}B_m-\frac{1}{6}D_m.$$

Therefore,

$$\lim A_m=\lim\frac{2C_m-B_m-D_m}{6}=\frac{2\ln 2-\frac{\pi}{4}-\frac{1}{2}\ln 2}{6}=\frac{1}{4}\ln 2-\frac{\pi}{24}.$$

Problem 3. Define the sequence $x_1, x_2, \ldots$ inductively by $x_1 = \sqrt{5}$ and $x_{n+1} = x_n^2 - 2$ for each $n \geq 1$. Compute

$$\lim_{n\to\infty} \frac{x_1 \cdot x_2 \cdot x_3 \cdots x_n}{x_{n+1}}.$$

Solution. Let $y_n = x_n^2$. Then $y_{n+1} = (y_n - 2)^2$ and $y_{n+1} - 4 = y_n(y_n - 4)$. Since $y_2 = 9 > 5$, we have $y_3 = (y_2 - 2)^2 > 5$ and inductively $y_n > 5, n \geq 2$. Hence, $y_{n+1} - y_n = y_n^2 - 5y_n + 4 > 4$ for all $n \geq 2$, so $y_n \to \infty$.

By $y_{n+1} - 4 = y_n(y_n - 4)$,

$$\begin{aligned}
\left(\frac{x_1 \cdot x_2 \cdot x_3 \cdots x_n}{x_{n+1}}\right)^2 &= \frac{y_1 \cdot y_2 \cdot y_3 \cdots y_n}{y_{n+1}} \\
&= \frac{y_{n+1} - 4}{y_{n+1}} \cdot \frac{y_1 \cdot y_2 \cdot y_3 \cdots y_n}{y_{n+1} - 4} = \frac{y_{n+1} - 4}{y_{n+1}} \cdot \frac{y_1 \cdot y_2 \cdot y_3 \cdots y_{n-1}}{y_n - 4} = \cdots \\
&= \frac{y_{n+1} - 4}{y_{n+1}} \cdot \frac{1}{y_1 - 4} = \frac{y_{n+1} - 4}{y_{n+1}} \to 1.
\end{aligned}$$

Therefore,

$$\lim_{n\to\infty} \frac{x_1 \cdot x_2 \cdot x_3 \cdots x_n}{x_{n+1}} = 1.$$

Problem 4. Let a, b be two integers and suppose that n is a positive integer for which the set

$$\mathbb{Z} \setminus \left\{ax^n + by^n \mid x, y \in \mathbb{Z}\right\}$$

is finite. Prove that $n = 1$.

Solution. Assume that $n > 1$. Notice that n may be replaced by any prime divisor p of n. Moreover, a and b should be coprime, otherwise the numbers not divisible by the greatest common divisor of a, b cannot be represented as $ax^n + by^n$.

If $p = 2$, then the number of the form $ax^2 + by^2$ takes not all possible remainders modulo 8. If, say, b is even, then ax^2 takes at most three different remainders modulo 8, by^2 takes at most two, hence $ax^2 + by^2$ takes at most $3 \times 2 = 6$ different remainders. If both a and b are odd, then $ax^2 + by^2 \equiv x^2 \pm y^2 \pmod 4$; the expression $x^2 + y^2$ does not take the remainder 3 modulo 4 and $x^2 - y^2$ does not take the remainder 2 modulo 4.

Consider the case when $p \geq 3$. The pth powers take exactly p different remainders modulo p^2. Indeed, $(x + kp)^p$ and x^p have the same remainder modulo p^2, and all numbers 0^p, 1^p, …, $(p-1)^p$ are different even modulo p. So, $ax^p + by^p$ take at most p^2 different remainders modulo p^2. If it takes strictly less then p^2 different remainders modulo p^2, we get infinitely many non-representable numbers. If it takes exactly p^2 remainders, then $ax^p + by^p$ is divisible by p^2 only if both x and y are divisible by p. Hence if $ax^p + by^p$ is divisible by p^2, it is also divisible by p^p. Again we get infinitely many non-representable numbers, for example the numbers congruent to p^2 modulo p^3 are non-representable.

Problem 5. Suppose that a, b, c are real numbers in the interval $[-1, 1]$ such that

$$1 + 2abc \geq a^2 + b^2 + c^2\,.$$

Prove that

$$1 + 2(abc)^n \geq a^{2n} + b^{2n} + c^{2n}$$

for all positive integers n.

Solution 1. Consider the symmetric matrix

$$A = \begin{pmatrix} 1 & a & b \\ a & 1 & c \\ b & c & 1 \end{pmatrix}.$$

By the constraint we have $\det A \geq 0$ and $\det \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix}, \det \begin{pmatrix} 1 & b \\ b & 1 \end{pmatrix}, \det \begin{pmatrix} 1 & c \\ c & 1 \end{pmatrix} \geq 0$. Hence A is positive semidefinite, and $A = B^2$ for some symmetric real matrix B.

Let the rows of B be x, y, z. Then $|x| = |y| = |z| = 1$, $a = x \cdot y$, $b = y \cdot z$ and $c = z \cdot x$, where $|x|$ and $x \cdot y$ denote the Euclidean norm and scalar product. Denote by $X = \otimes^n x$, $Y = \otimes^n y$, $Z = \otimes^n z$ the nth tensor powers, which belong to $\mathbb{R}^{3^n}$. Then $|X| = |Y| = |Z| = 1$, $X \cdot Y = a^n$, $Y \cdot Z = b^n$ and $Z \cdot X = c^n$. So, the matrix $\begin{pmatrix} 1 & a^n & b^n \\ a^n & 1 & c^n \\ b^n & c^n & 1 \end{pmatrix}$, being the Gram matrix of three vectors in $\mathbb{R}^{3^n}$, is positive semidefinite, and its determinant, $1 + 2(abc)^n - a^{2n} - b^{2n} - c^{2n}$ is non-negative.

Solution 2. The constraint can be written as

$$(a - bc)^2 \leq (1 - b^2)(1 - c^2). \tag{1}$$

By the Cauchy-Schwarz inequality,

$$\left(a^{n-1} + a^{n-2}bc + \ldots + b^{n-1}c^{n-1}\right)^2 \leq \left(|a|^{n-1} + |a|^{n-2}|bc| + \ldots + |bc|^{n-1}\right)^2 \leq$$
$$\leq \left(1 + |bc| + \ldots + |bc|^{n-1}\right)^2 \leq \left(1 + |b|^2 + \ldots + |b|^{2(n-1)}\right)\left(1 + |c|^2 + \ldots + |c|^{2(n-1)}\right)$$

Multiplying by (1), we get

$$(a - bc)^2(a^{n-1} + a^{n-2}bc + \ldots + b^{n-1}c^{n-1})^2 \leq$$
$$\left((1 - b^2)\left(1 + |b|^2 + \ldots + |b|^{2(n-1)}\right)\right)\left((1 - c^2)\left(1 + |c|^2 + \ldots + |c|^{2(n-1)}\right)\right),$$
$$(a^n - b^n c^n)^2 \leq (1 - b^n)(1 - c^n),$$
$$1 + 2(abc)^n \geq a^{2n} + b^{2n} + b^{2n}.$$

IMC2010, Blagoevgrad, Bulgaria

Day 2, July 27, 2010

Problem 1. (a) A sequence $x_1, x_2, \ldots$ of real numbers satisfies

$$x_{n+1} = x_n \cos x_n \quad \text{for all} \quad n \geq 1\,.$$

Does it follow that this sequence converges for all initial values x_1?

(b) A sequence $y_1, y_2, \ldots$ of real numbers satisfies

$$y_{n+1} = y_n \sin y_n \quad \text{for all} \quad n \geq 1\,.$$

Does it follow that this sequence converges for all initial values y_1?

Solution 1. (a) NO. For example, for $x_1 = \pi$ we have $x_n = (-1)^{n-1}\pi$, and the sequence is divergent.

(b) YES. Notice that $|y_n|$ is nonincreasing and hence converges to some number $a \geq 0$.

If $a = 0$, then $\lim y_n = 0$ and we are done. If $a > 0$, then $a = \lim |y_{n+1}| = \lim |y_n \sin y_n| = a \cdot |\sin a|$, so $\sin a = \pm 1$ and $a = (k + \frac{1}{2})\pi$ for some nonnegative integer k.

Since the sequence $|y_n|$ is nonincreasing, there exists an index n_0 such that $(k + \frac{1}{2})\pi \leq |y_n| < (k+1)\pi$ for all $n > n_0$. Then all the numbers $y_{n_0+1}, y_{n_0+2}, \ldots$ lie in the union of the intervals $\big[(k + \frac{1}{2})\pi, (k+1)\pi\big)$ and $\big(-(k+1)\pi, -(k+\frac{1}{2})\pi\big]$.

Depending on the parity of k, in one of the intervals $\big[(k+\frac{1}{2})\pi, (k+1)\pi\big)$ and $\big(-(k+1)\pi, -(k+\frac{1}{2})\pi\big]$ the values of the sine function is positive; denote this interval by I_+. In the other interval the sine function is negative; denote this interval by I_-. If $y_n \in I_-$ for some $n > n_0$ then y_n and $y_{n+1} = y_n \sin y_n$ have opposite signs, so $y_{n+1} \in I_+$. On the other hand, if If $y_n \in I_+$ for some $n > n_0$ then y_n and y_{n+1} have the same sign, so $y_{n+1} \in I_+$. In both cases, $y_{n+1} \in I_+$.

We obtained that the numbers $y_{n_0+2}, y_{n_0+3}, \ldots$ lie in I_+, so they have the same sign. Since $|y_n|$ is convergent, this implies that the sequence (y_n) is convergent as well.

Solution 2 for part (b). Similarly to the first solution, $|y_n| \to a$ for some real number a.

Notice that $t \cdot \sin t = (-t)\sin(-t) = |t| \sin |t|$ for all real t, hence $y_{n+1} = |y_n| \sin |y_n|$ for all $n \geq 2$. Since the function $t \mapsto t \sin t$ is continuous, $y_{n+1} = |y_n| \sin |y_n| \to |a| \sin |a| = a$.

Problem 2. Let $a_0, a_1, \ldots, a_n$ be positive real numbers such that $a_{k+1} - a_k \geq 1$ for all $k = 0, 1, \ldots, n-1$. Prove that

$$1 + \frac{1}{a_0}\left(1 + \frac{1}{a_1 - a_0}\right) \cdots \left(1 + \frac{1}{a_n - a_0}\right) \leq \left(1 + \frac{1}{a_0}\right)\left(1 + \frac{1}{a_1}\right) \cdots \left(1 + \frac{1}{a_n}\right).$$

Solution. Apply induction on n. Considering the empty product as 1, we have equality for $n = 0$.

Now assume that the statement is true for some n and prove it for $n+1$. For $n+1$, the statement can be written as the sum of the inequalities

$$1 + \frac{1}{a_0}\left(1 + \frac{1}{a_1 - a_0}\right) \cdots \left(1 + \frac{1}{a_n - a_0}\right) \leq \left(1 + \frac{1}{a_0}\right) \cdots \left(1 + \frac{1}{a_n}\right)$$

(which is the induction hypothesis) and

$$\frac{1}{a_0}\left(1 + \frac{1}{a_1 - a_0}\right) \cdots \left(1 + \frac{1}{a_n - a_0}\right) \cdot \frac{1}{a_{n+1} - a_0} \leq \left(1 + \frac{1}{a_0}\right) \cdots \left(1 + \frac{1}{a_n}\right) \cdot \frac{1}{a_{n+1}}. \qquad (1)$$

Hence, to complete the solution it is sufficient to prove (1).

To prove (1), apply a second induction. For $n = 0$, we have to verify

$$\frac{1}{a_0} \cdot \frac{1}{a_1 - a_0} \le \left(1 + \frac{1}{a_0}\right)\frac{1}{a_1}.$$

Multiplying by $a_0 a_1 (a_1 - a_0)$, this is equivalent with

$$\begin{aligned} a_1 &\le (a_0 + 1)(a_1 - a_0) \\ a_0 &\le a_0 a_1 - a_0^2 \\ 1 &\le a_1 - a_0. \end{aligned}$$

For the induction step it is sufficient that

$$\left(1 + \frac{1}{a_{n+1} - a_0}\right) \cdot \frac{a_{n+1} - a_0}{a_{n+2} - a_0} \le \left(1 + \frac{1}{a_{n+1}}\right) \cdot \frac{a_{n+1}}{a_{n+2}}.$$

Multiplying by $(a_{n+2} - a_0)a_{n+2}$,

$$\begin{aligned} (a_{n+1} - a_0 + 1)a_{n+2} &\le (a_{n+1} + 1)(a_{n+2} - a_0) \\ a_0 &\le a_0 a_{n+2} - a_0 a_{n+1} \\ 1 &\le a_{n+2} - a_{n+1}. \end{aligned}$$

Remark 1. It is easy to check from the solution that equality holds if and only if $a_{k+1} - a_k = 1$ for all k.

Remark 2. The statement of the problem is a direct corollary of the identity

$$1 + \sum_{i=0}^{n} \left(\frac{1}{x_i} \prod_{j \ne i} \left(1 + \frac{1}{x_j - x_i}\right)\right) = \prod_{i=0}^{n} \left(1 + \frac{1}{x_i}\right).$$

Problem 3. Denote by S_n the group of permutations of the sequence $(1, 2, \ldots, n)$. Suppose that G is a subgroup of S_n, such that for every $\pi \in G \setminus \{e\}$ there exists a unique $k \in \{1, 2, \ldots, n\}$ for which $\pi(k) = k$. (Here e is the unit element in the group S_n.) Show that this k is the same for all $\pi \in G \setminus \{e\}$.

Solution. Let us consider the action of G on the set $X = \{1, \ldots, n\}$. Let

$$G_x = \{\, g \in G\colon\ g(x) = x\} \quad \text{and} \quad Gx = \{\, g(x)\colon\ g \in G\}$$

be the stabilizer and the orbit of $x \in X$ under this action, respectively. The condition of the problem states that

$$G = \bigcup_{x \in X} G_x \tag{1}$$

and

$$G_x \cap G_y = \{e\} \quad \text{for all} \quad x \ne y. \tag{2}$$

We need to prove that $G_x = G$ for some $x \in X$.

Let $Gx_1, \ldots, Gx_k$ be the distinct orbits of the action of G. Then one can write (1) as

$$G = \bigcup_{i=1}^{k} \bigcup_{y \in Gx_i} G_y. \tag{3}$$

It is well known that

$$|Gx| = \frac{|G|}{|G_x|}. \tag{4}$$

Also note that if $y \in Gx$ then $Gy = Gx$ and thus $|Gy| = |Gx|$. Therefore,

$$|G_x| = \frac{|G|}{|Gx|} = \frac{|G|}{|Gy|} = |G_y| \quad \text{for all} \quad y \in Gx. \tag{5}$$

Combining (3), (2), (4) and (5) we get

$$|G| - 1 = |G \setminus \{e\}| = \left| \bigcup_{i=1}^{k} \bigcup_{y \in Gx_i} G_y \setminus \{e\} \right| = \sum_{i=1}^{k} \frac{|G|}{|G_{x_i}|}(|G_{x_i}| - 1),$$

hence

$$1 - \frac{1}{|G|} = \sum_{i=1}^{k} \left(1 - \frac{1}{|G_{x_i}|}\right). \tag{6}$$

If for some $i, j \in \{1, \ldots, k\}$ $|G_{x_i}|, |G_{x_i}| \geq 2$ then

$$\sum_{i=1}^{k} \left(1 - \frac{1}{|G_{x_i}|}\right) \geq \left(1 - \frac{1}{2}\right) + \left(1 - \frac{1}{2}\right) = 1 > 1 - \frac{1}{|G|}$$

which contradicts with (6), thus we can assume that

$$|G_{x_1}| = \ldots = |G_{x_{k-1}}| = 1.$$

Then from (6) we get $|G_{x_k}| = |G|$, hence $G_{x_k} = G$.

Problem 4. Let A be a symmetric $m \times m$ matrix over the two-element field all of whose diagonal entries are zero. Prove that for every positive integer n each column of the matrix A^n has a zero entry.

Solution. Denote by e_k ($1 \leq k \leq m$) the m-dimensional vector over F_2, whose k-th entry is 1 and all the other elements are 0. Furthermore, let u be the vector whose all entries are 1. The k-th column of A^n is $A^n e_k$. So the statement can be written as $A^n e_k \neq u$ for all $1 \leq k \leq m$ and all $n \geq 1$.

For every pair of vectors $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_m)$, define the bilinear form $(x, y) = x^T y = x_1 y_1 + ... + x_m y_m$. The product (x, y) has all basic properties of scalar products (except the property that $(x, x) = 0$ implies $x = 0$). Moreover, we have $(x, x) = (x, u)$ for every vector $x \in F_2^m$.

It is also easy to check that $(w, Aw) = w^T A w = 0$ for all vectors w, since A is symmetric and its diagonal elements are 0.

Lemma. Suppose that $v \in F_2^m$ a vector such that $A^n v = u$ for some $n \geq 1$. Then $(v, v) = 0$.

Proof. Apply induction on n. For odd values of n we prove the lemma directly. Let $n = 2k + 1$ and $w = A^k v$. Then

$$(v, v) = (v, u) = (v, A^n v) = v^T A^n v = v^T A^{2k+1} v = (A^k v, A^{k+1} v) = (w, Aw) = 0.$$

Now suppose that n is even, $n = 2k$, and the lemma is true for all smaller values of n. Let $w = A^k v$; then $A^k w = A^n v = u$ and thus we have $(w, w) = 0$ by the induction hypothesis. Hence,

$$(v, v) = (v, u) = v^T A^n v = v^T A^{2k} v = (A^k v)^T (A^k v) = (A^k v, A^k v) = (w, w) = 0.$$

The lemma is proved.

Now suppose that $A^n e_k = u$ for some $1 \le k \le m$ and positive integer n. By the Lemma, we should have $(e_k, e_k) = 0$. But this is impossible because $(e_k, e_k) = 1 \ne 0$.

Problem 5. Suppose that for a function $f\colon \mathbb{R} \to \mathbb{R}$ and real numbers $a < b$ one has $f(x) = 0$ for all $x \in (a, b)$. Prove that $f(x) = 0$ for all $x \in \mathbb{R}$ if

$$\sum_{k=0}^{p-1} f\left(y + \frac{k}{p}\right) = 0$$

for every prime number p and every real number y.

Solution. Let $N > 1$ be some integer to be defined later, and consider set of real polynomials

$$\mathcal{J}_N = \left\{ c_0 + c_1 x + \ldots + c_n x^n \in \mathbb{R}[x] \;\middle|\; \forall x \in \mathbb{R} \;\; \sum_{k=0}^{n} c_k f\Big(x + \frac{k}{N}\Big) = 0 \right\}.$$

Notice that $0 \in \mathcal{J}_N$, any linear combinations of any elements in $\mathcal{J}_N$ is in $\mathcal{J}_N$, and for every $P(x) \in \mathcal{J}_N$ we have $xP(x) \in \mathcal{J}_N$. Hence, $\mathcal{J}_N$ is an ideal of the ring $\mathbb{R}[x]$.

By the problem's conditions, for every prime divisors of N we have $\dfrac{x^N - 1}{x^{N/p} - 1} \in \mathcal{J}_N$. Since $\mathbb{R}[x]$ is a principal ideal domain (due to the Euclidean algorithm), the greatest common divisor of these polynomials is an element of $\mathcal{J}_N$. The complex roots of the polynomial $\dfrac{x^N - 1}{x^{N/p} - 1}$ are those Nth roots of unity whose order does not divide N/p. The roots of the greatest common divisor is the intersection of such sets; it can be seen that the intersection consist of the primitive Nth roots of unity. Therefore,

$$\gcd\left\{ \frac{x^N - 1}{x^{N/p} - 1} \;\middle|\; p \big| N \right\} = \Phi_N(x)$$

is the Nth cyclotomic polynomial. So $\Phi_N \in \mathcal{J}_N$, which polynomial has degree $\varphi(N)$.

Now choose N in such a way that $\dfrac{\varphi(N)}{N} < b - a$. It is well-known that $\liminf\limits_{N\to\infty} \dfrac{\varphi(N)}{N} = 0$, so there exists such a value for N. Let $\Phi_N(x) = a_0 + a_1 x + \ldots + a_{\varphi(N)} x^{\varphi(N)}$ where $a_{\varphi(N)} = 1$ and $|a_0| = 1$. Then, by the definition of $\mathcal{J}_N$, we have $\sum\limits_{k=0}^{\varphi(N)} a_k f\big(x + \frac{k}{N}\big) = 0$ for all $x \in \mathbb{R}$.

If $x \in [b, b + \frac{1}{N})$, then

$$f(x) = -\sum_{k=0}^{\varphi(N)-1} a_k f\big(x - \tfrac{\varphi(N)-k}{N}\big).$$

On the right-hand side, all numbers $x - \frac{\varphi(N)-k}{N}$ lie in (a, b). Therefore the right-hand side is zero, and $f(x) = 0$ for all $x \in [b, b + \frac{1}{N})$. It can be obtained similarly that $f(x) = 0$ for all $x \in (a - \frac{1}{N}, a]$ as well. Hence, $f = 0$ in the interval $(a - \frac{1}{N}, b + \frac{1}{N})$. Continuing in this fashion we see that f must vanish everywhere.

IMC2011, Blagoevgrad, Bulgaria

Day 1, July 30, 2011

Problem 1. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function. A point x is called a *shadow point* if there exists a point $y \in \mathbb{R}$ with $y > x$ such that $f(y) > f(x)$. Let $a < b$ be real numbers and suppose that

- all the points of the open interval $I = (a, b)$ are shadow points;
- a and b are not shadow points.

Prove that
a) $f(x) \le f(b)$ for all $a < x < b$;
b) $f(a) = f(b)$.

(José Luis Díaz-Barrero, Barcelona)

Solution. (a) We prove by contradiction. Suppose that exists a point $c \in (a, b)$ such that $f(c) > f(b)$.

By Weierstrass' theorem, f has a maximal value m on $[c, b]$; this value is attained at some point $d \in [c, b]$. Since $f(d) = \max_{[c.b]} f \ge f(c) > f(b)$, we have $d \ne b$, so $d \in [c, b) \subset (a, b)$. The point d, lying in (a, b), is a shadow point, therefore $f(y) > f(d)$ for some $y > d$. From combining our inequalities we get $f(y) > f(d) > f(b)$.

Case 1: $y > b$. Then $f(y) > f(b)$ contradicts the assumption that b is not a shadow point.

Case 2: $y \le b$. Then $y \in (d, b] \subset [c, b]$, therefore $f(y) > f(d) = m = \max_{[c,b]} f \ge f(y)$, contradiction again.

(b) Since $a < b$ and a is not a shadow point, we have $f(a) \ge f(b)$.

By part (a), we already have $f(x) \le f(b)$ for all $x \in (a, b)$. By the continuity at a we have

$$f(a) = \lim_{x\to a+0} f(x) \le \lim_{x\to a+0} f(b) = f(b)$$

Hence we have both $f(a) \ge f(b)$ and $f(a) \le f(b)$, so $f(a) = f(b)$.

Problem 2. Does there exist a real 3×3 matrix A such that $\mathrm{tr(A)} = 0$ and $A^2 + A^t = I$? ($\mathrm{tr(A)}$ denotes the trace of A, A^t is the transpose of A, and I is the identity matrix.)

(Moubinool Omarjee, Paris)

Solution. The answer is NO.

Suppose that $\mathrm{tr(A)} = 0$ and $A^2 + A^t = I$. Taking the transpose, we have

$$A = I - (A^2)^t = I - (A^t)^2 = I - (I - A^2)^2 = 2A^2 - A^4,$$

$$A^4 - 2A^2 + A = 0.$$

The roots of the polynomial $x^4 - 2x^2 + x = x(x - 1)(x^2 + x - 1)$ are $0, 1, \frac{-1\pm\sqrt{5}}{2}$ so these numbers can be the eigenvalues of A; the eigenvalues of A^2 can be $0, 1, \frac{1\pm\sqrt{5}}{2}$.

By $\mathrm{tr(A)} = 0$, the sum of the eigenvalues is 0, and by $\mathrm{tr(A^2)} = \mathrm{tr(I - A^t)} = 3$ the sum of squares of the eigenvalues is 3. It is easy to check that this two conditions cannot be satisfied simultaneously.

Problem 3. Let p be a prime number. Call a positive integer n *interesting* if

$$x^n - 1 = (x^p - x + 1)f(x) + pg(x)$$

for some polynomials f and g with integer coefficients.
a) Prove that the number $p^p - 1$ is interesting.
b) For which p is $p^p - 1$ the minimal interesting number?

(Eugene Goryachko and Fedor Petrov, St. Petersburg)

Solution. (a) Let's reformulate the property of being interesting: n is interesting if $x^n - 1$ is divisible by $x^p - x + 1$ in the ring of polynomials over $\mathbb{F}_p$ (the field of residues modulo p). All further congruences are modulo $x^p - x + 1$ in this ring. We have $x^p \equiv x - 1$, then $x^{p^2} = (x^p)^p \equiv (x - 1)^p \equiv x^p - 1 \equiv x - 2$, $x^{p^3} = (x^{p^2})^p \equiv (x - 2)^p \equiv x^p - 2^p \equiv x - 2^p - 1 \equiv x - 3$ and so on by Fermat's little theorem, finally $x^{p^p} \equiv x - p \equiv x$,

$$x(x^{p^p-1} - 1) \equiv 0.$$

Since the polynomials $x^p - x + 1$ and x are coprime, this implies $x^{p^p-1} - 1 \equiv 0$.

(b) We write

$$x^{1+p+p^2+\dots+p^{p-1}} = x \cdot x^p \cdot x^{p^2} \cdot \ldots \cdot x^{p^{p-1}} \equiv x(x-1)(x-2)\ldots(x-(p-1)) = x^p - x \equiv -1,$$

hence $x^{2(1+p+p^2+\dots+p^{p-1})} \equiv 1$ and $a = 2(1+p+p^2+\dots+p^{p-1})$ is an interesting number.

If $p > 3$, then $a = \frac{2}{p-1}(p^p - 1) < p^p - 1$, so we have an interesting number less than $p^p - 1$. On the other hand, we show that $p = 2$ and $p = 3$ do satisfy the condition. First notice that by $gcd(x^m - 1, x^k - 1) = x^{gcd(m,k)} - 1$, for every fixed p the greatest common divisors of interesting numbers is also an interesting number. Therefore the minimal interesting number divides all interesting numbers. In particular, the minimal interesting number is a divisor of $p^p - 1$.

For $p = 2$ we have $p^p - 1 = 3$, so the minimal interesting number is 1 or 3. But $x^2 - x + 1$ does not divide $x - 1$, so 1 is not interesting. Then the minimal interesting number is 3.

For $p = 3$ we have $p^p - 1 = 26$ whose divisors are $1, 2, 13, 26$. The numbers 1 and 2 are too small and $x^{13} \equiv -1 \not\equiv +1$ as shown above, so none of 1,2 and 13 is interesting. So 26 is the minimal interesting number.

Hence, $p^p - 1$ is the minimal interesting number if and only if $p = 2$ or $p = 3$.

Problem 4. Let $A_1, A_2, \ldots, A_n$ be finite, nonempty sets. Define the function

$$f(t) = \sum_{k=1}^{n} \sum_{1 \le i_1 < i_2 < \ldots < i_k \le n} (-1)^{k-1} t^{|A_{i_1} \cup A_{i_2} \cup \ldots \cup A_{i_k}|}.$$

Prove that f is nondecreasing on $[0, 1]$.

($|A|$ denotes the number of elements in A.)

(Levon Nurbekyan and Vardan Voskanyan, Yerevan)

Solution 1. Let $\Omega = \bigcup_{i=1}^{n} A_i$. Consider a random subset X of Ω which chosen in the following way: for each $x \in \Omega$, choose the element x for the set X with probability t, independently from the other elements.

Then for any set $C \subset \Omega$, we have

$$P(C \subset X) = t^{|C|}.$$

By the *inclusion-exclusion principle*,

$$P\big((A_1 \subset X) \text{ or } (A_2 \subset X) \text{ or } \ldots \text{ or } (A_n \subset X)\big) =$$

$$= \sum_{k=1}^{n} \sum_{1 \le i_1 < i_2 < \ldots < i_k \le n} (-1)^{k-1} P\big(A_{i_1} \cup A_{i_2} \cup \ldots \cup A_{i_k} \subset X\big) =$$

$$= \sum_{k=1}^{n} \sum_{1 \le i_1 < i_2 < \ldots < i_k \le n} (-1)^{k-1} t^{|A_{i_1} \cup A_{i_2} \cup \ldots \cup A_{i_k}|}.$$

The probability $P\big((A_1 \subset X) \text{ or } \ldots \text{ or } (A_n \subset X)\big)$ is a nondecreasing function of the probability t.

Problem 5. Let n be a positive integer and let V be a $(2n-1)$-dimensional vector space over the two-element field. Prove that for arbitrary vectors $v_1, \ldots, v_{4n-1} \in V$, there exists a sequence $1 \le i_1 < \ldots < i_{2n} \le 4n - 1$ of indices such that $v_{i_1} + \ldots + v_{i_{2n}} = 0$.

(Ilya Bogdanov, Moscow and Géza Kós, Budapest)

Solution. Let $V = \text{aff}\{v_1, \ldots, v_{4n-1}\}$. The statement $v_{i_1} + \dots + v_{i_{2n}} = 0$ is translation-invariant (i.e. replacing the vectors by $v_1 - a, \ldots, v_{4n-1} - a$), so we may assume that $0 \in V$. Let $d = \dim V$.

Lemma. The vectors can be permuted in such a way that $v_1 + v_2, v_3 + v_4, \ldots, v_{2d-1} + v_{2d}$ form a basis of V.

Proof. We prove by induction on d. If $d = 0$ or $d = 1$ then the statement is trivial.

First choose the vector v_1 such a way that $\text{aff}(v_2, v_3, \ldots, v_{4n-1}) = V$; this is possible since V is generated by some $d+1$ vectors and we have $d + 1 \le 2n < 4n - 1$. Next, choose v_2 such that $v_2 \ne v_1$. (By $d > 0$, not all vectors are the same.)

Now let $\ell = \{0, v_1 + v_2\}$ and let $V' = V/\ell$. For any $w \in V$, let $\tilde{w} = \ell + w = \{w, w + v_1 + v_2\}$ be the class of the factor space V' containing w. Apply the induction hypothesis to the vectors $\tilde{v_3}, \ldots, \tilde{v_{4n-1}}$. Since $\dim V' = d - 1$, the vectors can permuted in such a way that $\tilde{v_3} + \tilde{v_4}, \ldots, \tilde{v_{2d-1}} + \tilde{v_{2d}}$ is a basis of V'. Then $v_1 + v_2, v_3 + v_4, \ldots, v_{2d-1} + v_{2d}$ is a basis of V.

Now we can assume that $v_1 + v_2, v_3 + v_4, \ldots, v_{2d-1} + v_{2d}$ is a basis of V. The vector $w = (v_1 + v_3 + \dots + v_{2d-1}) + (v_{2d+1} + v_{2d+2} + \dots + v_{2n+d})$ is the sum of $2n$ vectors, so $w \in V$. Hence, $w + \varepsilon_1(v_1 + v_2) + \dots + \varepsilon_d(v_{2d-1} + v_{2d}) = 0$ with some $\varepsilon_1, \ldots, \varepsilon_d \in \mathbb{F}_2$, therefore

$$\sum_{i=1}^{d} \Big((1 - \varepsilon_i)v_{2i-1} + \varepsilon_i v_{2i}\Big) + \sum_{i=2d+1}^{2n+d} v_i = 0.$$

The left-hand side is the sum of $2n$ vectors.

IMC2011, Blagoevgrad, Bulgaria

Day 2, July 31, 2011

Problem 1. Let $(a_n)_{n=0}^{\infty}$ be a sequence with $\frac{1}{2} < a_n < 1$ for all $n \geq 0$. Define the sequence $(x_n)_{n=0}^{\infty}$ by

$$x_0 = a_0, \qquad x_{n+1} = \frac{a_{n+1} + x_n}{1 + a_{n+1}x_n} \quad (n \geq 0).$$

What are the possible values of $\lim\limits_{n\to\infty} x_n$? Can such a sequence diverge?

Johnson Olaleru, Lagos

Solution 1. We prove by induction that

$$0 < 1 - x_n < \frac{1}{2^{n+1}}.$$

Then we will have $(1 - x_n) \to 0$ and therefore $x_n \to 1$.

The case $n = 0$ is true since $\frac{1}{2} < x_0 = a_0 < 1$.

Supposing that the induction hypothesis holds for n, from the recurrence relation we get

$$1 - x_{n+1} = 1 - \frac{a_{n+1} + x_n}{1 + a_{n+1}x_n} = \frac{1 - a_{n+1}}{1 + a_{n+1}x_n}(1 - x_n).$$

By

$$0 < \frac{1 - a_{n+1}}{1 + a_{n+1}x_n} < \frac{1 - \frac{1}{2}}{1 + 0} = \frac{1}{2}$$

we obtain

$$0 < 1 - x_{n+1} < \frac{1}{2}(1 - x_n) < \frac{1}{2} \cdot \frac{1}{2^{n+1}} = \frac{1}{2^{n+2}}.$$

Hence, the sequence converges in all cases and $x_n \to 1$.

Solution 2. As is well-known,

$$\tanh(u + v) = \frac{\tanh u + \tanh v}{1 + \tanh u \tanh v}$$

for all real numbers u and v.

Setting $u_n = \operatorname{ar\,tanh} a_n$ we have $x_n = \tanh(u_0 + u_1 + \cdots + u_n)$. Then $u_0 + u_1 + \cdots + u_n > (n+1)\operatorname{ar\,tanh} \frac{1}{2}$ and $\lim\limits_{n\to\infty} x_n = \lim\limits_{u\to\infty} \tanh u = 1$.

Remark. If the condition $a_n \in (\frac{1}{2}, 1)$ is replaced by $a_n \in (0, 1)$ then the sequence remains increasing and bounded, but the limit can be less than 1.

Problem 2. An alien race has three genders: male, female, and emale. A *married triple* consists of three persons, one from each gender, who all like each other. Any person is allowed to belong to at most one married triple. A special feature of this race is that feelings are always mutual — if x likes y, then y likes x.

The race is sending an expedition to colonize a planet. The expedition has n males, n females, and n emales. It is known that every expedition member likes at least k persons of each of the two other genders. The problem is to create as many married triples as possible to produce healthy offspring so the colony could grow and prosper.

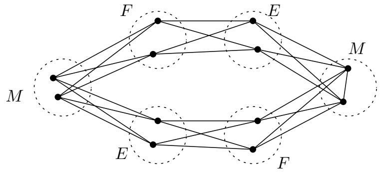
a) Show that if n is even and $k = \frac{n}{2}$, then it might be impossible to create even one married triple.

b) Show that if $k \geq \frac{3n}{4}$, then it is always possible to create n disjoint married triples, thus marrying all of the expedition members.

Fedor Duzhin and Nick Gravin, Singapore

Solution. (a) Let M be the set of males, F the set of females, and E the set of emales. Consider the (tripartite) graph G with vertices $M \cup F \cup E$ and edges for likes. A 3-cycle is then a possible family. We'll call G the *graph of likes.*

First, let $k = \frac{n}{2}$. Then n has to be even and we need to construct a graph of likes with no 3-cycles. We'll do the following: divide each of the sets M, F, and E into two equal parts and draw all edges between two parts as shown below:



Clearly, there is no 3-cycle.

(b) First divide the the expedition into male-emale-female triples arbitrarily. Let the *unhappiness* of such a subdivision be the number of pairs of aliens that belong to the same triple but don't like each other. We shall show that if unhappiness is positive, then the unhappiness can be decreased by a simple operation. It will follow that after several steps the unhappiness will be reduced to zero, which will lead to the happy marriage of everybody.

Assume that we have an emale which doesn't like at least one member of its triple (the other cases are similar). We perform the following operation: we swap this emale with another emale, so that each of these two emales will like the members of their new triples. Thus the unhappiness related to this emales will decrease, and the other pairs that contribute to the unhappiness remain unchanged, therefore the unhappiness will be decreased.

So, it remains to prove that such an operation is always possible. Enumerate the triples with $1, 2, \dots, n$ and denote by E_i, F_i, M_i the emale, female, and male members of the ith triple, respectively. Without loss of generality we may assume that E_1 doesn't like either F_1 or M_1 or both. We have to find an index $i > 1$ such that E_i likes the couple F_1, M_1 and E_1 likes the couple F_i, M_i; then we can swap E_1 and E_i.

There are at most $n/4$ indices i for which E_1 dislikes F_i and at most $n/4$ indices for which E_1 dislikes M_i, so there are no more than $n/2$ indices i for which E_1 dislikes someone from the couple M_i, F_i, and the set of these undesirable indexes includes 1. Similarly, there are no more than $n/2$ indices such that either M_1 or F_1 dislikes E_i. Since both undesirable sets of indices have at most $n/2$ elements and both contain 1, their union doesn't cover all indices, so we have some i which satisfies all conditions. Therefore we can always perform the operation that decreases unhappiness.

Solution 2 (for part b). Suppose that $k \geq \frac{3n}{4}$ and let's show that it's possible to marry all of the colonists. First, we'll prove that there exists a perfect matching between M and F. We need to check the condition of Hall's marriage theorem. In other words, for $A \subset M$, let $B \subset F$ be the set of all vertices of F adjacent to at least one vertex of A. Then we need to show that $|A| \leq |B|$. Let us assume the contrary, that is $|A| > |B|$. Clearly, $|B| \geq k$ if A is not empty. Let's consider any $f \in F \setminus B$. Then f is not adjacent to any vertex in A, therefore, f has degree in M not more than $n - |A| < n - |B| \leq n - k \leq \frac{n}{4}$, a contradiction.

Let's now construct a new bipartite graph, say H. The set of its vertices is $P \cup E$, where P is the set of pairs male–female from the perfect matching we just found. We will have an edge from $(m, f) = p \in P$ to $e \in E$ for each 3-cycle (m, f, e) of the graph G, where $(m, f) \in P$ and $e \in E$. Notice that the degree of each vertex of P in H is then at least $2k - n$.

What remains is to show that H satisfies the condition of Hall's marriage theorem and hence has a perfect matching. Assume, on the contrary, that the following happens. There is $A \subset P$ and $B \subset E$ such that $|A| = l$, $|B| < l$, and B is the set of all vertices of E adjacent to at least one vertex of A. Since the degree of each vertex of P is at least $2k - n$, we have $2k - n \leq |B| < l$. On the other hand, let $e \in E \setminus B$. Then for each pair $(m, f) = p \in P$, at most one of the pairs (e, m) and (e, f) is joined by an edge and hence the degree of e in G is at most $|M \setminus A| + |F \setminus A| + |A| = 2(n - l) + l = 2n - l$. But the degree of any vertex of G is $2k$ and thus we get $2k \leq 2n - l$, that is, $l \leq 2n - 2k$.

Finally, $2k - n < l \leq 2n - 2k$ implies that $k < \frac{3n}{4}$. This contradiction concludes the solution.

Problem 3. Determine the value of

$$\sum_{n=1}^{\infty} \ln\left(1 + \frac{1}{n}\right) \cdot \ln\left(1 + \frac{1}{2n}\right) \cdot \ln\left(1 + \frac{1}{2n+1}\right).$$

Gerhard Woeginger, Utrecht

Solution. Define $f(n) = \ln(\frac{n+1}{n})$ for $n \geq 1$, and observe that $f(2n)+f(2n+1) = f(n)$. The well-known inequality $\ln(1+x) \leq x$ implies $f(n) \leq 1/n$. Furthermore introduce

$$g(n) \;=\; \sum_{k=n}^{2n-1} f^3(k) \;<\; n\, f^3(n) \;\leq\; 1/n^2.$$

Then

$$\begin{aligned} g(n) - g(n+1) &= f^3(n) - f^3(2n) - f^3(2n+1) \\ &= (f(2n) + f(2n+1))^3 - f^3(2n) - f^3(2n+1) \\ &= 3\,(f(2n) + f(2n+1))\; f(2n)\, f(2n+1) \\ &= 3\, f(n)\, f(2n)\, f(2n+1), \end{aligned}$$

therefore

$$\sum_{n=1}^{N} f(n)\, f(2n)\, f(2n+1) \;=\; \frac{1}{3} \sum_{n=1}^{N} g(n) - g(n+1) \;=\; \frac{1}{3}\,(g(1) - g(N+1))\,.$$

Since $g(N+1) \to 0$ as $N \to \infty$, the value of the considered sum hence is

$$\sum_{n=1}^{\infty} f(n)\, f(2n)\, f(2n+1) \;=\; \frac{1}{3}\, g(1) \;=\; \frac{1}{3}\, \ln^3(2).$$

Problem 4. Let $f(x)$ be a polynomial with real coefficients of degree n. Suppose that $\dfrac{f(k) - f(m)}{k - m}$ is an integer for all integers $0 \leq k < m \leq n$. Prove that $a - b$ divides $f(a) - f(b)$ for all pairs of distinct integers a and b.

Fedor Petrov, St. Petersburg

Solution 1. We need the following

Lemma. Denote the least common multiple of $1, 2, \ldots, k$ by $L(k)$, and define

$$h_k(x) = L(k) \cdot \binom{x}{k} \qquad (k = 1, 2, \ldots).$$

Then the polynomial $h_k(x)$ satisfies the condition, i.e. $a - b$ divides $h_k(a) - h_k(b)$ for all pairs of distinct integers a, b.

Proof. It is known that

$$\binom{a}{k} = \sum_{j=0}^{k} \binom{a-b}{j} \binom{b}{k-j}.$$

(This formula can be proved by comparing the coefficient of x^k in $(1+x)^a$ and $(1+x)^{a-b}(1+x)^b$.) From here we get

$$h_k(a) - h_k(b) = L(K) \left(\binom{a}{k} - \binom{b}{k} \right) = L(K) \sum_{j=1}^{k} \binom{a-b}{j} \binom{b}{k-j} = (a-b) \sum_{j=1}^{k} \frac{L(k)}{j} \binom{a-b-1}{j-1} \binom{b}{k-j}.$$

On the right-hand side all fractions $\frac{L(k)}{j}$ are integers, so the right-hand side is a multiple of (a, b). The lemma is proved.

Expand the polynomial f in the basis $1, \binom{x}{1}, \binom{x}{2}, \ldots$ as

$$f(x) = A_0 + A_1 \binom{x}{1} + A_2 \binom{x}{2} + \cdots + A_n \binom{x}{n}. \tag{1}$$

We prove by induction on j that A_j is a multiple of $L(j)$ for $1 \leq j \leq n$. (In particular, A_j is an integer for $j \geq 1$.) Assume that $L(j)$ divides A_j for $1 \leq j \leq m-1$. Substituting m and some $k \in \{0, 1, \ldots, m-1\}$ in (1),

$$\frac{f(m) - f(k)}{m - k} = \sum_{j=1}^{m-1} \frac{A_j}{L(j)} \cdot \frac{h_j(m) - h_j(k)}{m - k} + \frac{A_m}{m - k}.$$

Since all other terms are integers, the last term $\frac{A_m}{m-k}$ is also an integer. This holds for all $0 \le k < m$, so A_m is an integer that is divisible by $L(m)$.

Hence, A_j is a multiple of $L(j)$ for every $1 \le j \le n$. By the lemma this implies the problem statement.

Solution 2. The statement of the problem follows immediately from the following claim, applied to the polynomial $g(x,y) = \frac{f(x)-f(y)}{x-y}$.

Claim. Let $g(x,y)$ be a real polynomial of two variables with total degree less than n. Suppose that $g(k,m)$ is an integer whenever $0 \le k < m \le n$ are integers. Then $g(k,m)$ is a integer for every pair k,m of integers.

Proof. Apply induction on n. If $n=1$ then g is a constant. This constant can be read from $g(0,1)$ which is an integer, so the claim is true.

Now suppose that $n \ge 2$ and the claim holds for $n-1$. Consider the polynomials

$$g_1(x,y) = g(x+1,y+1) - g(x,y+1) \quad \text{and} \quad g_2(x,y) = g(x,y+1) - g(x,y). \tag{1}$$

For every pair $0 \le k < m \le n-1$ of integers, the numbers $g(k,m)$, $g(k,m+1)$ and $g(k+1,m+1)$ are all integers, so $g_1(k,m)$ and $g_2(k,m)$ are integers, too. Moreover, in (1) the maximal degree terms of g cancel out, so $\deg g_1, \deg g_2 < \deg g$. Hence, we can apply the induction hypothesis to the polynomials g_1 and g_2 and we thus have $g_1(k,m), g_2(k,m) \in \mathbb{Z}$ for all $k,m \in \mathbb{Z}$.

In view of (1), for all $k,m \in \mathbb{Z}$, we have that

(a) $g(0,1) \in \mathbb{Z}$;

(b) $g(k,m) \in \mathbb{Z}$ if and only if $g(k+1,m+1) \in \mathbb{Z}$;

(c) $g(k,m) \in \mathbb{Z}$ if and only if $g(k,m+1) \in \mathbb{Z}$.

For arbitrary integers k,m, apply *(b)* $|k|$ times then apply *(c)* $|m-k-1|$ times as

$$g(k,m) \in \mathbb{Z} \Leftrightarrow \ldots \Leftrightarrow g(0,m-k) \in \mathbb{Z} \Leftrightarrow \ldots \Leftrightarrow g(0,1) \in \mathbb{Z}.$$

Hence, $g(k,m) \in \mathbb{Z}$. The claim has been proved.

Problem 5. Let $F = A_0A_1\ldots A_n$ be a convex polygon in the plane. Define for all $1 \le k \le n-1$ the operation f_k which replaces F with a new polygon

$$f_k(F) = A_0 \ldots A_{k-1}A'_kA_{k+1}\ldots A_n \ ,$$

where A'_k is the point symmetric to A_k with respect to the perpendicular bisector of $A_{k-1}A_{k+1}$. Prove that $(f_1 \circ f_2 \circ \ldots \circ f_{n-1})^n(F) = F$. We suppose that all operations are well-defined on the polygons, to which they are applied, i.e. results are convex polygons again. (A_0, $A_1, \ldots, A_n$ are the vertices of F in consecutive order.)

Mikhail Khristoforov, St. Petersburg

Solution. The operations f_i are rational maps on the $2(n-1)$-dimensional phase space of coordinates of the vertices $A_1,\ldots,A_{n-1}$. To show that $(f_1 \circ f_2 \circ \ldots \circ f_{n-1})^n$ is the identity, it is sufficient to verify this on some open set. For example, we can choose a neighborhood of the regular polygon, then all intermediate polygons in the proof will be convex.

Consider the operations f_i. Notice that (i) $f_i \circ f_i = id$ and (ii) $f_i \circ f_j = f_j \circ f_i$ for $|i-j| \ge 2$. We also show that (iii) $(f_i \circ f_{i+1})^3 = id$ for $1 \le i \le n-1$.

The operations f_i and f_{i+1} change the order of side lengths by interchanging two consecutive sides; after performing $(f_i \circ f_{i+1})^3$, the side lengths are in the original order. Moreover, the sums of opposite angles in the convex quadrilateral $A_{i-1}A_iA_{i+1}A_{i+2}$ are preserved in all operations. These quantities uniquely determine the quadrilateral, because with fixed sides, both angles $\angle A_1A_2A_3$ and $\angle A_1A_4A_3$ decrease when A_1A_3 increases. Hence, property (iii) is proved.

In the symmetric group S_n, the transpositions $\sigma_i = (i,i+1)$, which from a generator system, satisfy the same properties (i–iii). It is well-known that S_n is the maximal group with $n-1$ generators, satisfying (i–iii). In S_n we have $(\sigma_1 \circ \sigma_2 \circ \ldots \circ \sigma_{n-1})^n = (1,2,3,\ldots,n)^n = id$, so this implies $(f_1 \circ f_2 \circ \ldots \circ f_{n-1})^n = id$.

IMC 2012, Blagoevgrad, Bulgaria
Day 1, July 28, 2012

Problem 1. For every positive integer n, let $p(n)$ denote the number of ways to express n as a sum of positive integers. For instance, $p(4) = 5$ because

$$4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1.$$

Also define $p(0) = 1$.

Prove that $p(n) - p(n-1)$ is the number of ways to express n as a sum of integers each of which is strictly greater than 1.

(Proposed by Fedor Duzhin, Nanyang Technological University)

Solution 1. The statement is true for $n = 1$, because $p(0) = p(1) = 1$ and the only partition of 1 contains the term 1. In the rest of the solution we assume $n \geq 2$.

Let $\mathcal{P}_n = \{(a_1, \ldots, a_k) : k \in \mathbb{N},\ a_1 \geq \ldots \geq a_k,\ a_1 + \ldots + a_k = n\}$ be the set of partitions of n, and let $\mathcal{Q}_n = \{(a_1, \ldots, a_k) \in \mathcal{P}_n : a_k = 1\}$ the set of those partitions of n that contain the term 1. The set of those partitions of n that do not contain 1 as a term, is $\mathcal{P}_n \setminus \mathcal{Q}_n$. We have to prove that $|P_n \setminus \mathcal{Q}_n| = |\mathcal{P}_n| - |\mathcal{P}_{n-1}|$.

Define the map $\varphi\colon \mathcal{P}_{n-1} \to \mathcal{Q}_n$ as

$$\varphi(a_1, \ldots, a_k) = (a_1, \ldots, a_k, 1).$$

This is a partition of n containing 1 as a term (so indeed $\varphi(a_1, \ldots, a_k) \in \mathcal{Q}_n$). Moreover, each partition $(a_1, \ldots, a_k, 1) \in \mathcal{Q}_n$ uniquely determines $(a_1, \ldots, a_k)$. Therefore the map φ is a bijection between the sets $\mathcal{P}_{n-1}$ and $\mathcal{Q}_n$. Then $|\mathcal{P}_{n-1}| = |\mathcal{Q}_n|$. Since $\mathcal{Q}_n \subset \mathcal{P}_n$,

$$|\mathcal{P}_n \setminus \mathcal{Q}_n| = |\mathcal{P}_n| - |\mathcal{Q}_n| = |\mathcal{P}_n| - |\mathcal{P}_{n-1}| = p(n) - p(n-1).$$

Solution 2 (outline). Denote by $q(n)$ the number of partitions of n not containing 1 as term ($q(0) = 1$ as the only partition of 0 is the empty sum), and define the generating functions

$$F(x) = \sum_{n=0}^{\infty} p(n)x^n \quad \text{and} \quad G(x) = \sum_{n=0}^{\infty} q(n)x^n.$$

Since $q(n) \leq p(n) < 2^n$, these series converge in some interval, say for $|x| < \frac{1}{2}$, and the values uniquely determine the coefficients.

According to Euler's argument, we have

$$F(x) = \sum_{n=0}^{\infty} p(n)x^n = \prod_{k=1}^{\infty}(1 + x^k + x^{2k} + \ldots) = \prod_{k=1}^{\infty} \frac{1}{1 - x^k}$$

and

$$G(x) = \sum_{n=0}^{\infty} q(n)x^n = \prod_{k=2}^{\infty}(1 + x^k + x^{2k} + \ldots) = \prod_{k=2}^{\infty} \frac{1}{1 - x^k}.$$

Then $G(x) = (1-x)F(x)$. Comparing the coefficient of x^n in this identity we get $q(n) = p(n) - p(n-1)$.

Problem 2. Let n be a fixed positive integer. Determine the smallest possible rank of an $n \times n$ matrix that has zeros along the main diagonal and strictly positive real numbers off the main diagonal.

(Proposed by Ilya Bogdanov and Grigoriy Chelnokov, MIPT, Moscow)

Solution. For $n = 1$ the only matrix is (0) with rank 0. For $n = 2$ the determinant of such a matrix is negative, so the rank is 2. We show that for all $n \geq 3$ the minimal rank is 3.

Notice that the first three rows are linearly independent. Suppose that some linear combination of them, with coefficients c_1, c_2, c_3, vanishes. Observe that from the first column one deduces that c_2 and c_3 either have opposite signs or both zero. The same applies to the pairs (c_1, c_2) and (c_1, c_3). Hence they all must be zero.

It remains to give an example of a matrix of rank (at most) 3. For example, the matrix

$$\begin{pmatrix} 0^2 & 1^2 & 2^2 & \dots & (n-1)^2 \\ (-1)^2 & 0^2 & 1^2 & \dots & (n-2)^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (-n+1)^2 & (-n+2)^2 & (-n+3)^2 & \dots & 0^2 \end{pmatrix} = \Big((i-j)^2\Big)_{i,j=1}^{n} =$$

$$= \begin{pmatrix} 1^2 \\ 2^2 \\ \vdots \\ n^2 \end{pmatrix} (1, 1, \dots, 1) - 2 \begin{pmatrix} 1 \\ 2 \\ \vdots \\ n \end{pmatrix} (1, 2, \dots, n) + \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} (1^2, 2^2, \dots, n^2)$$

is the sum of three matrices of rank 1, so its rank cannot exceed 3.

Problem 3. Given an integer $n > 1$, let S_n be the group of permutations of the numbers $1, 2, \dots, n$. Two players, A and B, play the following game. Taking turns, they select elements (one element at a time) from the group S_n. It is forbidden to select an element that has already been selected. The game ends when the selected elements generate the whole group S_n. The player who made the last move loses the game. The first move is made by A. Which player has a winning strategy?

(Proposed by Fedor Petrov, St. Petersburg State University)

Solution. Player A can win for $n = 2$ (by selecting the identity) and for $n = 3$ (selecting a 3-cycle).

We prove that B has a winning strategy for $n \geq 4$. Consider the moment when all permitted moves lose immediately, and let H be the subgroup generated by the elements selected by the players. Choosing another element from H would not lose immediately, so all elements of H must have been selected. Since H and any other element generate S_n, H must be a maximal subgroup in S_n.

If $|H|$ is even, then the next player is A, so B wins. Denote by n_i the order of the subgroup generated by the first i selected elements; then $n_1 | n_2 | n_3 | \dots$. We show that B can achieve that n_2 is even and $n_2 < n!$; then $|H|$ will be even and A will be forced to make the final – losing – move.

Denote by g the element chosen by A on his first move. If the order n_1 of g is even, then B may choose the identical permutation id and he will have $n_2 = n_1$ even and $n_2 = n_1 < n!$.

If n_1 is odd, then g is a product of disjoint odd cycles, so it is an even permutation. Then B can chose the permutation $h = (1, 2)(3, 4)$ which is another even permutation. Since g and h are elements of the alternating group A_n, they cannot generate the whole S_n. Since the order of h is 2, B achieves $2 | n_2$.

Remark. If $n \geq 4$, all subgrups of odd order are subgroups of A_n which has even order. Hence, all maximal subgroups have even order and B is never forced to lose.

Problem 4. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function that satisfies $f'(t) > f(f(t))$ for all $t \in \mathbb{R}$. Prove that $f(f(f(t))) \leq 0$ for all $t \geq 0$.

(Proposed by Tomáš Bárta, Charles University, Prague)

Solution.

Lemma 1. Either $\lim_{t\to+\infty} f(t)$ does not exist or $\lim_{t\to+\infty} f(t) \neq +\infty$.

Proof. Assume that the limit is $+\infty$. Then there exists $T_1 > 0$ such that for all $t > T_1$ we have $f(t) > 2$. There exists $T_2 > 0$ such that $f(t) > T_1$ for all $t > T_2$. Hence, $f'(t) > f(f(t)) > 2$ for $t > T_2$. Hence, there exists T_3 such that $f(t) > t$ for $t > T_3$. Then $f'(t) > f(f(t)) > f(t)$, $f'(t)/f(t) > 1$, after integration $\ln f(t) - \ln T_3 > t - T_3$, i.e. $f(t) > T_3 e^{t-T_3}$ for all $t > T_3$. Then $f'(t) > f(f(t)) > T_3 e^{f(t)-T_3}$ and $f'(t)e^{-f(t)} > T_3 e^{-T_3}$. Integrating from T_3 to t yields $e^{-f(T_3)} - e^{-f(t)} > (t - T_3)T_3 e^{-T_3}$. The right-hand side tends to infinity, but the left-hand side is bounded from above, a contradiction. □

Lemma 2. For all $t > 0$ we have $f(t) < t$.

Proof. By Lemma 1, there are some positive real numbers t with $f(t) < t$. Hence, if the statement is false then there is some $t_0 > 0$ with $f(t_0) = t_0$.

Case I: There exist some value $t \geq t_0$ with $f(t) < t_0$. Let $T = \inf\{t \geq t_0 : f(t) < t_0\}$. By the continuity of f, $f(T) = t_0$. Then $f'(T) > f(f(T)) = f(t_0) = t_0 > 0$. This implies $f > f(T) = t_0$ in a right neighbourhood, contradicting the definition of T.

Case II: $f(t) \geq t_0$ for all $t \geq t_0$. Now we have $f'(t) > f(f(t)) \geq t_0 > 0$. So, f' has a positive lower bound over (t_0, ∞), which contradicts Lemma 1. □

Lemma 3. (a) If $f(s_1) > 0$ and $f(s_2) \geq s_1$, then $f(s) > s_1$ for all $s > s_2$.

(b) In particular, if $s_1 \leq 0$ and $f(s_1) > 0$, then $f(s) > s_1$ for all $s > s_1$.

Proof. Suppose that there are values $s > s_2$ with $f(s) \leq s_1$ and let $S = \inf\{s > s_2 : f(s) \leq s_1\}$. By the continuity we have $f(S) = s_1$. Similarly to Lemma 2, we have $f'(S) > f(f(S)) = f(s_1) > 0$. If $S > s_2$ then in a left neighbourhood of S we have $f < s_1$, contradicting the definition of S. Otherwise, if $S = s_2$ then we have $f > s_1$ in a right neighbourhood of s_2, contradiction again.

Part (b) follows if we take $s_2 = s_1$. □

With the help of these lemmas the proof goes as follows. Assume for contradiction that there exists some $t_0 > 0$ with $f(f(f(t_0))) > 0$. Let $t_1 = f(t_0)$, $t_2 = f(t_1)$ and $t_3 = f(t_2) > 0$. We show that $0 < t_3 < t_2 < t_1 < t_0$. By lemma 2 it is sufficient to prove that t_1 and t_2 are positive. If $t_1 < 0$, then $f(t_1) \leq 0$ (if $f(t_1) > 0$ then taking $s_1 = t_1$ in Lemma 3(b) yields $f(t_0) > t_1$, contradiction). If $t_1 = 0$ then $f(t_1) \leq 0$ by lemma 2 and the continuity of f. Hence, if $t_1 \leq 0$, then also $t_2 \leq 0$. If $t_2 = 0$ then $f(t_2) \leq 0$ by lemma 2 and the continuity of f (contradiction, $f(t_2) = t_3 > 0$). If $t_2 < 0$, then by lemma 3(b), $f(t_0) > t_2$, so $t_1 > t_2$. Applying lemma 3(a) we obtain $f(t_1) > t_2$, contradiction. We have proved $0 < t_3 < t_2 < t_1 < t_0$.

By lemma 3(a) ($f(t_1) > 0$, $f(t_0) \geq t_1$) we have $f(t) > t_1$ for all $t > t_0$ and similarly $f(t) > t_2$ for all $t > t_1$. It follows that for $t > t_0$ we have $f'(t) > f(f(t)) > t_2 > 0$. Hence, $\lim_{t\to+\infty} f(t) = +\infty$, which is a contradiction. This contradiction proves that $f(f(f(t))) \leq 0$ for all $t > 0$. For $t = 0$ the inequality follows from the continuity of f.

Problem 5. Let a be a rational number and let n be a positive integer. Prove that the polynomial $X^{2^n}(X+a)^{2^n} + 1$ is irreducible in the ring $\mathbb{Q}[X]$ of polynomials with rational coefficients.

(Proposed by Vincent Jugé, École Polytechnique, Paris)

Solution. First let us consider the case $a = 0$. The roots of $X^{2^{n+1}} + 1$ are exactly all primitive roots of unity of order 2^{n+2}, namely $e^{2\pi i \frac{k}{2^{n+2}}}$ for odd $k = 1, 3, 5, \ldots, 2^{n+2} - 1$. It is a cyclotomic polynomial, hence irreducible in $\mathbb{Q}[X]$.

Let now $a \neq 0$ and suppose that the polynomial in the question is reducible. Substituting $X = Y - \frac{a}{2}$ we get a polynomial $(Y - \frac{a}{2})^{2^n}(Y + \frac{a}{2})^{2^n} + 1 = (Y^2 - \frac{a^2}{4})^{2^n} + 1$. It is again a cyclotomic polynomial in the variable $Z = Y^2 - \frac{a^2}{4}$, and therefore it is not divisible by any polynomial in Y^2 with rational coefficients. Let us write this polynomial as the product of irreducible monic polynomials in Y with appropriate multiplicities, i.e.

$$\left(Y^2 - \frac{a^2}{4}\right)^{2^n} + 1 = \prod_{i=1}^{r} f_i(Y)^{m_i} \qquad f_i \text{ monic, irreducible, all different.}$$

Since the left-hand side is a polynomial in Y^2 we must have $\prod_i f_i(Y)^{m_i} = \prod_i f_i(-Y)^{m_i}$. By the above argument non of the f_i is a polynomial in Y^2, i.e. $f_i(-Y) \neq f_i(Y)$. Therefore for every i there is $i' \neq i$ such that $f_i(-Y) = \pm f_{i'}(Y)$. In particular r is even and irreducible factors f_i split into pairs. Let us renumber them so that $f_1, \dots, f_{\frac{r}{2}}$ belong to different pairs and we have $f_{i+\frac{r}{2}}(-Y) = \pm f_i(Y)$. Consider the polynomial $f(Y) = \prod_{i=1}^{r/2} f_i(Y)^{m_i}$. This polynomial is monic of degree 2^n and $(Y^2 - \frac{a^2}{4})^{2^n} + 1 = f(Y)f(-Y)$. Let us write $f(Y) = Y^{2^n} + \cdots + b$ where $b \in \mathbb{Q}$ is the constant term, i.e. $b = f(0)$. Comparing constant terms we then get $\left(\frac{a}{2}\right)^{2^{n+1}} + 1 = b^2$. Denote $c = \left(\frac{a}{2}\right)^{2^{n-1}}$. This is a nonzero rational number and we have $c^4 + 1 = b^2$.

It remains to show that there are no rational solutions $c, b \in \mathbb{Q}$ to the equation $c^4+1 = b^2$ with $c \neq 0$ which will contradict our assumption that the polynomial under consideration is reducible. Suppose there is a solution. Without loss of generality we can assume that $c, b > 0$. Write $c = \frac{u}{v}$ with u and v coprime positive integers. Then $u^4 + v^4 = (bv^2)^2$. Let us denote $w = bv^2$, this must be a positive integer too since u, v are positive integers. Let us show that the set $\mathcal{T} = \{(u, v, w) \in \mathbb{N}^3 \mid u^4 + v^4 = w^2 \text{ and } u, v, w \geq 1\}$ is empty. Suppose the contrary and consider some triple $(u, v, w) \in \mathcal{T}$ such that w is minimal. Without loss of generality, we may assume that u is odd. (u^2, v^2, w) is a primitive Pythagorean triple and thus there exist relatively prime integers $d > e \geq 1$ such that $u^2 = d^2 - e^2$, $v^2 = 2de$ and $w = d^2 + e^2$. In particular, considering the equation $u^2 = d^2 - e^2$ in $\mathbb{Z}/4\mathbb{Z}$ proves that d is odd and e is even. Therefore, we can write $d = f^2$ and $e = 2g^2$. Moreover, since $u^2 + e^2 = d^2$, (u, e, d) is also a primitive Pythagorean triple: there exist relatively prime integers $h > i \geq 1$ such that $u = h^2 - i^2$, $e = 2hi = 2g^2$ and $d = h^2 + i^2$. Once again, we can write $h = k^2$ and $i = l^2$, so that we obtain the relation $f^2 = d = h^2 + i^2 = k^4 + l^4$ and $(k, l, f) \in \mathcal{T}$. Then, the inequality $w > d^2 = f^4 \geq f$ contradicts the minimality of w.

Remark 1. One can also use Galois theory arguments in order to solve this question. Let us denote the polynomial in the question by $P(X) = X^{2^n}(X + a)^{2^n} + 1$ and we will also need the cyclotomic polynomial $T(X) = X^{2^n} + 1$. As we already said, when $a = 0$ then $P(X)$ is itself cyclotomic and hence irreducible. Let now $a \neq 0$ and x be any complex root of $P(x) = 0$. Then $\zeta = x(x+a)$ satisfies $T(\zeta) = 0$, hence it is a primitive root of unity of order 2^{n+1}. The field $\mathbb{Q}[x]$ is then an extension of $\mathbb{Q}[\zeta]$. The latter field is cyclotomic and its degree over $\mathbb{Q}$ is $\dim_{\mathbb{Q}}(\mathbb{Q}[\zeta]) = 2^n$. Since the polynomial in the question has degree 2^{n+1} we see that it is reducible if and only if the above mentioned extension is trivial, i.e. $\mathbb{Q}[x] = \mathbb{Q}[\zeta]$. For the sake of contradiction we will now assume that this is indeed the case. Let $S(X)$ be the minimal polynomial of x over $\mathbb{Q}$. The degree of S is then 2^n and we can number its roots by odd numbers in the set $I = \{1, 3, \dots, 2^{n+1} - 1\}$ so that $S(X) = \prod_{k \in I}(X - x_k)$ and $x_k(x_k + a) = \zeta^k$ because Galois automorphisms of $\mathbb{Q}[\zeta]$ map ζ to $\zeta^k, k \in I$. Then one has

$$S(X)S(-a - X) = \prod_{k \in I}(X - x_k)(-a - X - x_k) = (-1)^{|I|} \prod_{k \in I}\Big(X(X + a) - \zeta^k\Big) = T\big(X(X + a)\big) = P(X)\,.$$

In particular $P(-\frac{a}{2}) = S(-\frac{a}{2})^2$, i.e. $\left(\frac{a}{2}\right)^{2^{n+1}} + 1 = \left(\left(\frac{a}{2}\right)^{2^n} + 1\right)^2$. Therefore the rational numbers $c = \left(\frac{a}{2}\right)^{2^{n-1}} \neq 0$ and $b = \left(\frac{a}{2}\right)^{2^n} + 1$ satisfy $c^4 + 1 = b^2$ which is a contradiction as it was shown in the first proof.

Remark 2. It is well-known that the Diophantine equation $x^4 + y^4 = z^2$ has only trivial solutions (i.e. with $x = 0$ or $y = 0$). This implies immediately that $c^4 + 1 = b^2$ has no rational solution with nonzero c.

IMC 2012, Blagoevgrad, Bulgaria
Day 2, July 29, 2012

Problem 1. Consider a polynomial

$$f(x) = x^{2012} + a_{2011}\,x^{2011} + \ldots + a_1\,x + a_0\,.$$

Albert Einstein and Homer Simpson are playing the following game. In turn, they choose one of the coefficients $a_0, \ldots, a_{2011}$ and assign a real value to it. Albert has the first move. Once a value is assigned to a coefficient, it cannot be changed any more. The game ends after all the coefficients have been assigned values.

Homer's goal is to make $f(x)$ divisible by a fixed polynomial $m(x)$ and Albert's goal is to prevent this.

(a) Which of the players has a winning strategy if $m(x) = x - 2012$?

(b) Which of the players has a winning strategy if $m(x) = x^2 + 1$?

(Proposed by Fedor Duzhin, Nanyang Technological University)

Solution. We show that Homer has a winning strategy in both part (a) and part (b).

(a) Notice that the last move is Homer's, and only the last move matters. Homer wins if and only if $f(2012) = 0$, i.e.

$$2012^{2012} + a_{2011}\,2012^{2011} + \ldots + a_k\,2012^k + \ldots + a_1\,2012 + a_0\; = 0. \tag{1}$$

Suppose that all of the coefficients except for a_k have been assigned values. Then Homer's goal is to establish (1) which is a linear equation on a_k. Clearly, it has a solution and hence Homer can win.

(b) Define the polynomials

$$g(y) = a_0 + a_2 y + a_4 y^2 + \ldots + a_{2010} y^{1005} + y^{1006} \quad \text{and} \quad h(y) = a_1 + a_3 y + a_5 y^2 + \ldots + a_{2011} y^{1005},$$

so $f(x) = g(x^2) + h(x^2)\cdot x$. Homer wins if he can achieve that $g(y)$ and $h(y)$ are divisible by $y+1$, i.e. $g(-1) = h(-1) = 0$.

Notice that both $g(y)$ and $h(y)$ have an even number of undetermined coefficients in the beginning of the game. A possible strategy for Homer is to follow Albert: whenever Albert assigns a value to a coefficient in g or h, in the next move Homer chooses the value for a coefficient in the same polynomial. This way Homer defines the last coefficient in g and he also chooses the last coefficient in h. Similarly to part (a), Homer can choose these two last coefficients in such a way that both $g(-1) = 0$ and $h(-1) = 0$ hold.

Problem 2. Define the sequence $a_0, a_1, \ldots$ inductively by $a_0 = 1$, $a_1 = \frac{1}{2}$ and

$$a_{n+1} = \frac{n a_n^2}{1 + (n+1) a_n} \quad \text{for } n \ge 1.$$

Show that the series $\displaystyle\sum_{k=0}^{\infty} \frac{a_{k+1}}{a_k}$ converges and determine its value.

(Proposed by Christophe Debry, KU Leuven, Belgium)

Solution. Observe that

$$ka_k = \frac{(1+(k+1)a_k)a_{k+1}}{a_k} = \frac{a_{k+1}}{a_k} + (k+1)a_{k+1} \quad \text{for all } k \ge 1,$$

and hence

$$\sum_{k=0}^{n} \frac{a_{k+1}}{a_k} = \frac{a_1}{a_0} + \sum_{k=1}^{n} (ka_k - (k+1)a_{k+1}) = \frac{1}{2} + 1 \cdot a_1 - (n+1)a_{n+1} = 1 - (n+1)a_{n+1} \qquad (1)$$

for all $n \ge 0$.

By (1) we have $\sum_{k=0}^{n} \frac{a_{k+1}}{a_k} < 1$. Since all terms are positive, this implies that the series $\sum_{k=0}^{\infty} \frac{a_{k+1}}{a_k}$ is convergent. The sequence of terms, $\frac{a_{k+1}}{a_k}$ must converge to zero. In particular, there is an index n_0 such that $\frac{a_{k+1}}{a_k} < \frac{1}{2}$ for $n \ge n_0$. Then, by induction on n, we have $a_n < \frac{C}{2^n}$ with some positive constant C. From $na_n < \frac{Cn}{2^n} \to 0$ we get $na_n \to 0$, and therefore

$$\sum_{k=0}^{\infty} \frac{a_{k+1}}{a_k} = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{a_{k+1}}{a_k} = \lim_{n\to\infty} \Big(1 - (n+1)a_{n+1}\Big) = 1.$$

Remark. The inequality $a_n \le \frac{1}{2^n}$ can be proved by a direct induction as well.

Problem 3. Is the set of positive integers n such that $n! + 1$ divides $(2012n)!$ finite or infinite?

(Proposed by Fedor Petrov, St. Petersburg State University)

Solution 1. Consider a positive integer n with $n! + 1 \big| (2012n)!$. It is well-known that for arbitrary nonnegative integers $a_1, \ldots, a_k$, the number $(a_1 + \ldots + a_k)!$ is divisible by $a_1! \cdot \ldots \cdot a_k!$. (The number of sequences consisting of a_1 digits 1, …, a_k digits k, is $\frac{(a_1+\ldots+a_k)!}{a_1!\cdot\ldots\cdot a_k!}$.) In particular, $(n!)^{2012}$ divides $(2012n)!$.

Since $n! + 1$ is co-prime with $(n!)^{2012}$, their product $(n! + 1)(n!)^{2012}$ also divides $(2012n)!$, and therefore

$$(n! + 1) \cdot (n!)^{2012} \le (2012n)!\,.$$

By the known inequalities $\left(\frac{n+1}{e}\right)^n < n! \le n^n$, we get

$$\left(\frac{n}{e}\right)^{2013n} < (n!)^{2013} < (n! + 1) \cdot (n!)^{2012} \le (2012n)! < (2012n)^{2012n}$$

$$n < 2012^{2012} e^{2013}.$$

Therefore, there are only finitely many such integers n.

Remark. Instead of the estimate $\left(\frac{n+1}{e}\right)^n < n!$, we may apply the *Multinomial theorem*:

$$(x_1 + \cdots + x_\ell)^N = \sum_{k_1+\ldots+k_\ell=N} \frac{N!}{k_1! \cdot \ldots \cdot k_\ell!} x_1^{k_1} \ldots x_\ell^{k_\ell}.$$

Applying this to $N = 2012n$, $\ell = 2012$ and $x_1 = \ldots = x_\ell = 1$,

$$\frac{(2012n)!}{(n!)^{2012}} < (\underbrace{1 + 1 + \ldots + 1}_{2012})^{2012n} = 2012^{2012n},$$

$$n! < n! + 1 \le \frac{(2012n)!}{(n!)^{2012}} < 2012^{2012n}.$$

On the right-hand side we have a geometric progression which increases slower than the factorial on the left-hand side, so this is true only for finitely many n.

Solution 2. Assume that $n > 2012$ is an integer with $n!+1 \big| (2012n)!$. Notice that all prime divisors of $n!+1$ are greater than n, and all prime divisors of $(2012n)!$ are smaller than $2012n$.

Consider a prime p with $n < p < 2012n$. Among $1, 2, \ldots, 2012n$ there are $\left[\frac{2012n}{p}\right] < 2012$ numbers divisible by p; by $p^2 > n^2 > 2012n$, none of them is divisible by p^2. Therefore, the exponent of p in the prime factorization of $(2012n)!$ is at most 2011. Hence,

$$n!+1 = \gcd\big(n!+1, (2012n)!\big) < \prod_{n<p<2012p} p^{2011}.$$

Applying the inequality $\prod_{p \le X} p < 4^X$,

$$n! < \prod_{n<p<2012p} p^{2011} < \left(\prod_{p<2012n} p\right)^{2011} < \left(4^{2012n}\right)^{2011} = \left(4^{2012\cdot 2011}\right)^n. \tag{2}$$

Again, we have a factorial on the left-and side and a geometric progression on the right-hand side.

Problem 4. Let $n \ge 2$ be an integer. Find all real numbers a such that there exist real numbers x_1, $\ldots$, x_n satisfying

$$x_1(1-x_2) = x_2(1-x_3) = \ldots = x_{n-1}(1-x_n) = x_n(1-x_1) = a. \tag{1}$$

(Proposed by Walther Janous and Gerhard Kirchner, Innsbruck)

Solution. Throughout the solution we will use the notation $x_{n+1} = x_1$.

We prove that the set of possible values of a is

$$\left(-\infty, \frac{1}{4}\right] \bigcup \left\{\frac{1}{4\cos^2\frac{k\pi}{n}};\ k \in \mathbb{N},\ 1 \le k < \frac{n}{2}\right\}.$$

In the case $a \le \frac{1}{4}$ we can choose x_1 such that $x_1(1-x_1) = a$ and set $x_1 = x_2 = \ldots = x_n$. Hence we will now suppose that $a > \frac{1}{4}$.

The system (1) gives the recurrence formula

$$x_{i+1} = \varphi(x_i) = 1 - \frac{a}{x_i} = \frac{x_i - a}{x_i}, \quad i = 1, \ldots, n.$$

The fractional linear transform φ can be interpreted as a projective transform of the real projective line $\mathbb{R} \cup \{\infty\}$; the map φ is an element of the group $\mathrm{PGL}_2(\mathbb{R})$, represented by the linear transform $M = \begin{pmatrix} 1 & -a \\ 1 & 0 \end{pmatrix}$. (Note that $\det M \ne 0$ since $a \ne 0$.) The transform φ^n can be represented by M^n. A point $[u, v]$ (written in homogenous coordinates) is a fixed point of this transform if and only if $(u, v)^T$ is an eigenvector of M^n. Since the entries of M^n and the coordinates u, v are real, the corresponding eigenvalue is real, too.

The characteristic polynomial of M is $x^2 - x + a$, which has no real root for $a > \frac{1}{4}$. So M has two conjugate complex eigenvalues $\lambda_{1,2} = \frac{1}{2}\big(1 \pm \sqrt{4a-1}i\big)$. The eigenvalues of M^n are $\lambda_{1,2}^n$, they are real if and only if $\arg \lambda_{1,2} = \pm\frac{k\pi}{n}$ with some integer k; this is equivalent with

$$\pm\sqrt{4a-1} = \tan\frac{k\pi}{n},$$

$$a = \frac{1}{4}\left(1 + \tan^2\tfrac{k\pi}{n}\right) = \frac{1}{4\cos^2\frac{k\pi}{n}}.$$

If $\arg\lambda_1 = \frac{k\pi}{n}$ then $\lambda_1^n = \lambda_2^n$, so the eigenvalues of M^n are equal. The eigenvalues of M are distinct, so M and M^n have two linearly independent eigenvectors. Hence, M^n is a multiple of the identity. This means that the projective transform φ^n is the identity; starting from an arbitrary point $x_1 \in \mathbb{R}\cup\{\infty\}$, the cycle $x_1, x_2, \dots, x_n$ closes at $x_{n+1} = x_1$. There are only finitely many cycles $x_1, x_2, \dots, x_n$ containing the point ∞; all other cycles are solutions for (1).

Remark. If we write $x_j = P + Q\tan t_j$ where P, Q and $t_1, \dots, t_n$ are real numbers, the recurrence relation re-writes as

$$(P + Q\tan t_j)(1 - P - Q\tan t_{j+1}) = a$$
$$(1-P)Q\tan t_j - PQ\tan t_{j+1} = a + P(P-1) + Q^2\tan t_j \tan t_{j+1} \quad (j = 1, 2, \dots, n).$$

In view of the identity $\tan(\alpha-\beta) = \dfrac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}$, it is reasonable to choose $P = \frac{1}{2}$, and $Q = \sqrt{a - \frac{1}{4}}$. Then the recurrence leads to

$$t_j - t_{j+1} \equiv \arctan\sqrt{4a-1} \pmod{\pi}.$$

Problem 5. Let $c \geq 1$ be a real number. Let G be an abelian group and let $A \subset G$ be a finite set satisfying $|A+A| \leq c|A|$, where $X + Y := \{x + y \mid x \in X,\ y \in Y\}$ and $|Z|$ denotes the cardinality of Z. Prove that

$$|\underbrace{A + A + \ldots + A}_{k \text{ times}}| \leq c^k|A|$$

for every positive integer k. (Plünnecke's inequality)

(Proposed by Przemyslaw Mazur, Jagiellonian University)

Solution. Let B be a nonempty subset of A for which the value of the expression $c_1 = \frac{|A+B|}{|B|}$ is the least possible. Note that $c_1 \leq c$ since A is one of the possible choices of B.

Lemma 1. For any finite set $D \subset G$ we have $|A + B + D| \leq c_1|B + D|$.

Proof. Apply induction on the cardinality of D. For $|D| = 1$ the Lemma is true by the definition of c_1. Suppose it is true for some D and let $x \notin D$. Let $B_1 = \{y \in B \mid x + y \in B + D\}$. Then $B + (D \cup \{x\})$ decomposes into the union of two disjoint sets:

$$B + (D \cup \{x\}) = (B + D) \cup \big((B \setminus B_1) + \{x\}\big)$$

and therefore $|B + (D \cup \{x\})| = |B + D| + |B| - |B_1|$. Now we need to deal with the cardinality of the set $A + B + (D \cup \{x\})$. Writing $A + B + (D \cup \{x\}) = (A + B + D) \cup (A + B + \{x\})$ we count some of the elements twice: for example if $y \in B_1$, then $A + \{y\} + \{x\} \subset (A + B + D) \cap (A + B + \{x\})$. Therefore all the elements of the set $A + B_1 + \{x\}$ are counted twice and thus

$$|A + B + (D \cup \{x\})| \leq |A + B + D| + |A + B + \{x\}| - |A + B_1 + \{x\}| =$$
$$= |A + B + D| + |A + B| - |A + B_1| \leq c_1(|B + D| - |B| - |B_1|) = c_1|B + (D \cup \{x\})|,$$

where the last inequality follows from the inductive hypothesis and the observation that $\frac{|A+B|}{|B|} = c_1 \leq \frac{|A+B_1|}{|B_1|}$ (or B_1 is the empty set). □

Lemma 2. For every $k \geq 1$ we have $|\underbrace{A + \ldots + A}_{k \text{ times}} + B| \leq c_1^k|B|$.

Proof. Induction on k. For $k = 1$ the statement is true by definition of c_1. For greater k set $D = \underbrace{A + \ldots + A}_{k-1 \text{ times}}$ in the previous lemma: $|\underbrace{A + \ldots + A}_{k \text{ times}} + B| \leq c_1|\underbrace{A + \ldots + A}_{k-1 \text{ times}} + B| \leq c_1^k|B|$. □

Now notice that

$$|\underbrace{A + \ldots + A}_{k \text{ times}}| \leq |\underbrace{A + \ldots + A}_{k \text{ times}} + B| \leq c_1^k|B| \leq c^k|A|.$$

Remark. The proof above due to Giorgios Petridis and can be found at `http://gowers.wordpress.com/2011/02/10/a-new-way-of-proving-sumset-estimates/`

IMC 2012, Blagoevgrad, Bulgaria
Day 1, July 28, 2012

Problem 1. Let A and B be real symmetric matrices with all eigenvalues strictly greater than 1. Let λ be a real eigenvalue of matrix AB. Prove that $|\lambda| > 1$.

(Proposed by Pavel Kozhevnikov, MIPT, Moscow)

Solution. The transforms given by A and B strictly increase the length of every nonzero vector, this can be seen easily in a basis where the matrix is diagonal with entries greater than 1 in the diagonal. Hence their product AB also strictly increases the length of any nonzero vector, and therefore its real eigenvalues are all greater than 1 or less than -1.

Problem 2. Let $f: \mathbb{R} \to \mathbb{R}$ be a twice differentiable function. Suppose $f(0) = 0$. Prove that there exists $\xi \in (-\pi/2, \pi/2)$ such that

$$f''(\xi) = f(\xi)(1 + 2\tan^2 \xi).$$

(Proposed by Karen Keryan, Yerevan State University, Yerevan, Armenia)

Solution. Let $g(x) = f(x)\cos x$. Since $g(-\pi/2) = g(0) = g(\pi/2) = 0$, by Rolle's theorem there exist some $\xi_1 \in (-\pi/2, 0)$ and $\xi_2 \in (0, \pi/2)$ such that

$$g'(\xi_1) = g'(\xi_2) = 0.$$

Now consider the function

$$h(x) = \frac{g'(x)}{\cos^2 x} = \frac{f'(x)\cos x - f(x)\sin x}{\cos^2 x}.$$

We have $h(\xi_1) = h(\xi_2) = 0$, so by Rolle's theorem there exist $\xi \in (\xi_1, \xi_2)$ for which

$$0 = h'(\xi) = \frac{g''(\xi)\cos^2 \xi + 2\cos\xi \sin\xi g'(\xi)}{\cos^4 \xi} =$$

$$= \frac{(f''(\xi)\cos\xi - 2f'(\xi)\sin\xi - f(\xi)\cos\xi)\cos\xi + 2\sin\xi(f'(\xi)\cos\xi - f(\xi)\sin\xi)}{\cos^3 \xi} =$$

$$= \frac{f''(\xi)\cos^2\xi - f(\xi)(\cos^2\xi + 2\sin^2\xi)}{\cos^3\xi} = \frac{1}{\cos\xi}(f''(\xi) - f(\xi)(1 + 2\tan^2\xi)).$$

The last yields the desired equality.

Problem 3. There are $2n$ students in a school ($n \in \mathbb{N}$, $n \geq 2$). Each week n students go on a trip. After several trips the following condition was fulfilled: every two students were together on at least one trip. What is the minimum number of trips needed for this to happen?

(Proposed by Oleksandr Rybak, Kiev, Ukraine)

Solution. We prove that for any $n \geq 2$ the answer is 6.

First we show that less than 6 trips is not sufficient. In that case the total quantity of students in all trips would not exceed $5n$. A student meets $n - 1$ other students in each trip, so he or she takes part on at least 3 excursions to meet all of his or her $2n - 1$ schoolmates. Hence the total quantity of students during the trips is not less then $6n$ which is impossible.

Now let's build an example for 6 trips.

If n is even, we may divide $2n$ students into equal groups A, B, C, D. Then we may organize the trips with groups (A,B), (C,D), (A,C), (B,D), (A,D) and (B,C), respectively.

If n is odd and divisible by 3, we may divide all students into equal groups E, F, G, H, I, J. Then the members of trips may be the following: (E,F,G), (E,F,H), (G,H,I), (G,H,J), (E,I,J), (F,I,J).

In the remaining cases let $n=2x+3y$ be, where x and y are natural numbers. Let's form the groups A, B, C, D of x students each, and E, F, G, H, I, J of y students each. Then we apply the previous cases and organize the following trips: (A,B,E,F,G), (C,D,E,F,H), (A,C,G,H,I), (B,D,G,H,J), (A,D,E,I,J), (B,C,F,I,J).

Problem 4. Let $n\ge 3$ and let $x_1,\dots,x_n$ be nonnegative real numbers. Define $A=\sum\limits_{i=1}^{n}x_i$, $B=\sum\limits_{i=1}^{n}x_i^2$ and $C=\sum\limits_{i=1}^{n}x_i^3$. Prove that

$$(n+1)A^2B+(n-2)B^2\ge A^4+(2n-2)AC.$$

(Proposed by Géza Kós, Eötvös University, Budapest)

Solution. Let

$$p(X)=\prod_{i=1}^{n}(X-x_i)=X^n-AX^{n-1}+\frac{A^2-B}{2}X^{n-2}-\frac{A^3-3AB+2C}{6}X^{n-3}+\dots.$$

The $(n-3)$th derivative of p has three nonnegative real roots $0\le u\le v\le w$. Hence,

$$\frac{6}{n!}p^{(n-3)}(X)=X^3-\frac{3A}{n}X^2+\frac{3(A^2-B)}{n(n-1)}X-\frac{A^3-3AB+2C}{n(n-1)(n-2)}=(X-u)(X-v)(X-w),$$

so

$$u+v+w=\frac{3A}{n},\quad uv+vw+wu=\frac{3(A^2-B)}{n(n-1)}\quad\text{and}\quad uvw=\frac{A^3-3AB+2C}{n(n-1)(n-2)}.$$

From these we can see that

$$\frac{n^2(n-1)^2(n-2)}{9}\big((n+1)A^2B+(n-2)B^2-A^4-(2n-2)AC\big)=\ldots=$$
$$=u^2v^2+v^2w^2+w^2u^2-uvw(u+v+w)=uv(u-w)(v-w)+vw(v-u)(w-u)+wu(w-v)(u-v)\ge$$
$$\ge 0+uw(v-u)(w-v)+wu(w-v)(u-v)=0.$$

Problem 5. Does there exist a sequence (a_n) of complex numbers such that for every positive integer p we have that $\sum_{n=1}^{\infty}a_n^p$ converges if and only if p is not a prime?

(Proposed by Tomáš Bárta, Charles University, Prague)

Solution. The answer is YES. We prove a more general statement; suppose that $N=C\cup D$ is an arbitrary decomposition of N into two disjoint sets. Then there exists a sequence $(a_n)_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty}a_n^p$ is convergent for $p\in C$ and divergent for $p\in D$.

Define $C_k=C\cap[1,k]$ and $D_k\cap[1,k]$.

Lemma. For every positive integer k there exists a positive integer N_k and a sequence $X_k = (x_{k,1}, \dots, x_{k,N_k})$ of complex numbers with the following properties:

(a) For $p \in D_k$, we have $\left|\sum_{j=1}^{N_k} x_{k,j}^p\right| \ge 1$.

(b) For $p \in C_k$, we have $\sum_{j=1}^{N_k} x_{k,j}^p = 0$; moreover, $\left|\sum_{j=1}^{m} x_{k,j}^p\right| \le \frac{1}{k}$ holds for $1 \le m \le N_k$.

Proof. First we find some complex numbers $z_1 \dots, z_k$ with

$$\sum_{j=1}^{k} z_j^p = \begin{cases} 0 & p \in C_k \\ 1 & p \in D_k \end{cases} \tag{1}$$

As is well-known, this system of equations is equivalent to another system $\sigma_\nu(z_1, \dots, z_k) = w_\nu$ ($\nu = 1, 2, \dots, k$) where σ_ν is the νth elementary symmetric polynomial, and the constants w_ν are uniquely determined by the Newton-Waring-Girard formulas. Then the numbers $z_1, \dots, z_k$ are the roots of the polynomial $z^k - w_1 z^{k-1} + - \dots + (-1)^k w_k$ in some order.

Now let

$$M = \left\lceil \max_{1\le m\le k,\ p\in C_k} \left|\sum_{j=1}^{m} z_j^p\right| \right\rceil$$

and let $N_k = k\cdot(kM)^k$. We define the numbers $x_{k,1} \dots, x_{k,N_k}$ by repeating the sequence $(\frac{z_1}{kM}, \frac{z_2}{kM}, \dots, \frac{z_k}{kM})$ $(kM)^k$ times, i.e. $x_{k,\ell} = \frac{z_j}{kM}$ if $\ell \equiv j \pmod{k}$. Then we have

$$\sum_{j=1}^{N_k} x_{k,j}^p = (kM)^k \sum_{j=1}^{k} \left(\tfrac{z_j}{kM}\right)^p = (kM)^{k-p} \sum_{j=1}^{k} z_j^p;$$

then from (1) the properties (a) and the first part of (b) follows immediately. For the second part of (b), suppose that $p \in C_k$ and $1 \le m \le N_k$; then $m = kr + s$ with some integers r and $1 \le s \le k$ and hence

$$\left|\sum_{j=1}^{m} x_{k,j}^p\right| = \left|\sum_{j=1}^{kr} + \sum_{j=kr+1}^{kr+s}\right| = \left|\sum_{j=1}^{s} \left(\frac{z_j}{kM}\right)^p\right| \le \frac{M}{(kM)^p} \le \frac{1}{k}.$$

The lemma is proved.

Now let $S_k = N_1 \dots, N_k$ (we also define $S_0 = 0$). Define the sequence (a) by simply concatenating the sequences $X_1, X_2, \dots$:

$$(a_1, a_2, \dots) = (x_{1,1}, \dots, x_{1,N_1}, x_{2,1}, \dots, x_{2,N_2}, \dots, x_{k,1}, \dots, x_{k,N_k}, \dots); \tag{1}$$
$$a_{S_k+j} = x_{k+1,j} \quad (1 \le j \le N_{k+1}). \tag{2}$$

If $p \in D$ and $k \ge p$ then

$$\left|\sum_{j=S_k+1}^{S_{k+1}} a_j^p\right| = \left|\sum_{j=1}^{N_{k+1}} x_{k+1,j}^p\right| \ge 1;$$

By Cauchy's convergence criterion it follows that $\sum a_n^p$ is divergent.

If $p \in C$ and $S_u < n \le S_{u+1}$ with some $u \ge p$ then

$$\left|\sum_{j=S_p+1}^{n} a_n^p\right| = \left|\sum_{k=p+1}^{u-1} \sum_{j=1}^{N_k} x_{k,j}^p + \sum_{j=1}^{n-S_{u-1}} x_{u,j}^p\right| = \left|\sum_{j=1}^{n-S_{u-1}} x_{u,j}^p\right| \le \frac{1}{u}.$$

Then it follows that $\sum_{n=S_p+1}^{\infty} a_n^p = 0$, and thus $\sum_{n=1}^{\infty} a_n^p = 0$ is convergent.

IMC 2013, Blagoevgrad, Bulgaria
Day 2, August 9, 2013

Problem 1. Let z be a complex number with $|z+1|>2$. Prove that $|z^3+1|>1$.

(Proposed by Walther Janous and Gerhard Kirchner, Innsbruck)

Solution. Since $z^3+1=(z+1)(z^2-z+1)$, it suffices to prove that $|z^2-z+1|\ge \frac{1}{2}$.

Assume that $z+1=re^{\varphi i}$, where $r=|z+1|>2$, and $\varphi=\arg(z+1)$ is some real number. Then

$$z^2-z+1=(re^{\varphi i}-1)^2-(re^{\varphi i}-1)+1=r^2e^{2\varphi i}-3re^{\varphi i}+3,$$

and

$$\begin{aligned}|z^2-z+1|^2&=\big(r^2e^{2\varphi i}-3re^{\varphi i}+3\big)\big(r^2e^{-2\varphi i}-3re^{-\varphi i}+3\big)=\\&=r^4+9r^2+9-(6r^3+18r)\cos\varphi+6r^2\cos 2\varphi=\\&=r^4+9r^2+9-(6r^3+18r)\cos\varphi+6r^2(2\cos^2\varphi-1)=\\&=12\left(r\cos\varphi-\frac{r^2+3}{4}\right)^2+\frac{1}{4}(r^2-3)^2>0+\frac{1}{4}=\frac{1}{4}.\end{aligned}$$

This finishes the proof.

Problem 2. Let p and q be relatively prime positive integers. Prove that

$$\sum_{k=0}^{pq-1}(-1)^{\left\lfloor\frac{k}{p}\right\rfloor+\left\lfloor\frac{k}{q}\right\rfloor}=\begin{cases}0 & \text{if } pq \text{ is even,}\\ 1 & \text{if } pq \text{ is odd.}\end{cases}\qquad (*)$$

(Here $\lfloor x\rfloor$ denotes the integer part of x.)

(Proposed by Alexander Bolbot, State University, Novosibirsk)

Solution. Suppose first that pq is even (which implies that p and q have opposite parities), and let $a_k=(-1)^{\left\lfloor\frac{k}{p}\right\rfloor+\left\lfloor\frac{k}{q}\right\rfloor}$. We show that $a_k+a_{pq-1-k}=0$, so the terms on the left-and side of $(*)$ cancel out in pairs.

For every positive integer k we have $\left\{\frac{k}{p}\right\}+\left\{\frac{p-1-k}{p}\right\}=\frac{p-1}{p}$, hence

$$\left\lfloor\frac{k}{p}\right\rfloor+\left\lfloor\frac{pq-1-k}{p}\right\rfloor=\left(\frac{k}{p}-\left\{\frac{k}{p}\right\}\right)+\left(\frac{pq-1-k}{p}-\left\{\frac{-1-k}{p}\right\}\right)=\frac{pq-1}{p}-\frac{p-1}{p}=q-1$$

and similarly

$$\left\lfloor\frac{pq-1-k}{q}\right\rfloor+\left\lfloor\frac{k}{q}\right\rfloor=p-1.$$

Since p and q have opposite parities, it follows that $\left\lfloor\frac{k}{p}\right\rfloor+\left\lfloor\frac{k}{q}\right\rfloor$ and $\left\lfloor\frac{pq-1-k}{p}\right\rfloor+\left\lfloor\frac{pq-1-k}{q}\right\rfloor$ have opposite parities and therefore $a_{pq-1-k}=-a_k$.

Now suppose that pq is odd. For every index k, denote by p_k and q_k the remainders of k modulo p and q, respectively. (I.e., $0\le p_k<p$ and $0\le q_k<q$ such that $k\equiv p_k \pmod p$ and $k\equiv q_k \pmod q$.)

Notice that

$$\left\lfloor\frac{k}{p}\right\rfloor+\left\lfloor\frac{k}{q}\right\rfloor\equiv p\left\lfloor\frac{k}{p}\right\rfloor+q\left\lfloor\frac{k}{q}\right\rfloor=(k-p_k)+(k-q_k)\equiv p_k+q_k \pmod 2.$$

Since p and q are co-prime, by the Chinese remainder theorem the map $k \mapsto (p_k, q_k)$ is a bijection between the sets $\{0, 1, \dots, pq-1\}$ and $\{0, 1, \dots, p-1\} \times \{0, 1, \dots, q-1\}$. Hence

$$\sum_{k=0}^{pq-1} (-1)^{\left\lfloor \frac{k}{p} \right\rfloor + \left\lfloor \frac{k}{q} \right\rfloor} = \sum_{k=0}^{pq-1} (-1)^{p_k + q_k} = \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} (-1)^{i+j} = \left(\sum_{i=0}^{p-1} (-1)^i \right) \cdot \left(\sum_{j=0}^{q-1} (-1)^j \right) = 1.$$

Problem 3. Suppose that $v_1, \dots, v_d$ are unit vectors in $\mathbb{R}^d$. Prove that there exists a unit vector u such that

$$|u \cdot v_i| \le 1/\sqrt{d}$$

for $i = 1, 2, \dots, d$.

(Here $\cdot$ denotes the usual scalar product on $\mathbb{R}^d$.)

(Proposed by Tomasz Tkocz, University of Warwick)

Solution. If $v_1, \dots, v_d$ are linearly dependent then we can simply take a unit vector u perpendicular to $\operatorname{span}(v_1, \dots, v_d)$. So assume that $v_1, \dots, v_d$ are linearly independent. Let $w_1, \dots, w_d$ be the dual basis of $(v_1, \dots, v_d)$, i.e.

$$w_i \cdot v_j = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \ne j \end{cases} \quad \text{for } 1 \le i, j \le d.$$

From $w_i \cdot v_i = 1$ we have $|w_i| \ge 1$.

For every sequence $\varepsilon = (\varepsilon_1, \dots, \varepsilon_d) \in \{+1, -1\}^d$ of signs define $u_\varepsilon = \sum_{i=1}^d \varepsilon_i w_i$. Then we have

$$|u_\varepsilon \cdot v_k| = \left| \sum_{i=1}^d \varepsilon_i (w_i \cdot v_k) \right| = \left| \sum_{i=1}^d \varepsilon_i \delta_{ik} \right| = |\varepsilon_k| = 1 \quad \text{for } k = 1, \dots, d.$$

Now estimate the average of $|u_\varepsilon|^2$.

$$\frac{1}{2^d} \sum_{\varepsilon \in \{+1,-1\}^n} |u_\varepsilon|^2 = \frac{1}{2^d} \sum_{\varepsilon \in \{+1,-1\}^n} \left(\sum_{i=1}^d \varepsilon_i w_i \right) \cdot \left(\sum_{j=1}^d \varepsilon_j w_j \right) =$$

$$= \sum_{i=1}^d \sum_{j=1}^d (w_i \cdot w_j) \left(\frac{1}{2^d} \sum_{\varepsilon \in \{+1,-1\}^n} \varepsilon_i \varepsilon_j \right) = \sum_{i=1}^d \sum_{j=1}^d (w_i \cdot w_j) \delta_{ij} = \sum_{i=1}^d |w_i|^2 \ge d.$$

It follows that there is a ε such that $|u_\varepsilon|^2 \ge d$. For that ε the vector $u = \dfrac{u_\varepsilon}{|u_\varepsilon|}$ satisfies the conditions.

Problem 4. Does there exist an infinite set M consisting of positive integers such that for any $a, b \in M$, with $a < b$, the sum $a + b$ is square-free?

(A positive integer is called square-free if no perfect square greater than 1 divides it.)

(Proposed by Fedor Petrov, St. Petersburg State University)

Solution. The answer is yes. We construct an infinite sequence $1 = n_1 < 2 = n_2 < n_3 < \dots$ so that $n_i + n_j$ is square-free for all $i < j$. Suppose that we already have some numbers $n_1 < \dots < n_k$ ($k \ge 2$), which satisfy this condition and find a suitable number n_{k+1} to be the next element of the sequence.

We will choose n_{k+1} of the form $n_{k+1} = 1 + Mx$, with $M = ((n_1 + \dots + n_k + 2k)!)^2$ and some positive integer x. For $i = 1, 2, \dots, k$ we have $n_i + n_{k+1} = 1 + Mx + n_i = (1 + n_i)m_i$, where m_i and M are co-prime, so any perfect square dividing $1 + Mx + n_i$ is co-prime with M.

In order to find a suitable x, take a large N and consider the values $x = 1, 2, \dots, N$. If a value $1 \le x \le N$ is not suitable, this means that there is an index $1 \le i \le k$ and some prime p such that $p^2 | 1 + Mx + n_i$. For $p \le 2k$ this is impossible because $p|M$. Moreover, we also have $p^2 \le 1 + Mx + n_i < M(N+1)$, so $2k < p < \sqrt{M(N+1)}$.

For any fixed i and p, the values for x for which $p^2 | 1 + Mx + n_i$ form an arithmetic progression with difference p^2. Therefore, there are at most $\dfrac{N}{p^2} + 1$ such values. In total, the number of unsuitable values x is less than

$$\sum_{i=1}^{k} \sum_{2k<p<\sqrt{M(N+1)}} \left(\frac{N}{p^2} + 1\right) < k \cdot \left(N \sum_{p>2k} \frac{1}{p^2} + \sum_{p<\sqrt{M(N+1)}} 1 \right) <$$

$$< kN \sum_{p>2k} \left(\frac{1}{p-1} - \frac{1}{p} \right) + k\sqrt{M(N+1)} < \frac{N}{2} + k\sqrt{M(N+1)}.$$

If N is big enough then this is less than N, and there exist a suitable choice for x.

Problem 5. Consider a circular necklace with 2013 beads. Each bead can be painted either white or green. A painting of the necklace is called *good*, if among any 21 successive beads there is at least one green bead. Prove that the number of good paintings of the necklace is odd.

(Two paintings that differ on some beads, but can be obtained from each other by rotating or flipping the necklace, are counted as different paintings.)

(Proposed by Vsevolod Bykov and Oleksandr Rybak, Kiev)

Solution 1. For $k = 0, 1, \dots$ denote by N_k be the number of *good open laces*, consisting of k (white and green) beads in a row, such that among any 21 successive beads there is at least one green bead. For $k \le 21$ all laces have this property, so $N_k = 2^k$ for $0 \le k \le 20$; in particular, N_0 is odd, and $N_1, \dots, N_{20}$ are even.

For $k \ge 21$, there must be a green bead among the last 21 ones. Suppose that the last green bead is at the ℓth position; then $\ell \ge k - 20$. The previous $\ell - 1$ beads have $N_{\ell-1}$ good colorings, and every such good coloring provides a good lace of length k. Hence,

$$N_k = N_{k-1} + N_{k-2} + \dots + N_{k-21} \quad \text{for } k \ge 21. \tag{1}$$

From (1) we can see that $N_{21} = N_0 + \dots + N_{20}$ is odd, and $N_{22} = N_1 + \dots + N_{21}$ is also odd.

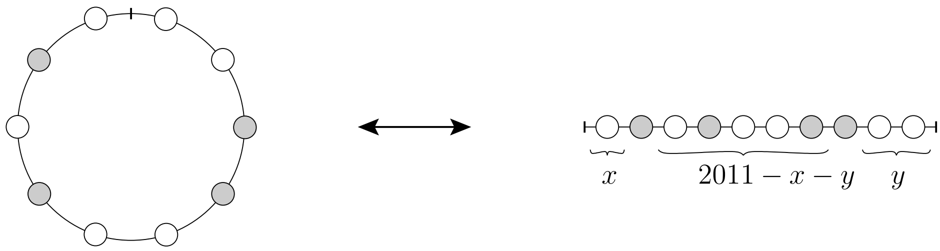
Applying (1) again to the term N_{k-1},

$$N_k = N_{k-1} + \dots + N_{k-21} = \Big(N_{k-2} + \dots + N_{k-22}\Big) + N_{k-2} + \dots + N_{k-21} \equiv N_{k-22} \pmod 2$$

so the sequence of parities in (N_k) is periodic with period 22. We conclude that

- N_k is odd if $k \equiv 0 \pmod{22}$ or $k \equiv 21 \pmod{22}$;
- N_k is even otherwise.

Now consider the good circular necklaces of 2013 beads. At a fixed point between two beads cut each. The resulting open lace may have some consecutive white beads at the two ends, altogether at most 20. Suppose that there are x white beads at the beginning and y white beads at the end; then we have $x, y \ge$ and $x + y \le 20$, and we have a good open lace in the middle, between the first and the last green beads. That middle lace consist of $2011 - x - y$ beads. So, for any fixed values of x and y the number of such cases is $N_{2011-x-y}$.



It is easy to see that from such a good open lace we can reconstruct the original circular lace. Therefore, the number of good circular necklaces is

$$\sum_{x+y\le 20} N_{2011-x-y} = N_{2011}+2N_{2010}+3N_{2009}+\ldots+21N_{1991} \equiv N_{2011}+N_{2009}+N_{2007}+\ldots+N_{1991} \pmod 2.$$

By $91\cdot 22-1=2001$ the term N_{2001} is odd, the other terms are all even, so the number of the good circular necklaces is odd.

Solution 2 (by Yoav Krauz, Israel). There is just one good monochromatic necklace. Let us count the parity of good necklaces having both colors.

For each necklace, we define an *adjusted necklace*, so that at position 0 we have a white bead and at position 1 we have a green bead. If the necklace is satisfying the condition, it corresponds to itself; if both beads 0 and 1 are white we rotate it (so that the bead 1 goes to place 0) until bead 1 becomes green; if bead 1 is green, we rotate it in the opposite direction until the bead 0 will be white. This procedure is called adjusting, and the place between the white and green bead which are rotated into places 0 and 1 will be called *distinguished place*. The interval consisting of the subsequent green beads after the distinguished place and subsequent white beads before it will be called *distinguished interval*.

For each adjusted necklace we have several necklaces corresponding to it, and the number of them is equal to the length of distinguished interval (the total number of beads in it) minus 1. Since we count only the parity, we can disregard the adjusted necklaces with even distinguished intervals and count once each adjusted necklace with odd distinguished interval.

Now we shall prove that the number of necklaces with odd distinguished intervals is even by grouping them in pairs. The pairing is the following. If the number of white beads with in the distinguished interval is even, we turn the last white bead (at the distinguished place) into green. The white interval remains, since a positive even number minus 1 is still positive. If the number of white beads in the distinguished interval is odd, we turn the green bead next to the distinguished place into white. The green interval remains since it was even; the white interval was odd and at most 19 so it will become even and at most 20, so we still get a good necklace.

This pairing on good necklaces with distinguished intervals of odd length shows, that the number of such necklaces is even; hence the total number of all good necklaces using both colors is even. Therefore, together with monochromatic green necklace, the number of good necklaces is odd.

IMC 2014, Blagoevgrad, Bulgaria

Day 1, July 31, 2014

Problem 1. Determine all pairs (a, b) of real numbers for which there exists a unique symmetric 2×2 matrix M with real entries satisfying $\operatorname{trace}(M) = a$ and $\det(M) = b$.

(Proposed by Stephan Wagner, Stellenbosch University)

Solution 1. Let the matrix be

$$M = \begin{bmatrix} x & z \\ z & y \end{bmatrix}.$$

The two conditions give us $x + y = a$ and $xy - z^2 = b$. Since this is symmetric in x and y, the matrix can only be unique if $x = y$. Hence $2x = a$ and $x^2 - z^2 = b$. Moreover, if (x, y, z) solves the system of equations, so does $(x, y, -z)$. So M can only be unique if $z = 0$. This means that $2x = a$ and $x^2 = b$, so $a^2 = 4b$.

If this is the case, then M is indeed unique: if $x + y = a$ and $xy - z^2 = b$, then

$$(x - y)^2 + 4z^2 = (x + y)^2 + 4z^2 - 4xy = a^2 - 4b = 0,$$

so we must have $x = y$ and $z = 0$, meaning that

$$M = \begin{bmatrix} a/2 & 0 \\ 0 & a/2 \end{bmatrix}$$

is the only solution.

Solution 2. Note that $\operatorname{trace}(M) = a$ and $\det(M) = b$ if and only if the two eigenvalues λ_1 and λ_2 of M are solutions of $x^2 - ax + b = 0$. If $\lambda_1 \neq \lambda_2$, then

$$M_1 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \qquad \text{and} \qquad M_2 = \begin{bmatrix} \lambda_2 & 0 \\ 0 & \lambda_1 \end{bmatrix}$$

are two distinct solutions, contradicting uniqueness. Thus $\lambda_1 = \lambda_2 = \lambda = a/2$, which implies $a^2 = 4b$ once again. In this case, we use the fact that M has to be diagonalisable as it is assumed to be symmetric. Thus there exists a matrix T such that

$$M = T^{-1} \cdot \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \cdot T,$$

however this reduces to $M = \lambda(T^{-1} \cdot I \cdot T) = \lambda I$, which shows again that M is unique.

Problem 2. Consider the following sequence

$$(a_n)_{n=1}^{\infty} = (1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, 1, \dots).$$

Find all pairs (α, β) of positive real numbers such that $\lim\limits_{n\to\infty} \dfrac{\sum\limits_{k=1}^{n} a_k}{n^\alpha} = \beta.$

(Proposed by Tomas Barta, Charles University, Prague)

Solution. Let $N_n = \binom{n+1}{2}$ (then a_{N_n} is the first appearance of number n in the sequence) and consider limit of the subsequence

$$b_{N_n} := \frac{\sum_{k=1}^{N_n} a_k}{N_n^\alpha} = \frac{\sum_{k=1}^{n} 1 + \cdots + k}{\binom{n+1}{2}^\alpha} = \frac{\sum_{k=1}^{n} \binom{k+1}{2}}{\binom{n+1}{2}^\alpha} = \frac{\binom{n+2}{3}}{\binom{n+1}{2}^\alpha} = \frac{\frac{1}{6}n^3(1+2/n)(1+1/n)}{(1/2)^\alpha n^{2\alpha}(1+1/n)^\alpha}.$$

We can see that $\lim\limits_{n\to\infty} b_{N_n}$ is positive and finite if and only if $\alpha = 3/2$. In this case the limit is equal to $\beta = \frac{\sqrt{2}}{3}$. So, this pair $(\alpha, \beta) = (\frac{3}{2}, \frac{\sqrt{2}}{3})$ is the only candidate for solution. We will show convergence of the original sequence for these values of α and β.

Let N be a positive integer in $[N_n+1, N_{n+1}]$, i.e., $N = N_n + m$ for some $1 \le m \le n+1$. Then we have

$$b_N = \frac{\binom{n+2}{3} + \binom{m+1}{2}}{\left(\binom{n+1}{2} + m\right)^{3/2}}$$

which can be estimated by

$$\frac{\binom{n+2}{3}}{\left(\binom{n+1}{2} + n\right)^{3/2}} \le b_N \le \frac{\binom{n+2}{3} + \binom{n+1}{2}}{\binom{n+1}{2}^{3/2}}.$$

Since both bounds converge to $\frac{\sqrt{2}}{3}$, the sequence b_N has the same limit and we are done.

Problem 3. Let n be a positive integer. Show that there are positive real numbers $a_0, a_1, \dots, a_n$ such that for each choice of signs the polynomial

$$\pm a_n x^n \pm a_{n-1}x^{n-1} \pm \cdots \pm a_1 x \pm a_0$$

has n distinct real roots.

(Proposed by Stephan Neupert, TUM, München)

Solution. We proceed by induction on n. The statement is trivial for $n = 1$. Thus assume that we have some $a_n, \dots, a_0$ which satisfy the conditions for some n. Consider now the polynomials

$$\tilde{P}(x) = \pm a_n x^{n+1} \pm a_{n-1}x^n \pm \ldots \pm a_1 x^2 \pm a_0 x$$

By induction hypothesis and $a_0 \ne 0$, each of these polynomials has $n+1$ distinct zeros, including the n nonzero roots of $\pm a_n x^n \pm a_{n-1}x^{n-1} \pm \ldots \pm a_1 x \pm a_0$ and 0. In particular none of the polynomials has a root which is a local extremum. Hence we can choose some $\varepsilon > 0$ such that for each such polynomial $\tilde{P}(x)$ and each of its local extrema s we have $|\tilde{P}(s)| > \varepsilon$. We claim that then each of the polynomials

$$P(x) = \pm a_n x^{n+1} \pm a_{n-1}x^n \pm \ldots \pm a_1 x^2 \pm a_0 x \pm \varepsilon$$

has exactly $n+1$ distinct zeros as well. As $\tilde{P}(x)$ has $n+1$ distinct zeros, it admits a local extremum at n points. Call these local extrema $-\infty = s_0 < s_1 < s_2 < \ldots < s_n < s_{n+1} = \infty$. Then for each $i \in \{0, 1, \ldots, n\}$ the values $\tilde{P}(s_i)$ and $\tilde{P}(s_{i+1})$ have opposite signs (with the obvious convention at infinity). By choice of ε the same holds true for $P(s_i)$ and $P(s_{i+1})$. Hence there is at least one real zero of $P(x)$ in each interval (s_i, s_{i+1}), i.e. $P(x)$ has at least (and therefore exactly) $n+1$ zeros. This shows that we have found a set of positive reals $a'_{n+1} = a_n, a'_n = a_{n-1}, \ldots, a'_1 = a_0, a'_0 = \varepsilon$ with the desired properties.

Problem 4. Let $n > 6$ be a perfect number, and let $n = p_1^{e_1} \cdots p_k^{e_k}$ be its prime factorisation with $1 < p_1 < \ldots < p_k$. Prove that e_1 is an even number.

A number n is *perfect* if $s(n) = 2n$, where $s(n)$ is the sum of the divisors of n.

(Proposed by Javier Rodrigo, Universidad Pontificia Comillas)

Solution. Suppose that e_1 is odd, contrary to the statement.

We know that $s(n) = \prod_{i=1}^{k}(1 + p_i + p_i^2 + \cdots + p_i^{e_i}) = 2n = 2p_1^{e_1} \cdots p_k^{e_k}$. Since e_1 is an odd number, $p_1 + 1$ divides the first factor $1 + p_1 + p_1^2 + \cdots + p_1^{e_1}$, so $p_1 + 1$ divides $2n$. Due to $p_1 + 1 > 2$, at least one of the primes $p_1, \ldots, p_k$ divides $p_1 + 1$. The primes $p_3, \ldots, p_k$ are greater than $p_1 + 1$ and p_1 cannot divide $p_1 + 1$, so p_2 must divide $p_1 + 1$. Since $p_1 + 1 < 2p_2$, this possible only if $p_2 = p_1 + 1$, therefore $p_1 = 2$ and $p_2 = 3$. Hence, $6|n$.

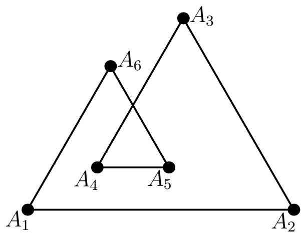
Now $n, \frac{n}{2}, \frac{n}{3}, \frac{n}{6}$ and 1 are distinct divisors of n, so

$$s(n) \geq n + \frac{n}{2} + \frac{n}{3} + \frac{n}{6} + 1 = 2n + 1 > 2n,$$

contradiction.

Remark. It is well-known that all even perfect numbers have the form $n = 2^{p-1}(2^p - 1)$ such that p and $2^p - 1$ are primes. So if e_1 is odd then $k = 2$, $p_1 = 2$, $p_2 = 2^p - 1$, $e_1 = p - 1$ and $e_2 = 1$. If $n > 6$ then $p > 2$ so p is odd and $e_1 = p - 1$ should be even.

Problem 5. Let $A_1A_2 \ldots A_{3n}$ be a closed broken line consisting of $3n$ line segments in the Euclidean plane. Suppose that no three of its vertices are collinear, and for each index $i = 1, 2, \ldots, 3n$, the triangle $A_iA_{i+1}A_{i+2}$ has counterclockwise orientation and $\angle A_iA_{i+1}A_{i+2} = 60°$, using the notation $A_{3n+1} = A_1$ and $A_{3n+2} = A_2$. Prove that the number of self-intersections of the broken line is at most $\frac{3}{2}n^2 - 2n + 1$.

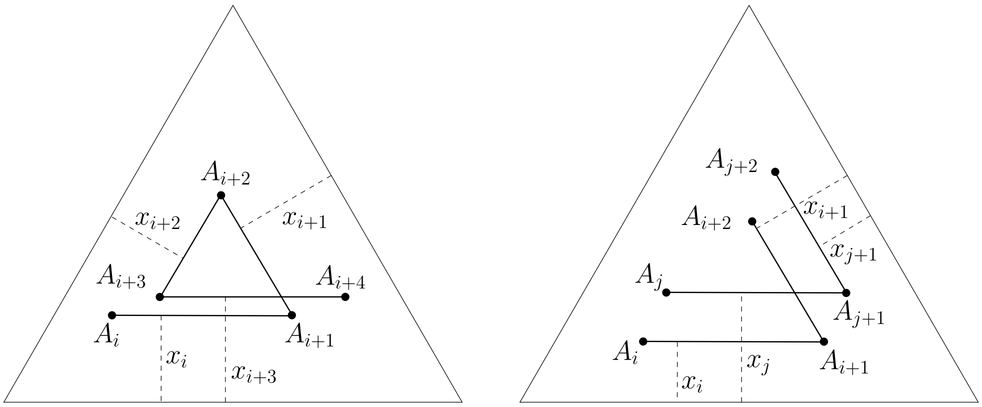


(Proposed by Martin Langer)

Solution. Place the broken line inside an equilateral triangle T such that their sides are parallel to the segments of the broken line. For every $i = 1, 2, \ldots, 3n$, denote by x_i the distance between the segment A_iA_{i+1} and that side of T which is parallel to A_iA_{i+1}. We will use indices modulo $3n$ everywhere.

It is easy to see that if $i \equiv j \pmod 3$ then the polylines $A_iA_{i+1}A_{i+2}$ and $A_jA_{j+1}A_{j+2}$ intersect at most once, and this is possible only if either $x_i < x_{i+1}$ and $x_j > x_{j+1}$ or $x_i < x_{i+1}$ and $x_j > x_{j+1}$. Moreover, such cases cover all self-intersections. So, the number of self-intersections cannot exceed number of pairs (i, j) with the property

$$(*)\ \ i \equiv j \pmod 3, \quad \text{and} \quad (x_i < x_{i+1} \text{ and } x_j > x_{j+1}) \quad \text{or} \quad (x_i > x_{i+1} \text{ and } x_j < x_{j+1}).$$



Grouping the indices $1, 2, \ldots, 3n$, by remainders modulo 3, we have n indices in each residue class. Altogether there are $3\binom{n}{2}$ index pairs (i, j) with $i \equiv j \pmod 3$. We will show that for every integer k with $1 \le k < \frac{n}{2}$, there is some index i such that the pair $(i, i + 6k)$ does not satisfy $(*)$. This is already $\left[\frac{n-1}{2}\right]$ pair; this will prove that there are at most

$$3\binom{n}{2} - \left[\frac{n-1}{2}\right] \ge \frac{3}{2}n^2 - 2n + 1$$

self-intersections.

Without loss of generality we may assume that $x_{3n} = x_0$ is the smallest among $x_1, \ldots, x_{3n}$. Suppose that all of the pairs

$$(-6k, 0), \quad (-6k+1, 1), \quad (-6k+2, 2), \quad \ldots, \quad (-1, 6k-1), \quad (0, 6k) \qquad (**)$$

satisfy $(*)$. Since x_0 is minimal, we have $x_{-6k} > x_0$. The pair $(-6k, 0)$ satisfies $(*)$, so $x_{-6k+1} < x_1$. Then we can see that $x_{-6k+2} > x_2$, and so on; finally we get $x_0 > x_{6k}$. But this contradicts the minimality of x_0. Therefore, there is a pair in $(**)$ that does not satisfy $(*)$.

Remark. The bound $3\binom{n}{2} - \left[\frac{n-1}{2}\right] = \left[\frac{3}{2}n^2 - 2n + 1\right]$ is sharp.

IMC 2014, Blagoevgrad, Bulgaria

Day 2, August 1, 2014

Problem 1. For a positive integer x, denote its n^{th} decimal digit by $d_n(x)$, i.e. $d_n(x) \in \{0, 1, \dots, 9\}$ and $x = \sum_{n=1}^{\infty} d_n(x) 10^{n-1}$. Suppose that for some sequence $(a_n)_{n=1}^{\infty}$, there are only finitely many zeros in the sequence $(d_n(a_n))_{n=1}^{\infty}$. Prove that there are infinitely many positive integers that do not occur in the sequence $(a_n)_{n=1}^{\infty}$.

(Proposed by Alexander Bolbot, State University, Novosibirsk)

Solution 1. By the assumption there is some index n_0 such that $d_n(a_n) \neq 1$ for $n \geq n_0$. We show that

$$a_{n+1}, a_{n+2}, \ldots > 10^n \quad \text{for} \quad n \geq n_0. \tag{1}$$

Notice that in the sum $a_n = \sum_{k=1}^{\infty} d_k(a_n) 10^{k-1}$ we have the term $d_n(a_n) 10^{n-1}$ with $d_n(a_n) \geq 1$. Therefore, $a_n \geq 10^{n-1}$. Then for $m > n$ we have $a_m \geq 10^m > 10^n$. This proves (1).

From (1) we know that only the first n elements, $a_1, a_2, \ldots, a_n$ may lie in the interval $[1, 10^n]$. Hence, at least $10^n - n$ integers in this interval do not occur in the sequence at all. As $\lim(10^n - n) = \infty$, this shows that there are infinitely many numbers that do not appear among $a_1, a_2, \ldots$.

Solution 2. We will use Cantor's diagonal method to construct infinitely many positive integers that do not occur in the sequence (a_n)

Assume that $d_n(a_n) \neq 0$ for $n > n_0$. Define the sequence of digits

$$g_n = \begin{cases} 2 & d_n(x_n) = 1 \\ 1 & d_n(x_n) \neq 1. \end{cases}$$

Hence $g_n \neq d_n(a_n)$ for every positive integer n. Let

$$x_k = \sum_{n=1}^{k} g_n \cdot 10^{n-1} \quad \text{for} \quad k = 1, 2, \ldots.$$

As $x_{k+1} \geq 10^k > x_k$, the sequence (x_k) is increasing and so it contains infinitely many distinct positive integers. We show that the numbers $x_{n_0}, x_{n_0+1}, x_{n_0+2}, \ldots$ no not occur in the sequence (a_n); in other words, $x_k \neq a_n$ for every pair $n \geq 1$ and $k \geq n_0$ of integers.

Indeed, if $k \geq n$ then $d_n(x_k) = g_n \neq d_n(a_n)$, so $x_k \neq a_n$.

If $n > k \geq n_0$ then $d_n(x_k) = 0 \neq d_n(a_n)$, so $x_k \neq a_n$.

Problem 2. Let $A = (a_{ij})_{i,j=1}^{n}$ be a symmetric $n \times n$ matrix with real entries, and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ denote its eigenvalues. Show that

$$\sum_{1 \leq i < j \leq n} a_{ii} a_{jj} \geq \sum_{1 \leq i < j \leq n} \lambda_i \lambda_j,$$

and determine all matrices for which equality holds.

(Proposed by Martin Niepel, Comenius University, Bratislava)

Solution. Eigenvalues of a real symmetric matrix are real, hence the inequality makes sense. Similarly, for Hermitian matrices diagonal entries as well as eigenvalues have to be real.

Since the trace of a matrix is the sum of its eigenvalues, for A we have

$$\sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} \lambda_i,$$

and consequently

$$\sum_{i=1}^{n} a_{ii}^2 + 2\sum_{i<j} a_{ii}a_{jj} = \sum_{i=1}^{n} \lambda_i^2 + 2\sum_{i<j} \lambda_i\lambda_j.$$

Therefore our inequality is equivalent to

$$\sum_{i=1}^{n} a_{ii}^2 \leq \sum_{i=1}^{n} \lambda_i^2.$$

Matrix A^2, which is equal to A^TA (or A^*A in Hermitian case), has eigenvalues $\lambda_1^2, \lambda_2^2, \dots, \lambda_n^2$. On the other hand, the trace of A^TA gives the square of the Frobenius norm of A, so we have

$$\sum_{i=1}^{n} a_{ii}^2 \leq \sum_{i,j=1}^{n} |a_{ij}|^2 = \operatorname{tr}(A^TA) = \operatorname{tr}(A^2) = \sum_{i=1}^{n} \lambda_i^2.$$

The inequality follows, and it is clear that the equality holds for diagonal matrices only.

Remark. Same statement is true for Hermitian matrices.

Problem 3. Let $f(x) = \dfrac{\sin x}{x}$, for $x > 0$, and let n be a positive integer. Prove that $\left|f^{(n)}(x)\right| < \dfrac{1}{n+1}$, where $f^{(n)}$ denotes the n^{th} derivative of f.

(Proposed by Alexander Bolbot, State University, Novosibirsk)

Solution 1. Putting $f(0) = 1$ we can assume that the function is analytic in $\mathbb{R}$. Let $g(x) = x^{n+1}(f^n(x) - \frac{1}{n+1})$. Then $g(0) = 0$ and

$$g'(x) = (n+1)x^n \left(f^{(n)}(x) - \frac{1}{n+1} \right) + x^{n+1} f^{(n+1)}(x) =$$

$$= x^n \left((n+1)f^{(n)}(x) + xf^{(n+1)}(x) - 1 \right) = x^n \left((xf(x))^{(n+1} - 1 \right) = x^n (\sin^{(n+1)}(x) - 1) \leq 0.$$

Hence $g(x) \leq 0$ for $x > 0$. Taking into account that $g'(x) < 0$ for $0 < x < \frac{\pi}{2}$ we obtain the desired (strict) inequality for $x > 0$.

Solution 2.

$$\left(\frac{\sin x}{x}\right)^{(n)} = \frac{\mathrm{d}^n}{\mathrm{d}x^n}\int_0^1 -\cos(xt)\mathrm{d}t = \int_0^1 \frac{\partial^n}{\partial x^n}\left(-\cos(xt)\right)\mathrm{d}t = \int_0^1 t^n g_n(xt)\mathrm{d}t$$

where the function $g_n(u)$ can be $\pm\sin u$ or $\pm\cos u$, depending on n. We only need that $|g_n| \leq 1$ and equality occurs at finitely many points. So,

$$\left|\left(\frac{\sin x}{x}\right)^{(n)}\right| \leq \int_0^1 t^n \big|g_n(xt)\big|\mathrm{d}t < \int_0^1 t^n \mathrm{d}t = \frac{1}{n+1}.$$

Problem 4. We say that a subset of $\mathbb{R}^n$ is k-*almost contained* by a hyperplane if there are less than k points in that set which do not belong to the hyperplane. We call a finite set of points k-*generic* if there is no hyperplane that k-almost contains the set. For each pair of positive integers k and n, find the minimal number $d(k,n)$ such that every finite k-generic set in $\mathbb{R}^n$ contains a k-generic subset with at most $d(k,n)$ elements.
(Proposed by Shachar Carmeli, Weizmann Inst. and Lev Radzivilovsky, Tel Aviv Univ.)

Solution. The answer is: $d(k,n) = \begin{cases} k \cdot n & k,n > 1 \\ k+n & \text{otherwise} \end{cases}$

Throughout the solution, we shall often say that a hyperplanes **skips** a point to signify that the plane does not contain that point.

For $n = 1$ the claim is obvious.

For $k = 1$ we have an arbitrary finite set of points in $\mathbb{R}^n$ such that neither hyperplane contains it entirely. We can build a subset of $n+1$ points step by step: on each step we add a point, not contained in the minimal plane spanned by the previous points. Thus any 1-generic set contains a non-degenerate simplex of $n+1$ points, and obviously a non-degenerate simplex of $n+1$ points cannot be reduced without loosing 1-generality.

In the case $k,n > 1$ we shall give an example of $k \cdot n$ points. On each of the Cartesian axes choose k distinct points, different from the origin. Let's show that this set is k-generic. There are two types of planes: containing the origin and skipping it. If a plane contains the origin, it either contains all the chose points of a axis or skips all of them. Since no plane contains all axes, it skips the k chosen points on one of the axes. If a plane skips the origin, it it contains at most one point of each axis. Therefore it skips at least $n(k-1)$ points. It remains to verify a simple inequality $n(k-1) \geq k$ which is equivalent to $(n-1)(k-1) \geq 1$ which holds for $n,k > 1$.

The example we have shown is minimal by inclusion: if any point is removed, say a point from axis i, then the hyperplane $x_i = 0$ skips only $k-1$ points, and our set stops being k-generic. Hence $d(k,n) \geq kn$.

It remains to prove that Hence $d(k,n) \geq kn$ for $k,n > 1$, meaning: for each k-generic finite set of points, it is possible to choose a k-generic subset of at most kn points. Let us call a subset of points **minimal** if by taking out any point, we loose k-generality. It suffices to prove that any minimal k-generic subset in $\mathbb{R}^n$ has at most kn points. A hyperplane will be called **ample** if it skips precisely k points. A point cannot be removed from a k-generic set, if and only if it is skipped by an ample hyperplane. Thus, in a minimal set each point is skipped by an ample hyperplane.

Organise the following process: on each step we choose an ample hyperplane, and paint blue all the points which are skipped by it. Each time we choose an ample hyperplane, which skips one of the unpainted points. The unpainted points at each step (after the beginning) is the intersection of all chosen hyperplanes. The intersection set of chosen hyperplanes is reduced with each step (since at least one point is being painted on each step).

Notice, that on each step we paint at most k points. So if we start with a minimal set of more then nk points, we can choose n planes and still have at least one unpainted points. The intersection of the chosen planes is a point (since on each step the dimension of the intersection plane was reduced), so there are at most $nk+1$ points in the set. The last unpainted point will be denoted by O. The last unpainted line (which was formed on the step before the last) will be denoted by ℓ_1.

This line is an intersection of all the chosen hyperplanes except the last one. If we have more than nk points, then ℓ_1 contains exactly $k+1$ points from the set, one of which is O.

We could have executed the same process with choosing the same hyperplanes, but in different order. Anyway, at each step we would paint at most k points, and after n steps only O would remain unpainted; so it was precisely k points on each step. On step before the last, we might get a different line, which is intersection of all planes except the last one. The lines obtained in this way will be denoted $\ell_1, \ell_2, ..., \ell_n$, and each contains exactly k points except O. Since we have O and k points on n lines, that is the entire set. Notice that the vectors spanning these lines are linearly independent (since for each line we have a hyperplane containing all the other lines except that line). So by removing O we obtain the example that we've described already, which is k-generic.

Remark. From the proof we see, that the example is unique.

Problem 5. For every positive integer n, denote by D_n the number of permutations $(x_1, \dots, x_n)$ of $(1, 2, \dots, n)$ such that $x_j \neq j$ for every $1 \leq j \leq n$. For $1 \leq k \leq \frac{n}{2}$, denote by $\Delta(n, k)$ the number of permutations $(x_1, \dots, x_n)$ of $(1, 2, \dots, n)$ such that $x_i = k + i$ for every $1 \leq i \leq k$ and $x_j \neq j$ for every $1 \leq j \leq n$. Prove that

$$\Delta(n, k) = \sum_{i=0}^{k-1} \binom{k-1}{i} \frac{D_{(n+1)-(k+i)}}{n-(k+i)}.$$

(Proposed by Combinatorics; Ferdowsi University of Mashhad, Iran; Mirzavaziri)

Solution. Let $a_r \in \{i_1, \dots, i_k\} \cap \{a_1, \dots, a_k\}$. Thus $a_r = i_s$ for some $s \neq r$. Now there are two cases:

Case 1. $a_s \in \{i_1, \dots, i_k\}$. Let $a_s = i_t$. In this case a derangement $x = (x_1, \dots, x_n)$ satisfies the condition $x_{i_j} = a_j$ if and only if the derangement $x' = (x'_1, \dots, x'_{i_t-1}, x'_{i_t+1}, x'_n)$ of the set $[n] \setminus \{i_t\}$ satisfies the condition $x'_{i_j} = a'_j$ for all $j \neq t$, where $a'_j = a_j$ for $j \neq s$ and $a'_s = a_t$. This provides a one to one correspondence between the derangements $x = (x_1, \dots, x_n)$ of $[n]$ with $x_{i_j} = a_j$ for the given sets $\{i_1, \dots, i_k\}$ and $\{a_1, \dots, a_k\}$ with ℓ elements in their intersections, and the derangements $x' = (x'_1, \dots, x'_{i_t-1}, x'_{i_t+1}, x'_n)$ of $[n] \setminus \{i_t\}$ with $x_{i_j} = a'_j$ for the given sets $\{i_1, \dots, i_k\} \setminus \{i_t\}$ and $\{a'_1, \dots, a'_k\} \setminus \{a'_t\}$ with $\ell - 1$ elements in their intersections.

Case 2. $a_s \notin \{i_1, \dots, i_k\}$. In this case a derangement $x = (x_1, \dots, x_n)$ satisfies the condition $x_{i_j} = a_j$ if and only if the derangement $x' = (x'_1, \dots, x'_{a_s-1}, x'_{a_s+1}, x'_n)$ of the set $[n] \setminus \{a_s\}$ satisfies the condition $x'_{i_j} = a_j$ for all $j \neq s$. This provides a one to one correspondence between the derangements $x = (x_1, \ldots, x_n)$ of $[n]$ with $x_{i_j} = a_j$ for the given sets $\{i_1, \ldots, i_k\}$ and $\{a_1, \ldots, a_k\}$ with ℓ elements in their intersections, and the derangements $x' = (x'_1, \ldots, x'_{a_s-1}, x'_{a_s+1}, x'_n)$ of $[n] \setminus \{a_s\}$ with $x_{i_j} = a_j$ for the given sets $\{i_1, \ldots, i_k\} \setminus \{i_s\}$ and $\{a_1, \ldots, a_k\} \setminus \{a_s\}$ with $\ell - 1$ elements in their intersections.

These considerations show that $\Delta(n, k, \ell) = \Delta(n-1, k-1, \ell-1)$. Iterating this argument we have

$$\Delta(n, k, \ell) = \Delta(n - \ell, k - \ell, 0).$$

We can therefore assume that $\ell = 0$. We thus evaluate $\Delta(n, k, 0)$, where $2k \leqslant n$. For $k = 0$, we obviously have $\Delta(n, 0, 0) = D_n$. For $k \geqslant 1$, we claim that

$$\Delta(n, k, 0) = \Delta(n-1, k-1, 0) + \Delta(n-2, k-1, 0).$$

For a derangement $x = (x_1, \ldots, x_n)$ satisfying $x_{i_j} = a_j$ there are two cases: $x_{a_1} = i_1$ or $x_{a_1} \neq i_1$.

If the first case occurs then we have to evaluate the number of derangements of the set $[n] \setminus \{i_1, a_1\}$ for the given sets $\{i_2, \ldots, i_k\}$ and $\{a_2, \ldots, a_k\}$ with 0 elements in their intersections. The number is equal to $\Delta(n-2, k-1, 0)$.

If the second case occurs then we have to evaluate the number of derangements of the set $[n] \setminus \{a_1\}$ for the given sets $\{i_2, \ldots, i_k\}$ and $\{a_2, \ldots, a_k\}$ with 0 elements in their intersections. The number is equal to $\Delta(n-1, k-1, 0)$.

We now use induction on k to show that

$$\Delta(n, k, 0) = \sum_{i=0}^{k-1} \binom{k-1}{i} \frac{D_{(n+1)-(k+i)}}{n-(k+i)}, \quad 2 \leqslant 2k \leqslant n.$$

For $k = 1$ we have

$$\Delta(n, 1, 0) = \Delta(n-1, 0, 0) + \Delta(n-2, 0, 0) = D_{n-1} + D_{n-2} = \frac{D_n}{n-1}.$$

Now let the result be true for $k-1$. We can write

$$\begin{aligned}
\Delta(n,k,0) &= \Delta(n-1,k-1,0)+\Delta(n-2,k-1,0)\\
&= \sum_{i=0}^{k-2}\binom{k-2}{i}\frac{D_{n-(k-1+i)}}{(n-1)-(k-1+i)}+\sum_{i=0}^{k-2}\binom{k-2}{i}\frac{D_{(n-1)-(k-1+i)}}{(n-2)-(k-1+i)}\\
&= \sum_{i=0}^{k-2}\binom{k-2}{i}\frac{D_{(n+1)-(k+i)}}{n-(k+i)}+\sum_{i=1}^{k-1}\binom{k-2}{i-1}\frac{D_{n-(k+i-1)}}{(n-1)-(k+i-1)}\\
&= \frac{D_{(n+1)-k}}{n-k}+\sum_{i=1}^{k-2}\binom{k-2}{i}\frac{D_{(n+1)-(k+i)}}{n-(k+i)}\\
&\quad +\frac{D_{(n+1)-(2k-1)}}{n-(2k-1)}+\sum_{i=1}^{k-2}\binom{k-2}{i-1}\frac{D_{(n+1)-(k+i)}}{n-(k+i)}\\
&= \frac{D_{(n+1)-k}}{n-k}+\sum_{i=1}^{k-2}\left[\binom{k-2}{i}+\binom{k-2}{i-1}\right]\frac{D_{(n+1)-(k+i)}}{n-(k+i)}+\frac{D_{(n+1)-(2k-1)}}{n-(2k-1)}\\
&= \frac{D_{(n+1)-k}}{n-k}+\sum_{i=1}^{k-2}\binom{k-1}{i}\frac{D_{(n+1)-(k+i)}}{n-(k+i)}+\frac{D_{(n+1)-(2k-1)}}{n-(2k-1)}\\
&= \sum_{i=0}^{k-1}\binom{k-1}{i}\frac{D_{(n+1)-(k+i)}}{n-(k+i)}.
\end{aligned}$$

Remark. As a corollary of the above problem, we can solve the first problem. Let $n=2k$, $i_j=j$ and $a_j=k+j$ for $j=1,\ldots,k$. Then a derangement $x=(x_1,\ldots,x_n)$ satisfies the condition $x_{i_j}=a_j$ if and only if $x'=(x_{k+1},\ldots,x_n)$ is a permutation of $[k]$. The number of such permutations x' is $k!$. Thus $\sum_{i=0}^{k-1}\binom{k-1}{i}\frac{D_{k+1-i}}{k-i}=k!$.

IMC 2015, Blagoevgrad, Bulgaria

Day 1, July 29, 2015

Problem 1. For any integer $n \geq 2$ and two $n \times n$ matrices with real entries A, B that satisfy the equation

$$A^{-1} + B^{-1} = (A+B)^{-1}$$

prove that $\det(A) = \det(B)$.

Does the same conclusion follow for matrices with complex entries?

(Proposed by Zbigniew Skoczylas, Wrocław University of Technology)

Solution. Multiplying the equation by $(A+B)$ we get

$$I = (A+B)(A+B)^{-1} = (A+B)(A^{-1}+B^{-1}) =$$
$$= AA^{-1} + AB^{-1} + BA^{-1} + BB^{-1} = I + AB^{-1} + BA^{-1} + I$$
$$AB^{-1} + BA^{-1} + I = 0.$$

Let $X = AB^{-1}$; then $A = XB$ and $BA^{-1} = X^{-1}$, so we have $X + X^{-1} + I = 0$; multiplying by $(X-I)X$,

$$0 = (X-I)X \cdot (X + X^{-1} + I) = (X-I) \cdot (X^2 + X + I) = X^3 - I.$$

Hence,

$$X^3 = I$$
$$(\det X)^3 = \det(X^3) = \det I = 1$$
$$\det X = 1$$
$$\det A = \det(XB) = \det X \cdot \det B = \det B.$$

In case of complex matrices the statement is false. Let $\omega = \frac{1}{2}(-1 + i\sqrt{3})$. Obviously $\omega \notin \mathbb{R}$ and $\omega^3 = 1$, so $0 = 1 + \omega + \omega^2 = 1 + \omega + \overline{\omega}$.

Let $A = I$ and let B be a diagonal matrix with all entries along the diagonal equal to either ω or $\overline{\omega} = \omega^2$ such a way that $\det(B) \neq 1$ (if n is not divisible by 3 then one may set $B = \omega I$). Then $A^{-1} = I$, $B^{-1} = \overline{B}$. Obviously $I + B + \overline{B} = 0$ and

$$(A+B)^{-1} = (-\overline{B})^{-1} = -B = I + \overline{B} = A^{-1} + B^{-1}.$$

By the choice of A and B, $\det A = 1 \neq \det B$.

Problem 2. For a positive integer n, let $f(n)$ be the number obtained by writing n in binary and replacing every 0 with 1 and vice versa. For example, $n = 23$ is 10111 in binary, so $f(n)$ is 1000 in binary, therefore $f(23) = 8$. Prove that

$$\sum_{k=1}^{n} f(k) \le \frac{n^2}{4}.$$

When does equality hold?

(Proposed by Stephan Wagner, Stellenbosch University)

Solution. If r and k are positive integers with $2^{r-1} \le k < 2^r$ then k has r binary digits, so $k + f(k) = \overline{\underbrace{11\dots1}_{r}}^{(2)} = 2^r - 1$.

Assume that $2^{s-1} - 1 \le n \le 2^s - 1$. Then

$$\frac{n(n+1)}{2} + \sum_{k=1}^{n} f(k) = \sum_{k=1}^{n} (k + f(k)) =$$

$$= \sum_{r=1}^{s-1} \sum_{2^{r-1} \le k < 2^r} (k + f(k)) + \sum_{2^{s-1} \le k \le n} (k + f(k)) =$$

$$= \sum_{r=1}^{s-1} 2^{r-1} \cdot (2^r - 1) + (n - 2^{s-1} + 1) \cdot (2^s - 1) =$$

$$= \sum_{r=1}^{s-1} 2^{2r-1} - \sum_{r=1}^{s-1} 2^{r-1} + (n - 2^{s-1} + 1)(2^s - 1) =$$

$$= \tfrac{2}{3}(4^{s-1} - 1) - (2^{s-1} - 1) + (2^s - 1)n - 2^{2s-1} + 3 \cdot 2^{s-1} - 1 =$$

$$= (2^s - 1)n - \tfrac{1}{3}4^s + 2^s - \tfrac{2}{3}$$

and therefore

$$\frac{n^2}{4} - \sum_{k=1}^{n} f(k) = \frac{n^2}{4} - \left((2^s - 1)n - \tfrac{1}{3}4^s + 2^s - \tfrac{2}{3} - \frac{n(n+1)}{2} \right) =$$

$$= \tfrac{3}{4}n^2 - (2^s - \tfrac{3}{2})n + \tfrac{1}{3}4^s - 2^s + \tfrac{2}{3} =$$

$$= \frac{3}{4} \left(n - \frac{2^{s+1} - 2}{3} \right) \left(n - \frac{2^{s+1} - 4}{3} \right).$$

Notice that the difference of the last two factors is less than 1, and one of them must be an integer: $\frac{2^{s+1}-2}{3}$ is integer if s is even, and $\frac{2^{s+1}-4}{3}$ is integer if s is odd. Therefore, either one of them is 0, resulting a zero product, or both factors have the same sign, so the product is strictly positive. This solves the problem and shows that equality occurs if $n = \dfrac{2^{s+1} - 2}{3}$ (s is even) or $n = \dfrac{2^{s+1} - 4}{3}$ (s is odd).

Problem 3. Let $F(0) = 0$, $F(1) = \frac{3}{2}$, and $F(n) = \frac{5}{2}F(n-1) - F(n-2)$ for $n \geq 2$.

Determine whether or not $\sum_{n=0}^{\infty} \frac{1}{F(2^n)}$ is a rational number.

(Proposed by Gerhard Woeginger, Eindhoven University of Technology)

Solution 1. The characteristic equation of our linear recurrence is $x^2 - \frac{5}{2}x + 1 = 0$, with roots $x_1 = 2$ and $x_2 = \frac{1}{2}$. So $F(n) = a \cdot 2^n + b \cdot (\frac{1}{2})^n$ with some constants a, b. By $F(0) = 0$ and $F(1) = \frac{3}{2}$, these constants satisfy $a + b = 0$ and $2a + \frac{b}{2} = \frac{3}{2}$. So $a = 1$ and $b = -1$, and therefore

$$F(n) = 2^n - 2^{-n}.$$

Observe that

$$\frac{1}{F(2^n)} = \frac{2^{2^n}}{(2^{2^n})^2 - 1} = \frac{1}{2^{2^n} - 1} - \frac{1}{(2^{2^n})^2 - 1} = \frac{1}{2^{2^n} - 1} - \frac{1}{2^{2^{n+1}} - 1},$$

so

$$\sum_{n=0}^{\infty} \frac{1}{F(2^n)} = \sum_{n=0}^{\infty} \left(\frac{1}{2^{2^n} - 1} - \frac{1}{2^{2^{n+1}} - 1} \right) = \frac{1}{2^{2^0} - 1} = 1.$$

Hence the sum takes the value 1, which is rational.

Solution 2. As in the first solution we find that $F(n) = 2^n - 2^{-n}$. Then

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{1}{F(2^n)} &= \sum_{n=0}^{\infty} \frac{1}{2^{2^n} - 2^{-2^n}} = \sum_{n=0}^{\infty} \frac{(\frac{1}{2})^{2^n}}{1 - (\frac{1}{2})^{2^{n+1}}} \\
&= \sum_{n=0}^{\infty} (\tfrac{1}{2})^{2^n} \sum_{k=0}^{\infty} \left((\tfrac{1}{2})^{2^{n+1}} \right)^k = \sum_{n=0}^{\infty} (\tfrac{1}{2})^{2^n} \sum_{k=0}^{\infty} (\tfrac{1}{2})^{2k \cdot 2^n} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} (\tfrac{1}{2})^{2^n(2k+1)} = \sum_{m=1}^{\infty} (\tfrac{1}{2})^m = 1.
\end{aligned}$$

(Here we used the fact that every positive integer m has a unique representation $m = 2^n(2k+1)$ with non-negative integers n and k.)

This shows that the series converges to 1.

Problem 4. Determine whether or not there exist 15 integers $m_1, \ldots, m_{15}$ such that

$$\sum_{k=1}^{15} m_k \cdot \arctan(k) = \arctan(16). \qquad (1)$$

(Proposed by Gerhard Woeginger, Eindhoven University of Technology)

Solution. We show that such integers $m_1, \ldots, m_{15}$ do not exist.

Suppose that (1) is satisfied by some integers $m_1, \ldots, m_{15}$. Then the argument of the complex number $z_1 = 1 + 16i$ coincides with the argument of the complex number

$$z_2 = (1+i)^{m_1} (1+2i)^{m_2} (1+3i)^{m_3} \cdots\cdots (1+15i)^{m_{15}}.$$

Therefore the ratio $R = z_2/z_1$ is real (and not zero). As $\operatorname{Re} z_1 = 1$ and $\operatorname{Re} z_2$ is an integer, R is a nonzero integer.

By considering the squares of the absolute values of z_1 and z_2, we get

$$(1+16^2)R^2 = \prod_{k=1}^{15}(1+k^2)^{m_k}.$$

Notice that $p = 1+16^2 = 257$ is a prime (the fourth Fermat prime), which yields an easy contradiction through p-adic valuations: all prime factors in the right hand side are strictly below p (as $k < 16$ implies $1+k^2 < p$). On the other hand, in the left hand side the prime p occurs with an odd exponent.

Problem 5. Let $n \ge 2$, let $A_1, A_2, \dots, A_{n+1}$ be $n+1$ points in the n-dimensional Euclidean space, not lying on the same hyperplane, and let B be a point strictly inside the convex hull of $A_1, A_2, \dots, A_{n+1}$. Prove that $\angle A_iBA_j > 90°$ holds for at least n pairs (i,j) with $1 \le i < j \le n+1$.

(Proposed by Géza Kós, Eötvös University, Budapest)

Solution. Let $\mathbf{v}_i = \overrightarrow{BA_i}$. The condition $\angle A_iBA_j > 90°$ is equivalent with $\mathbf{v}_i \cdot \mathbf{v}_j < 0$. Since B is an interior point of the simplex, there are some weights $w_1, \dots, w_{n+1} > 0$ with $\sum_{i=1}^{n+1} w_i\mathbf{v}_i = \mathbf{0}$.

Let us build a graph on the vertices $1, \dots, n+1$. Let the vertices i and j be connected by an edge if $\mathbf{v}_i \cdot \mathbf{v}_j < 0$. We show that this graph is connected. Since every connected graph on $n+1$ vertices has at least n edges, this will prove the problem statement.

Suppose the contrary that the graph is not connected; then the vertices can be split in two disjoint nonempty sets, say V and W such that $V \cup W = \{1, 2, \dots, n+1\}$. Since there is no edge between the two vertex sets, we have $\mathbf{v}_i \cdot \mathbf{v}_j \ge 0$ for all $i \in V$ and $j \in W$.

Consider

$$0 = \left(\sum_{i\in V\cup W} w_i\mathbf{v}_i\right)^2 = \left(\sum_{i\in V} w_i\mathbf{v}_i\right)^2 + \left(\sum_{i\in W} w_i\mathbf{v}_i\right)^2 + 2\sum_{i\in V}\sum_{i\in W} w_iw_j(\mathbf{v}_i \cdot \mathbf{v}_j).$$

Notice that all terms are nonnegative on the right-hand side. Moreover, $\sum_{i\in V} w_i\mathbf{v}_i \ne \mathbf{0}$ and $\sum_{i\in W} w_i\mathbf{v}_i \ne \mathbf{0}$, so there are at least two strictly nonzero terms, contradiction.

Remark 1. The number n in the statement is sharp; if $\mathbf{v}_{n+1} = (1, 1, \dots, 1)$ and $v_i = (\underbrace{0, \dots, 0}_{i-1}, -1, \underbrace{0, \dots, 0}_{n-i})$ for $i = 1, \dots, n$ then $\mathbf{v}_i \cdot \mathbf{v}_j < 0$ holds only when $i = n+1$ or $j = n+1$.

Remark 2. The origin of the problem is here: `http://math.stackexchange.com/questions/476640/n-simplex-in-an-intersection-of-n-balls/789390`

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Problem 6. Prove that

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\,(n+1)} < 2.$$

(Proposed by Ivan Krijan, University of Zagreb)

Solution. We prove that

$$\frac{1}{\sqrt{n}\,(n+1)} < \frac{2}{\sqrt{n}} - \frac{2}{\sqrt{n+1}}. \tag{1}$$

Multiplying by $\sqrt{n}(n+1)$, the inequality (1) is equivalent with

$$1 < 2(n+1) - 2\sqrt{n(n+1)}$$

$$2\sqrt{n(n+1)} < n + (n+1)$$

which is true by the AM-GM inequality.

Applying (1) to the terms in the left-hand side,

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\,(n+1)} < \sum_{n=1}^{\infty} \left(\frac{2}{\sqrt{n}} - \frac{2}{\sqrt{n+1}} \right) = 2.$$

Problem 7. Compute

$$\lim_{A\to+\infty} \frac{1}{A} \int_1^A A^{\frac{1}{x}} \mathrm{d}x\,.$$

(Proposed by Jan Šustek, University of Ostrava)

Solution 1. We prove that

$$\lim_{A\to+\infty} \frac{1}{A} \int_1^A A^{\frac{1}{x}} \mathrm{d}x = 1\,.$$

For $A > 1$ the integrand is greater than 1, so

$$\frac{1}{A}\int_1^A A^{\frac{1}{x}} \mathrm{d}x > \frac{1}{A}\int_1^A 1 \mathrm{d}x = \frac{1}{A}(A-1) = 1 - \frac{1}{A}.$$

In order to find a tight upper bound, fix two real numbers, $\delta > 0$ and $K > 0$, and split the interval into three parts at the points $1+\delta$ and $K\log A$. Notice that for sufficiently large A (i.e., for $A > A_0(\delta, K)$ with some $A_0(\delta, K) > 1$) we have $1 + \delta < K \log A < A$.) For $A > 1$ the integrand is decreasing, so we can estimate it by its value at the starting points of the intervals:

$$\frac{1}{A}\int_1^A A^{\frac{1}{x}} \mathrm{d}x = \frac{1}{A}\left(\int_1^{1+\delta} + \int_{1+\delta}^{K\log A} + \int_{K\log A}^{A} \right) <$$

$$= \frac{1}{A}\left(\delta \cdot A + (K\log A - 1 - \delta)A^{\frac{1}{1+\delta}} + (A - K\log A)A^{\frac{1}{K\log A}} \right) <$$

$$< \frac{1}{A}\left(\delta A + K A^{\frac{1}{1+\delta}} \log A + A \cdot A^{\frac{1}{K\log A}} \right) = \delta + K A^{-\frac{\delta}{1+\delta}} \log A + e^{\frac{1}{K}}.$$

Hence, for $A > A_0(\delta, K)$ we have

$$1 - \frac{1}{A} < \frac{1}{A}\int_1^A A^{\frac{1}{x}} \mathrm{d}x < \delta + K A^{-\frac{\delta}{1+\delta}} \log A + e^{\frac{1}{K}}.$$

Taking the limit $A \to \infty$ we obtain

$$1 \le \liminf_{A\to\infty} \frac{1}{A} \int_1^A A^{\frac{1}{x}} \mathrm{d}x \le \limsup_{A\to\infty} \frac{1}{A} \int_1^A A^{\frac{1}{x}} \mathrm{d}x \le \delta + e^{\frac{1}{K}}.$$

Now from $\delta \to +0$ and $K \to \infty$ we get

$$1 \le \liminf_{A\to\infty} \frac{1}{A} \int_1^A A^{\frac{1}{x}} \mathrm{d}x \le \limsup_{A\to\infty} \frac{1}{A} \int_1^A A^{\frac{1}{x}} \mathrm{d}x \le 1,$$

so $\liminf\limits_{A\to\infty} \frac{1}{A} \int_1^A A^{\frac{1}{x}} \mathrm{d}x = \limsup\limits_{A\to\infty} \frac{1}{A} \int_1^A A^{\frac{1}{x}} \mathrm{d}x = 1$ and therefore

$$\lim_{A\to+\infty} \frac{1}{A} \int_1^A A^{\frac{1}{x}} \mathrm{d}x = 1\,.$$

Solution 2. We will employ l'Hospital's rule.

Let $f(A,x) = A^{\frac{1}{x}}$, $g(A,x) = \frac{1}{x} A^{\frac{1}{x}}$, $F(A) = \int_1^A f(A,x)\mathrm{d}x$ and $G(A) = \int_1^A g(A,x)\mathrm{d}x$. Since $\frac{\partial}{\partial A} f$ and $\frac{\partial}{\partial A} g$ are continuous, the parametric integrals $F(A)$ and $G(A)$ are differentiable with respect to A, and

$$F'(A) = f(A,A) + \int_1^A \frac{\partial}{\partial A} f(A,x)\mathrm{d}x = A^{\frac{1}{A}} + \int_1^A \frac{1}{x} A^{\frac{1}{x}-1} \mathrm{d}x = A^{\frac{1}{A}} + \frac{1}{A} G(A),$$

and

$$G'(A) = g(A,A) + \int_1^A \frac{\partial}{\partial A} g(A,x)\mathrm{d}x = \frac{A^{\frac{1}{A}}}{A} + \int_1^A \frac{1}{x^2} A^{\frac{1}{x}-1} \mathrm{d}x =$$

$$= A^{\frac{1}{A}} + \left[\frac{-1}{\log A} A^{\frac{1}{x}-1} \right]_1^A = \frac{A^{\frac{1}{A}}}{A} - \frac{A^{\frac{1}{A}}}{A\log A} + \frac{1}{\log A}.$$

Since $\lim_{A\to\infty} A^{\frac{1}{A}} = 1$, we can see that $\lim_{A\to\infty} G'(A) = 0$. Aplying l'Hospital's rule to $\lim\limits_{A\to\infty} \frac{G(A)}{A}$ we get

$$\lim_{A\to\infty} \frac{G(A)}{A} = \lim_{A\to\infty} \frac{G'(A)}{1} = 0,$$

so

$$\lim_{A\to\infty} F'(A) = \lim_{A\to\infty} \left(A^{\frac{1}{A}} + \frac{G(A)}{A} \right) = 1 + 0 = 1.$$

Now applying l'Hospital's rule to $\lim\limits_{A\to\infty} \frac{F(A)}{A}$ we get

$$\lim_{A\to+\infty} \frac{1}{A} \int_1^A A^{\frac{1}{x}} \mathrm{d}x = \lim_{A\to\infty} \frac{F(A)}{A} = \lim_{A\to\infty} \frac{F'(A)}{1} = 1.$$

Problem 8. Consider all 26^{26} words of length 26 in the Latin alphabet. Define the *weight* of a word as $1/(k+1)$, where k is the number of letters not used in this word. Prove that the sum of the weights of all words is 3^{75}.

(Proposed by Fedor Petrov, St. Petersburg State University)

Solution. Let $n = 26$, then $3^{75} = (n+1)^{n-1}$. We use the following well-known

Lemma. If $f(x)$ is a polynomial of degree at most n, then its $(n+1)$-st finite difference vanishes: $\Delta^{n+1} f(x) := \sum_{i=0}^{n+1} (-1)^i \binom{n+1}{i} f(x+i) \equiv 0$.

Proof. If Δ is the operator which maps $f(x)$ to $f(x+1) - f(x)$, then Δ^{n+1} is indeed $(n+1)$-st power of Δ and the claim follows from the observation that Δ decreases the power of a polynomial.

In other words, $f(x) = \sum_{i=1}^{n+1} (-1)^{i+1} \binom{n+1}{i} f(x+i)$. Applying this for $f(x) = (n-x)^n$, substituting $x = -1$ and denoting $i = j+1$ we get

$$(n+1)^n = \sum_{j=0}^{n} (-1)^j \binom{n+1}{j+1} (n-j)^n = (n+1) \sum_{j=0}^{n} \binom{n}{j} \cdot \frac{(-1)^j}{j+1} \cdot (n-j)^n.$$

The j-th summand $\binom{n}{j} \cdot \frac{(-1)^j}{j+1} \cdot (n-j)^n$ may be interpreted as follows: choose j letters, consider all $(n-j)^n$ words without those letters and sum up $\frac{(-1)^j}{j+1}$ over all those words. Now we change the order of summation, counting at first by words. For any fixed word W with k absent letters we get $\sum_{j=0}^{k} \binom{k}{j} \cdot \frac{(-1)^j}{j+1} = \frac{1}{k+1} \cdot \sum_{j=0}^{k} (-1)^j \cdot \binom{k+1}{j+1} = \frac{1}{k+1}$, since the alternating sum of binomial coefficients $\sum_{j=-1}^{k} (-1)^j \cdot \binom{k+1}{j+1}$ vanishes. That is, after changing order of summation we get exactly initial sum, and it equals $(n+1)^{n-1}$.

Problem 9. An $n \times n$ complex matrix A is called *t-normal* if $AA^t = A^tA$ where A^t is the transpose of A. For each n, determine the maximum dimension of a linear space of complex $n \times n$ matrices consisting of t-normal matrices.

(Proposed by Shachar Carmeli, Weizmann Institute of Science)

Solution.

Answer: The maximum dimension of such a space is $\frac{n(n+1)}{2}$.

The number $\frac{n(n+1)}{2}$ can be achieved, for example the symmetric matrices are obviously t-normal and they form a linear space with dimension $\frac{n(n+1)}{2}$. We shall show that this is the maximal possible dimension.

Let M_n denote the space of $n \times n$ complex matrices, let $S_n \subset M_n$ be the subspace of all symmetric matrices and let $A_n \subset M_n$ be the subspace of all anti-symmetric matrices, i.e. matrices A for which $A^t = -A$.

Let $V \subset M_n$ be a linear subspace consisting of t-normal matrices. We have to show that $\dim(V) \leq \dim(S_n)$. Let $\pi : V \to S_n$ denote the linear map $\pi(A) = A + A^t$. We have

$$\dim(V) = \dim(\mathrm{Ker}\,(\pi)) + \dim(\mathrm{Im}\,(\pi))$$

so we have to prove that $\dim(\mathrm{Ker}\,(\pi)) + \dim(\mathrm{Im}\,(\pi)) \leq \dim(S_n)$. Notice that $\mathrm{Ker}\,(\pi) \subseteq A_n$.

We claim that for every $A \in \mathrm{Ker}\,(\pi)$ and $B \in V$, $A\pi(B) = \pi(B)A$. In other words, $\mathrm{Ker}\,(\pi)$ and $\mathrm{Im}\,(\pi)$ commute. Indeed, if $A, B \in V$ and $A = -A^t$ then

$$\begin{gathered}(A+B)(A+B)^t = (A+B)^t(A+B) \Leftrightarrow \\ \Leftrightarrow AA^t + AB^t + BA^t + BB^t = A^tA + A^tB + B^tA + B^tB \Leftrightarrow \\ \Leftrightarrow AB^t - BA = -AB + B^tA \Leftrightarrow A(B+B^t) = (B+B^t)A \Leftrightarrow \\ \Leftrightarrow A\pi(B) = \pi(B)A.\end{gathered}$$

Our bound on the dimension on V follows from the following lemma:

Lemma. Let $X \subseteq S_n$ and $Y \subseteq A_n$ be linear subspaces such that every element of X commutes with every element of Y. Then

$$\dim(X) + \dim(Y) \leq \dim(S_n)$$

Proof. Without loss of generality we may assume $X = Z_{S_n}(Y) := \{x \in S_n : xy = yx \quad \forall y \in Y\}$. Define the bilinear map $B : S_n \times A_n \to \mathbb{C}$ by $B(x,y) = \mathrm{tr(d[x,y])}$ where $[x,y] = xy - yx$ and $d = diag(1,...,n)$ is the matrix with diagonal elements $1,...,n$ and zeros off the diagonal. Clearly $B(X,Y) = \{0\}$. Furthermore, if $y \in Y$ satisfies that $B(x,y) = 0$ for all $x \in S_n$ then $\mathrm{tr(d[x,y])} = -\mathrm{tr([d,x],y])} = 0$ for every $x \in S_n$.

We claim that $\{[d,x] : x \in S_n\} = A_n$. Let E_i^j denote the matrix with 1 in the entry (i,j) and 0 in all other entries. Then a direct computation shows that $[d, E_i^j] = (j-i)E_i^j$ and therefore $[d, E_i^j + E_j^i] = (j-i)(E_i^j - E_j^i)$ and the collection $\{(j-i)(E_i^j - E_j^i)\}_{1 \leq i < j \leq n}$ span A_n for $i \neq j$.

It follows that if $B(x,y) = 0$ for all $x \in S_n$ then $\mathrm{tr(yz)} = 0$ for every $z \in A_n$. But then, taking $z = \bar{y}$, where $\bar{y}$ is the entry-wise complex conjugate of y, we get $0 = \mathrm{tr(y\bar{y})} = -\mathrm{tr(y\bar{y}^t)}$ which is the sum of squares of all the entries of y. This means that $y = 0$.

It follows that if $y_1, ..., y_k \in Y$ are linearly independent then the equations

$$B(x, y_1) = 0, \quad \ldots, \quad B(x, y_k) = 0$$

are linearly independent as linear equations in x, otherwise there are $a_1, ..., a_k$ such that $B(x, a_1y_1 + ... + a_ky_k) = 0$ for every $x \in S_n$, a contradiction to the observation above. Since the solution of k linearly independent linear equations is of codimension k,

$$\dim(\{x \in S_n : [x, y_i] = 0, \text{ for } i = 1, .., k\}) \leq$$

$$\leq \dim(x \in S_n : B(x, y_i) = 0 \text{ for } i = 1, ..., k) = \dim(S_n) - k.$$

The lemma follows by taking $y_1, ..., y_k$ to be a basis of Y.

Since $\mathrm{Ker}\,(\pi)$ and $\mathrm{Im}\,(\pi)$ commute, by the lemma we deduce that

$$\dim(V) = \dim(\mathrm{Ker}\,(\pi)) + \dim(\mathrm{Im}\,(\pi)) \leq \dim(S_n) = \frac{n(n+1)}{2}.$$

Problem 10. Let n be a positive integer, and let $p(x)$ be a polynomial of degree n with integer coefficients. Prove that

$$\max_{0\leq x\leq 1} \big|p(x)\big| > \frac{1}{e^n}.$$

(Proposed by Géza Kós, Eötvös University, Budapest)

Solution. Let

$$M = \max_{0\leq x\leq 1} \big|p(x)\big|.$$

For every positive integer k, let

$$J_k = \int_0^1 \big(p(x)\big)^{2k} \mathrm{d}x.$$

Obviously $0 < J_k < M^{2k}$ is a rational number. If $(p(x))^{2k} = \sum_{i=0}^{2kn} a_{k,i}x^i$ then $J_k = \sum_{i=0}^{2kn} \frac{a_{k,i}}{i+1}$. Taking the least common denominator, we can see that $J_k \geq \dfrac{1}{\mathrm{lcm}(1, 2, \ldots, 2kn+1)}$.

An equivalent form of the prime number theorem is that $\log \mathrm{lcm}(1, 2, \ldots, N) \sim N$ if $N \to \infty$. Therefore, for every $\varepsilon > 0$ and sufficiently large k we have

$$\mathrm{lcm}(1, 2, \ldots, 2kn+1) < e^{(1+\varepsilon)(2kn+1)}$$

and therefore

$$M^{2k} > J_k \geq \frac{1}{\mathrm{lcm}(1, 2, \ldots, 2kn+1)} > \frac{1}{e^{(1+\varepsilon)(2kn+1)}},$$

$$M > \frac{1}{e^{(1+\varepsilon)(n+\frac{1}{2k})}}.$$

Taking $k \to \infty$ and then $\varepsilon \to +0$ we get

$$M \geq \frac{1}{e^n}.$$

Since e is transcendent, equality is impossible.

Remark. The constant $\frac{1}{e} \approx 0.3679$ is not sharp. It is known that the best constant is between 0.4213 and 0.4232. (See I. E. Pritsker, The Gelfond–Schnirelman method in prime number theory, Canad. J. Math. 57 (2005), 1080–1101.)

IMC 2016, Blagoevgrad, Bulgaria

Day 1, July 27, 2016

Problem 1. Let $f\colon [a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Suppose that f has infinitely many zeros, but there is no $x\in(a,b)$ with $f(x)=f'(x)=0$.

(a) Prove that $f(a)f(b)=0$.

(b) Give an example of such a function on $[0,1]$.

(Proposed by Alexandr Bolbot, Novosibirsk State University)

Solution. (a) Choose a convergent sequence z_n of zeros and let $c=\lim z_n\in[a,b]$. By the continuity of f we obtain $f(c)=0$. We want to show that either $c=a$ or $c=b$, so $f(a)=0$ or $f(b)=0$; then the statement follows.

If c was an interior point then we would have $f(c)=0$ and $f'(c)=\lim\dfrac{f(z_n)-f(c)}{z_n-c}=\lim\dfrac{0-0}{z_n-c}=0$ simultaneously, contradicting the conditions. Hence, $c=a$ or $c=b$.

(b) Let

$$f(x)=\begin{cases} x\sin\dfrac{1}{x} & \text{if } 0<x\le 1\\ 0 & \text{if } x=0.\end{cases}$$

This function has zeros at the points $\dfrac{1}{k\pi}$ for $k=1,2,\ldots$, and it is continuous at 0 as well.

In $(0,1)$ we have

$$f'(x)=\sin\frac{1}{x}-\frac{1}{x}\cos\frac{1}{x}.$$

Since $\sin\frac{1}{x}$ and $\cos\frac{1}{x}$ cannot vanish at the same point, we have either $f(x)\ne0$ or $f'(x)\ne0$ everywhere in $(0,1)$.

Problem 2. Let k and n be positive integers. A sequence $(A_1,\ldots,A_k)$ of $n\times n$ real matrices is *preferred* by Ivan the Confessor if $A_i^2\ne0$ for $1\le i\le k$, but $A_iA_j=0$ for $1\le i,j\le k$ with $i\ne j$. Show that $k\le n$ in all preferred sequences, and give an example of a preferred sequence with $k=n$ for each n.

(Proposed by Fedor Petrov, St. Petersburg State University)

Solution 1. For every $i=1,\ldots,n$, since $A_i\cdot A_i\ne0$, there is a column $v_i\in\mathbb{R}^n$ in A_i such that $A_iv_i\ne0$. We will show that the vectors $v_1,\ldots,v_k$ are linearly independent; this immediately proves $k\le n$.

Suppose that a linear combination of $v_1,\ldots,v_k$ vanishes:

$$c_1v_1+\ldots+c_kv_k=0,\quad c_1,\ldots,c_k\in\mathbb{R}.$$

For $i\ne j$ we have $A_iA_j=0$; in particular, $A_iv_j=0$. Now, for each $i=1,\ldots,n$, from

$$0=A_i(c_1v_1+\ldots+c_kv_k)=\sum_{j=1}^{k}c_j(A_iv_j)=c_i(A_iv_i)$$

we can see that $c_i=0$. Hence, $c_1=\ldots=c_k=0$.

The case $k=n$ is possible: if A_i has a single 1 in the main diagonal at the ith position and its other entries are zero then $A_i^2=A_i$ and $A_iA_j=0$ for $i\ne j$.

Remark. The solution above can be re-formulated using block matrices in the following way. Consider

$$\begin{pmatrix} A_1 & A_2 & \dots & A_k \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_k \end{pmatrix} = \begin{pmatrix} A_1^2 & 0 & \dots & 0 \\ 0 & A_2^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_k^2 \end{pmatrix}.$$

It is easy to see that the rank of the left-hand side is at most n; the rank of the right-hand side is at least k.

Solution 2. Let U_i and K_i be the image and the kernel of the matrix A_i (considered as a linear operator on $\mathbb{R}^n$), respectively. For every pair i, j of indices, we have $U_j \subset K_i$ if and only if $i \neq j$.

Let $X_0 = \mathbb{R}^n$ and let $X_i = K_1 \cap K_2 \cap \dots \cap K_i$ for $i = 1, \dots, k$, so $X_0 \supset X_1 \supset \dots \supset X_k$. Notice also that $U_i \subset X_{i-1}$ because $U_i \subset K_j$ for every $j < i$, and $U_i \not\subset X_i$ because $U_i \not\subset K_i$. Hence, $X_i \neq X_{i-1}$; X_i is a proper subspace of X_{i-1}.

Now, from

$$n = \dim X_0 > \dim X_1 > \dots > \dim X_k \geq 0$$

we get $k \geq n$.

Problem 3. Let n be a positive integer. Also let $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ be real numbers such that $a_i + b_i > 0$ for $i = 1, 2, \dots, n$. Prove that

$$\sum_{i=1}^{n} \frac{a_i b_i - b_i^2}{a_i + b_i} \leq \frac{\sum\limits_{i=1}^{n} a_i \cdot \sum\limits_{i=1}^{n} b_i - \left(\sum\limits_{i=1}^{n} b_i\right)^2}{\sum\limits_{i=1}^{n} (a_i + b_i)}.$$

(Proposed by Daniel Strzelecki, Nicolaus Copernicus University in TorÃžn, Poland)

Solution. By applying the identity

$$\frac{XY - Y^2}{X + Y} = Y - \frac{2Y^2}{X + Y}$$

with $X = a_i$ and $Y = b_i$ to the terms in the LHS and $X = \sum\limits_{i=1}^{n} a_i$ and $Y = \sum\limits_{i=1}^{n} b_i$ to the RHS,

$$LHS = \sum_{i=1}^{n} \frac{a_i b_i - b_i^2}{a_i + b_i} = \sum_{i=1}^{n} \left(b_i - \frac{2b_i^2}{a_i + b_i}\right) = \sum_{i=1}^{n} b_i - 2\sum_{i=1}^{n} \frac{b_i^2}{a_i + b_i},$$

$$RHS = \frac{\sum\limits_{i=1}^{n} a_i \cdot \sum\limits_{i=1}^{n} b_i - \left(\sum\limits_{i=1}^{n} b_i\right)^2}{\sum\limits_{i=1}^{n} a_i + \sum\limits_{i=1}^{n} b_i} = \sum_{i=1}^{n} b_i - 2\frac{\left(\sum\limits_{i=1}^{n} b_i\right)^2}{\sum\limits_{i=1}^{n} (a_i + b_i)}.$$

Therefore, the statement is equivalent with

$$\sum_{i=1}^{n} \frac{b_i^2}{a_i + b_i} \geq \frac{\left(\sum\limits_{i=1}^{n} b_i\right)^2}{\sum\limits_{i=1}^{n} (a_i + b_i)},$$

which is the same as the well-known variant of the Cauchy-Schwarz inequality,

$$\sum_{i=1}^{n} \frac{X_i^2}{Y_i} \geq \frac{(X_1 + \dots + X_n)^2}{Y_1 + \dots + Y_n} \quad (Y_1, \dots, Y_n > 0)$$

with $X_i = b_i$ and $Y_i = a_i + b_i$.

Problem 4. Let $n \geq k$ be positive integers, and let $\mathcal{F}$ be a family of finite sets with the following properties:

(i) $\mathcal{F}$ contains at least $\binom{n}{k}+1$ distinct sets containing exactly k elements;

(ii) for any two sets $A, B \in \mathcal{F}$, their union $A \cup B$ also belongs to $\mathcal{F}$.

Prove that $\mathcal{F}$ contains at least three sets with at least n elements.

(Proposed by Fedor Petrov, St. Petersburg State University)

Solution 1. If $n = k$ then we have at least two distinct sets in the family with exactly n elements and their union, so the statement is true. From now on we assume that $n > k$.

Fix $\binom{n}{k}+1$ sets of size k in $\mathcal{F}$, call them 'generators'. Let $V \in \mathcal{F}$ be the union of the generators. Since V has at least $\binom{n}{k}+1$ subsets of size k, we have $|V| > n$.

Call an element $v \in V$ 'appropriate' if v belongs to at most $\binom{n-1}{k-1}$ generators. Then there exist at least $\binom{n}{k}+1-\binom{n-1}{k-1} = \binom{n-1}{k}+1$ generators not containing v. Their union contains at least n elements, and the union does not contain v.

Now we claim that among any n elements $x_1, \ldots, x_n$ of V, there exists an appropriate element. Consider all pairs (G, x_i) such that G is a generator and $x_i \in G$. Every generator has exactly k elements, so the number of such pairs is at most $\left(\binom{n}{k}+1\right)\cdot k$. If some x_i is not appropriate then x_i is contained in at least $\binom{n-1}{k-1}+1$ generators; if none of $x_1, \ldots, x_n$ was appropriate then we wold have at least $n\cdot\left(\binom{n-1}{k-1}+1\right)$ pairs. But $n\cdot\left(\binom{n-1}{k-1}+1\right) > (\binom{n-1}{k-1}+1)\cdot k$, so this is not possible; at least one of $x_1, \ldots, x_n$ must be appropriate.

Since $|V| > n$, the set V contains some appropriate element v_1. Let $U_1 \in \mathcal{F}$ be the union of all generators not containing v_1. Then $|U_1| \geq n$ and $v_1 \notin U_1$. Now take an appropriate element v_2 from U_1 and let $U_2 \in \mathcal{F}$ be the union of all generators not containing v_2. Then $|U_2| \geq n$, so we have three sets, V, U_1 and U_2 in $\in \mathcal{F}$ with at least n elements: $V \neq U_1$ because $v_1 \in V$ and $v_1 \notin U_1$, and U_2 is different from V and U_1 because $v_2 \in V, U_1$ but $v_2 \notin U_2$.

Solution 2. We proceed by induction on k, so we can assume that the statement of the problem is known for smaller values of k. By contradiction, assume that $\mathcal{F}$ has less than 3 sets with at least n elements, that is the number of such sets is 0, 1 or 2. We can assume without loss of generality that $\mathcal{F}$ consists of exactly $N := \binom{n}{k}+1$ distinct sets of size k and all their possible unions. Denote the sets of size k by $S_1, S_2, \ldots$.

Consider a maximal set $I \subset \{1, \ldots, N\}$ such that $A := \bigcup_{i\in I} S_i$ has size less than n, $|A| < n$. This means that adding any S_j for $j \notin I$ makes the size at least n, $|S_j \cup A| \geq n$. First, let's prove that such j exist. Otherwise, all the sets S_i are contained in A. But there are only $\binom{|A|}{k} \leq \binom{n-1}{k} < N$ distinct k-element subsets of A, this is a contradiction. So there is at least one j such that $|S_j \cup A| \geq n$. Consider all possible sets that can be obtained as $S_j \cup A$ for $j \notin I$. Their size is at least n, so their number can be 1 or 2. If there are two of them, say B and C then $B \subset C$ or $C \subset B$, for otherwise the union of B and C would be different from both B and C, so we would have three sets B, C and $B \cup C$ of size at least n. We see that in any case there must exist $x \notin A$ such that $x \in S_j$ for all $j \notin I$. Consider sets $S'_j = S_j \setminus \{x\}$ for $j \notin I$. Their sizes are equal to $k-1$. Their number is at least

$$N - \binom{n-1}{k} = \binom{n-1}{k-1} + 1.$$

By the induction hypothesis, we can form 3 sets of size at least $n-1$ by taking unions of the sets S'_j for $j \notin I$. Adding x back we see that the corresponding unions of the sets S_j will have sizes at least n, so we are done proving the induction step.

The above argument allows us to decrease k all the way to $k = 0$, so it remains to check the statement for $k = 0$. The assumption says that we have at least $\binom{n}{0}+1 = 2$ sets of size 0. This is impossible, because there is only one empty set. Thus the statement trivially holds for $k = 0$.

Problem 5. Let S_n denote the set of permutations of the sequence $(1, 2, \ldots, n)$. For every permutation $\pi = (\pi_1, \ldots, \pi_n) \in S_n$, let $\mathrm{inv}(\pi)$ be the number of pairs $1 \leq i < j \leq n$ with $\pi_i > \pi_j$; i.e. the number of inversions in π. Denote by $f(n)$ the number of permutations $\pi \in S_n$ for which $\mathrm{inv}(\pi)$ is divisible by $n+1$.

Prove that there exist infinitely many primes p such that $f(p-1) > \frac{(p-1)!}{p}$, and infinitely many primes p such that $f(p-1) < \frac{(p-1)!}{p}$.

(Proposed by Fedor Petrov, St. Petersburg State University)

Solution. We will use the well-known formula

$$\sum_{\pi\in S_n} x^{\mathrm{inv}(\pi)} = 1\cdot(1+x)\cdot(1+x+x^2)\dots(1+x+\dots+x^{n-1}).$$

(This formula can be proved by induction on n. The cases $n=1,2$ are obvious. From each permutation of $(1,2,\dots,n-1)$, we can get a permutation of $(1,2,\dots,n)$ such that we insert the element n at one of the n possible positions before, between or after the numbers $1,2,\dots,n-1$; the number of inversions increases by $n-1, n-2, \dots, 1$ or 0, respectively.)

Now let

$$G_n(x) = \sum_{\pi\in S_n} x^{\mathrm{inv}(\pi)}.$$

and let $\varepsilon = e^{\frac{2\pi i}{n+1}}$. The sum of coefficients of the powers divisible by $n+1$ may be expressed as a trigonometric sum as

$$f(n) = \frac{1}{n+1}\sum_{k=0}^{n-1} G_n(\varepsilon^k) = \frac{n!}{n+1} + \frac{1}{n+1}\sum_{k=1}^{n-1} G_n(\varepsilon^k).$$

Hence, we are interested in the sign of

$$f(n) - \frac{n!}{n+1} = \sum_{k=1}^{n-1} G_n(\varepsilon^k)$$

with $n=p-1$ where p is a (large, odd) prime.

For every fixed $1\le k\le p-1$ we have

$$G_{p-1}(\varepsilon^k) = \prod_{j=1}^{p-1}(1+\varepsilon^k+\varepsilon^{2k}+\ldots+\varepsilon^{(j-1)k}) = \prod_{j=1}^{p-1}\frac{1-\varepsilon^{jk}}{1-\varepsilon^k} = \frac{(1-\varepsilon^k)(1-\varepsilon^{2k})\cdots(1-\varepsilon^{(p-1)k})}{(1-\varepsilon^k)^{p-1}}.$$

Notice that the factors in the numerator are $(1-\varepsilon)$, $(1-\varepsilon^2)$, $\dots$, $(1-\varepsilon^{p-1})$; only their order is different. So, by the identity $(z-\varepsilon)(z-\varepsilon^2)\dots(z-\varepsilon^{p-1}) = 1+z+\dots+z^{p-1}$,

$$G_{p-1}(\varepsilon^k) = \frac{p}{(1-\varepsilon^k)^{p-1}} = \frac{p}{\left(1-e^{\frac{2k\pi i}{p}}\right)^{p-1}}.$$

Hence, $f(p-1) - \frac{(p-1)!}{p}$ has the same sign as

$$\sum_{k=1}^{p-1}(1-e^{\frac{2k\pi i}{p}})^{1-p} = \sum_{k=1}^{p-1} e^{\frac{k(1-p)\pi i}{p}}\left(-2i\sin\frac{\pi k}{p}\right)^{1-p} =$$

$$= 2\cdot 2^{1-p}(-1)^{\frac{p-1}{2}}\sum_{k=1}^{\frac{p-1}{2}}\cos\frac{\pi k(p-1)}{p}\left(\sin\frac{\pi k}{p}\right)^{1-p}.$$

For large primes p the term with $k=1$ increases exponentially faster than all other terms, so this term determines the sign of the whole sum. Notice that $\cos\frac{\pi(p-1)}{p}$ converges to -1. So, the sum is positive if $p-1$ is odd and negative if $p-1$ is even. Therefore, for sufficiently large primes, $f(p-1) - \frac{(n-1)!}{p}$ is positive if $p\equiv 3 \pmod 4$ and it is negative if $p\equiv 1 \pmod 4$.

IMC 2016, Blagoevgrad, Bulgaria

Day 2, July 28, 2016

Problem 1. Let $(x_1, x_2, \ldots)$ be a sequence of positive real numbers satisfying $\displaystyle\sum_{n=1}^{\infty} \frac{x_n}{2n-1} = 1$. Prove that

$$\sum_{k=1}^{\infty} \sum_{n=1}^{k} \frac{x_n}{k^2} \le 2.$$

(Proposed by Gerhard J. Woeginger, The Netherlands)

Solution. By interchanging the sums,

$$\sum_{k=1}^{\infty} \sum_{n=1}^{k} \frac{x_n}{k^2} = \sum_{1 \le n \le k} \frac{x_n}{k^2} = \sum_{n=1}^{\infty} \left(x_n \sum_{k=n}^{\infty} \frac{1}{k^2} \right).$$

Then we use the upper bound

$$\sum_{k=n}^{\infty} \frac{1}{k^2} \le \sum_{k=n}^{\infty} \frac{1}{k^2 - \frac{1}{4}} = \sum_{k=n}^{\infty} \left(\frac{1}{k - \frac{1}{2}} - \frac{1}{k + \frac{1}{2}} \right) = \frac{1}{n - \frac{1}{2}}$$

and get

$$\sum_{k=1}^{\infty} \sum_{n=1}^{k} \frac{x_n}{k^2} = \sum_{n=1}^{\infty} \left(x_n \sum_{k=n}^{\infty} \frac{1}{k^2} \right) < \sum_{n=1}^{\infty} \left(x_n \cdot \frac{1}{n - \frac{1}{2}} \right) = 2 \sum_{n=1}^{\infty} \frac{x_n}{2n-1} = 2.$$

Problem 2. Today, Ivan the Confessor prefers continuous functions $f : [0,1] \to \mathbb{R}$ satisfying $f(x) + f(y) \ge |x - y|$ for all pairs $x, y \in [0,1]$. Find the minimum of $\int_0^1 f$ over all preferred functions.

(Proposed by Fedor Petrov, St. Petersburg State University)

Solution. The minimum of $\int_0^1 f$ is $\frac{1}{4}$.

Applying the condition with $0 \le x \le \frac{1}{2}$, $y = x + \frac{1}{2}$ we get

$$f(x) + f(x + \tfrac{1}{2}) \ge \tfrac{1}{2}.$$

By integrating,

$$\int_0^1 f(x)\,\mathrm{d}x = \int_0^{1/2} \left(f(x) + f(x + \tfrac{1}{2}) \right) \mathrm{d}x \ge \int_0^{1/2} \tfrac{1}{2}\,\mathrm{d}x = \tfrac{1}{4}.$$

On the other hand, the function $f(x) = \left| x - \frac{1}{2} \right|$ satisfies the conditions because

$$|x - y| = \left| \left(x - \tfrac{1}{2} \right) + \left(\tfrac{1}{2} - y \right) \right| \le \left| x - \tfrac{1}{2} \right| + \left| \tfrac{1}{2} - y \right| = f(x) + f(y),$$

and establishes

$$\int_0^1 f(x)\,\mathrm{d}x = \int_0^{1/2} \left(\tfrac{1}{2} - x \right) \mathrm{d}x + \int_{1/2}^1 \left(x - \tfrac{1}{2} \right) \mathrm{d}x = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}.$$

Problem 3. Let n be a positive integer, and denote by $\mathbb{Z}_n$ the ring of integers modulo n. Suppose that there exists a function $f : \mathbb{Z}_n \to \mathbb{Z}_n$ satisfying the following three properties:

(i) $f(x) \neq x$,

(ii) $f(f(x)) = x$,

(iii) $f(f(f(x+1)+1)+1) = x$ for all $x \in \mathbb{Z}_n$.

Prove that $n \equiv 2 \pmod 4$.

(Proposed by Ander Lamaison Vidarte, Berlin Mathematical School, Germany)

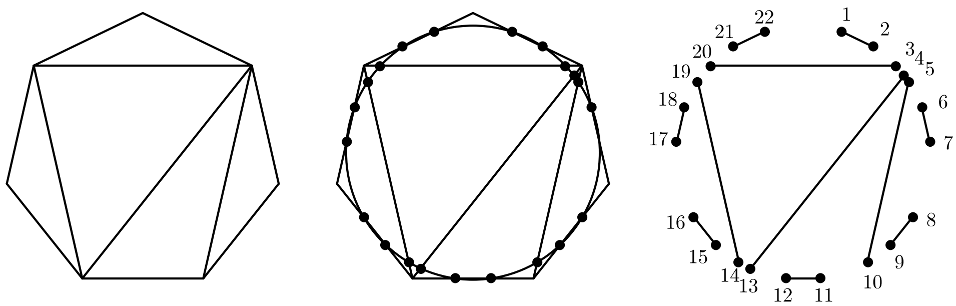
Solution. From property (ii) we can see that f is surjective, so f is a permutation of the elements in $\mathbb{Z}_n$, and its order is at most 2. Therefore, the permutation f is the product of disjoint transpositions of the form $\big(x, f(x)\big)$. Property (i) yields that this permutation has no fixed point, so n is be even, and the number of transpositions is precisely $n/2$.

Consider the permutation $g(x) = f(x+1)$. If g was odd then $g \circ g \circ g$ also would be odd. But property (iii) constraints that $g \circ g \circ g$ is the identity which is even. So g cannot be odd; g must be even. The cyclic permutation $h(x) = x - 1$ has order n, an even number, so h is odd. Then $f(x) = g \circ h$ is odd. Since f is the product of $n/2$ transpositions, this shows that $n/2$ must be odd, so $n \equiv 2 \pmod 4$.

Remark. There exists a function with properties (i–iii) for every $n \equiv 2 \pmod 4$. For $n = 2$ take $f(1) = 2$, $f(2) = 1$. Here we outline a possible construction for $n \geq 6$.

Let $n = 4k+2$, take a regular polygon with $k+2$ sides, and divide it into k triangles with $k-1$ diagonals. Draw a circle that intersects each side and each diagonal twice; altogether we have $4k+2$ intersections. Label the intersection points clockwise around the circle. On every side and diagonal we have two intersections; let f send them to each other.

This function f obviously satisfies properties (i) and (ii). For every x we either have $f(x+1) = x$ or the effect of adding 1 and taking f three times is going around the three sides of a triangle, so this function satisfies property (iii).



Problem 4. Let k be a positive integer. For each nonnegative integer n, let $f(n)$ be the number of solutions $(x_1, \ldots, x_k) \in \mathbb{Z}^k$ of the inequality $|x_1| + \ldots + |x_k| \leq n$. Prove that for every $n \geq 1$, we have $f(n-1)f(n+1) \leq f(n)^2$.

(Proposed by Esteban Arreaga, Renan Finder and José Madrid, IMPA, Rio de Janeiro)

Solution 1. We prove by induction on k. If $k = 1$ then we have $f(n) = 2n+1$ and the statement immediately follows from the AM-GM inequality.

Assume that $k \geq 2$ and the statement is true for $k-1$. Let $g(m)$ be the number of integer solutions of $|x_1| + \ldots + |x_{k-1}| \leq m$; by the induction hypothesis $g(m-1)g(m+1) \leq g(m)^2$ holds; this can be transformed to

$$\frac{g(0)}{g(1)} \leq \frac{g(1)}{g(2)} \leq \frac{g(2)}{g(3)} \leq \ldots .$$

For any integer constant c, the inequality $|x_1|+...+|x_{k-1}|+|c| \le n$ has $g\big(n-|c|\big)$ integer solutions. Therefore, we have the recurrence relation

$$f(n) = \sum_{c=-n}^{n} g\big(n-|c|\big) = g(n) + 2g(n-1) + ... + 2g(0).$$

It follows that

$$\frac{f(n-1)}{f(n)} = \frac{g(n-1)+2g(n-2)+...+2g(0)}{g(n)+2g(n-1)+...+2g(1)+2g(0)} \le$$

$$\le \frac{g(n)+g(n-1)+\big(g(n-1)+...+2g(0)+2\cdot 0\big)}{g(n+1)+g(n)+\big(g(n)+...+2g(1)+2g(0)\big)} = \frac{f(n)}{f(n+1)}$$

as required.

Solution 2. We first compute the generating function for $f(n)$:

$$\sum_{n=0}^{\infty} f(n)q^n = \sum_{(x_1,x_2,...,x_k)\in\mathbb{Z}^k} \sum_{c=0}^{\infty} q^{|x_1|+|x_2|+\cdots+|x_k|+c} = \left(\sum_{x\in\mathbb{Z}} q^{|x|}\right)^k \frac{1}{1-q} = \frac{(1+q)^k}{(1-q)^{k+1}}.$$

For each $a = 0, 1, \ldots$ denote by $g_a(n)$ $(n = 0, 1, 2, \ldots)$ the coefficients in the following expansion:

$$\frac{(1+q)^a}{(1-q)^{k+1}} = \sum_{n=0}^{\infty} g_a(n)q^n.$$

So it is clear that $g_{a+1}(n) = g_a(n) + g_a(n-1)$ $(n \ge 1)$, $g_a(0) = 1$. Call a sequence of positive numbers $g(0)$, $g(1)$, $g(2)$, $\ldots$ *good* if $\frac{g(n-1)}{g(n)}$ $(n = 1, 2, \ldots)$ is an increasing sequence. It is straightforward to check that g_0 is good:

$$g_0(n) = \binom{k+n}{k}, \qquad \frac{g_0(n-1)}{g_0(n)} = \frac{n}{k+n}.$$

If g is a good sequence then a new sequence g' defined by $g'(0) = g(0)$, $g'(n) = g(n) + g(n-1)$ $(n \ge 1)$ is also good:

$$\frac{g'(n-1)}{g'(n)} = \frac{g(n-1)+g(n-2)}{g(n)+g(n-1)} = \frac{1+\frac{g(n-2)}{g(n-1)}}{1+\frac{g(n)}{g(n-1)}},$$

where define $g(-1) = 0$. Thus we see that each of the sequences g_1, g_2, $\ldots$, $g_k = f$ are good. So the desired inequality holds.

Problem 5. Let A be a $n\times n$ complex matrix whose eigenvalues have absolute value at most 1. Prove that

$$\|A^n\| \le \frac{n}{\ln 2}\|A\|^{n-1}.$$

(Here $\|B\| = \sup_{\|x\|\le 1} \|Bx\|$ for every $n \times n$ matrix B and $\|x\| = \sqrt{\sum_{i=1}^{n} |x_i|^2}$ for every complex vector $x \in \mathbb{C}^n$.)

(Proposed by Ian Morris and Fedor Petrov, St. Petersburg State University)

Solution 1. Let $r = \|A\|$. We have to prove $\|A^n\| \le \frac{n}{\ln 2} r^{n-1}$.

As is well-known, the matrix norm satisfies $\|XY\| \le \|X\|\cdot\|Y\|$ for any matrices X, Y, and as a simple consequence, $\|A^k\| \le \|A\|^k = r^k$ for every positive integer k.

Let $\chi(t) = (t-\lambda_1)(t-\lambda_2)\ldots(t-\lambda_n) = t^n + c_1t^{n-1} + \cdots + c_n$ be the characteristic polynomial of A. From Vieta's formulas we get

$$|c_k| = \left|\sum_{1\le i_1<\ldots<i_k\le n} \lambda_{i_1}\cdots\lambda_{i_k}\right| \le \sum_{1\le i_1<\ldots<i_k\le n} |\lambda_{i_1}\cdots\lambda_{i_k}| \le \binom{n}{k} \quad (k = 1, 2, \ldots, n)$$

By the Cayley–Hamilton theorem we have $\chi(A) = 0$, so

$$\|A^n\| = \|c_1 A^{n-1} + \cdots + c_n\| \le \sum_{k=1}^{n} \binom{n}{k} \|A^k\| \le \sum_{k=1}^{n} \binom{n}{k} r^k = (1+r)^n - r^n.$$

Combining this with the trivial estimate $\|A^n\| \le r^n$, we have

$$\|A^n\| \le \min\big(r^n, (1+r)^n - r^n\big).$$

Let $r_0 = \frac{1}{\sqrt[n]{2}-1}$; it is easy to check that the two bounds are equal if $r = r_0$, moreover

$$r_0 = \frac{1}{e^{\ln 2/n} - 1} < \frac{n}{\ln 2}.$$

For $r \le r_0$ apply the trivial bound:

$$\|A^n\| \le r^n \le r_0 \cdot r^{n-1} < \frac{n}{\ln 2} r^{n-1}.$$

For $r > r_0$ we have

$$\|A^n\| \le (1+r)^n - r^n = r^{n-1} \cdot \frac{(1+r)^n - r^n}{r^{n-1}}.$$

Notice that the function $f(r) = \frac{(1+r)^n - r^n}{r^{n-1}}$ is decreasing because the numerator has degree $n-1$ and all coefficients are positive, so

$$\frac{(1+r)^n - r^n}{r^{n-1}} < \frac{(1+r_0)^n - r_0^n}{r_0^{n-1}} = r_0\big((1+1/r_0)^n - 1\big) = r_0 < \frac{n}{\ln 2},$$

so $\|A^n\| < \frac{n}{\ln 2} r^{n-1}$.

Solution 2. We will use the following facts which are easy to prove:

- For any square matrix A there exists a unitary matrix U such that UAU^{-1} is upper-triangular.
- For any matrices A, B we have $\|A\| \le \|(A|B)\|$ and $\|B\| \le \|(A|B)\|$ where $(A|B)$ is the matrix whose columns are the columns of A and the columns of B.
- For any matrices A, B we have $\|A\| \le \|\binom{A}{B}\|$ and $\|B\| \le \|\binom{A}{B}\|$ where $\binom{A}{B}$ is the matrix whose rows are the rows of A and the rows of B.
- Adding a zero row or a zero column to a matrix does not change its norm.

We will prove a stronger inequality

$$\|A^n\| \le n\|A\|^{n-1}$$

for any $n \times n$ matrix A whose eigenvalues have absolute value at most 1. We proceed by induction on n. The case $n = 1$ is trivial. Without loss of generality we can assume that the matrix A is upper-triangular. So we have

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & & \cdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}.$$

Note that the eigenvalues of A are precisely the diagonal entries. We split A as the sum of 3 matrices, $A = X + Y + Z$ as follows:

$$X = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & a_{12} & \cdots & a_{1n} \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & & \cdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}.$$

Denote by A' the matrix obtained from A by removing the first row and the first column:

$$A' = \begin{pmatrix} a_{22} & \cdots & a_{2n} \\ \cdots & & \cdots \\ 0 & \cdots & a_{nn} \end{pmatrix}.$$

We have $\|X\| \leq 1$ because $|a_{11}| \leq 1$. We also have

$$\|A'\| = \|Z\| \leq \|Y + Z\| \leq \|A\|.$$

Now we decompose A^n as follows:

$$A^n = XA^{n-1} + (Y + Z)A^{n-1}.$$

We substitute $A = X + Y + Z$ in the second term and expand the parentheses. Because of the following identities:

$$Y^2 = 0, \quad YX = 0, \quad ZY = 0, \quad ZX = 0$$

only the terms YZ^{n-1} and Z^n survive. So we have

$$A^n = XA^{n-1} + (Y + Z)Z^{n-1}.$$

By the induction hypothesis we have $\|A'^{n-1}\| \leq (n-1)\|A'\|^{n-2}$, hence $\|Z^{n-1}\| \leq (n-1)\|Z\|^{n-2} \leq (n-1)\|A\|^{n-2}$. Therefore

$$\|A^n\| \leq \|XA^{n-1}\| + \|(Y + Z)Z^{n-1}\| \leq \|A\|^{n-1} + (n-1)\|Y + Z\|\|A\|^{n-2} \leq n\|A\|^{n-1}.$$

IMC 2017, Blagoevgrad, Bulgaria

Day 1, August 2, 2017

Problem 1. Determine all complex numbers λ for which there exist a positive integer n and a real $n \times n$ matrix A such that $A^2 = A^T$ and λ is an eigenvalue of A.

(Proposed by Alexandr Bolbot, Novosibirsk State University)

Solution. By taking squares,

$$A^4 = (A^2)^2 = (A^T)^2 = (A^2)^T = (A^T)^T = A,$$

so

$$A^4 - A = 0;$$

it follows that all eigenvalues of A are roots of the polynomial $X^4 - X$.

The roots of $X^4 - X = X(X^3 - 1)$ are 0, 1 and $\frac{-1 \pm \sqrt{3}i}{2}$. In order to verify that these values are possible, consider the matrices

$$A_0 = \begin{pmatrix} 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \quad A_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}.$$

The numbers 0 and 1 are the eigenvalues of the 1×1 matrices A_0 and A_1, respectively. The numbers $\frac{-1 \pm \sqrt{3}i}{2}$ are the eigenvalues of A_2; it is easy to check that

$$A_2^2 = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} = A_2^T.$$

The matrix A_4 establishes all the four possible eigenvalues in a single matrix.

Remark. The matrix A_2 represents a rotation by $2\pi/3$.

Problem 2. Let $f\colon \mathbb{R} \to (0, \infty)$ be a differentiable function, and suppose that there exists a constant $L > 0$ such that

$$\left|f'(x) - f'(y)\right| \le L\left|x - y\right|$$

for all x, y. Prove that

$$\left(f'(x)\right)^2 < 2Lf(x)$$

holds for all x.

(Proposed by Jan Šustek, University of Ostrava)

Solution. Notice that f' satisfies the Lipschitz-property, so f' is continuous and therefore locally integrable.

Consider an arbitrary $x \in \mathbb{R}$ and let $d = f'(x)$. We need to prove $f(x) > \frac{d^2}{2L}$.

If $d = 0$ then the statement is trivial.

If $d > 0$ then the condition provides $f'(x-t) \geq d - Lt$; this estimate is positive for $0 \leq t < \frac{d}{L}$. By integrating over that interval,

$$f(x) > f(x) - f(x - \tfrac{d}{L}) = \int_0^{\frac{d}{L}} f'\big(x-t\big)\mathrm{d}t \geq \int_0^{\frac{d}{L}} (d - Lt)\mathrm{d}t = \frac{d^2}{2L}.$$

If $d < 0$ then apply $f'(x+t) \leq d + Lt = -|d| + Lt$ and repeat the same argument as

$$f(x) > f(x) - f(x + \tfrac{|d|}{L}) = \int_0^{\frac{|d|}{L}} \big(-f'(x+t)\big)\mathrm{d}t \geq \int_0^{\frac{|d|}{L}} (|d| - Lt)\mathrm{d}t = \frac{d^2}{2L}.$$

Problem 3. For any positive integer m, denote by $P(m)$ the product of positive divisors of m (e.g. $P(6) = 36$). For every positive integer n define the sequence

$$a_1(n) = n, \qquad a_{k+1}(n) = P(a_k(n)) \quad (k = 1, 2, \ldots, 2016).$$

Determine whether for every set $S \subseteq \{1, 2, \ldots, 2017\}$, there exists a positive integer n such that the following condition is satisfied:

For every k with $1 \leq k \leq 2017$, the number $a_k(n)$ is a perfect square if and only if $k \in S$.

(Proposed by Matko Ljulj , University of Zagreb)

Solution. We prove that the answer is yes; for every $S \subset \{1, 2, \ldots, 2017\}$ there exists a suitable n. Specially, n can be a power of 2: $n = 2^{w_1}$ with some nonnegative integer w_1. Write $a_k(n) = 2^{w_k}$; then

$$2^{w_{k+1}} = a_{k+1}(n) = P(a_k(n)) = P(2^{w_k}) = 1 \cdot 2 \cdot 4 \cdots 2^{w_k} = 2^{\frac{w_k(w_k+1)}{2}},$$

so

$$w_{k+1} = \frac{w_k(w_k+1)}{2}.$$

The proof will be completed if we prove that for each choice of S there exists an initial value w_1 such that w_k is even if and only if $k \in S$.

Lemma. Suppose that the sequences $(b_1, b_2, \ldots)$ and $(c_1, c_2, \ldots)$ satisfy $b_{k+1} = \frac{b_k(b_k+1)}{2}$ and $c_{k+1} = \frac{c_k(c_k+1)}{2}$ for $k \geq 1$, and $c_1 = b_1 + 2^m$. Then for each $k = 1, \ldots m$ we have $c_k \equiv b_k + 2^{m-k+1} \pmod{2^{m-k+2}}$.

As an immediate corollary, we have $b_k \equiv c_k \pmod 2$ for $1 \leq k \leq m$ and $b_{m+1} \equiv c_{m+1} + 1 \pmod 2$.

Proof. We prove the by induction. For $k = 1$ we have $c_1 = b_1 + 2^m$ so the statement holds. Suppose the statement is true for some $k < m$, then for $k+1$ we have

$$\begin{aligned} c_{k+1} &= \frac{c_k\,(c_k+1)}{2} \equiv \frac{\big(b_k + 2^{m-k+1}\big)\big(b_k + 2^{m-k+1} + 1\big)}{2} \\ &= \frac{b_k^2 + 2^{m-k+2}b_k + 2^{2m-2k+2} + b_k + 2^{m-k+1}}{2} = \\ &= \frac{b_k(b_k+1)}{2} + 2^{m-k} + 2^{m-k+1}b_k + 2^{2m-2k+1} \equiv \frac{b_k(b_k+1)}{2} + 2^{m-k} \pmod{2^{m-k+1}}, \end{aligned}$$

therefore $c_{k+1} \equiv b_{k+1} + 2^{m-(k+1)+1} \pmod{2^{m-(k+1)+2}}$.

Going back to the solution of the problem, for every $1 \le m \le 2017$ we construct inductively a sequence $(v_1, v_2, \ldots)$ such that $v_{k+1} = \frac{v_k(v_k+1)}{2}$, and for every $1 \le k \le m$, v_k is even if and only if $k \in S$.

For $m = 1$ we can choose $v_1 = 0$ if $1 \in S$ or $v_1 = 1$ if $1 \notin S$. If we already have such a sequence $(v_1, v_2, \ldots)$ for a positive integer m, we can choose either the same sequence or choose $v'_1 = v_1 + 2^m$ and apply the same recurrence $v'_{k+1} = \frac{v'_k(v'_k+1)}{2}$. By the Lemma, we have $v_k \equiv v'_k \pmod 2$ for $k \le m$, but v_{m+1} and v_{m+1} have opposite parities; hence, either the sequence (v_k) or the sequence (v'_k) satisfies the condition for $m+1$.

Repeating this process for $m = 1, 2, \ldots, 2017$, we obtain a suitable sequence (w_k).

Problem 4. There are n people in a city, and each of them has exactly 1000 friends (friendship is always symmetric). Prove that it is possible to select a group S of people such that at least $n/2017$ persons in S have exactly two friends in S.

(Proposed by Rooholah Majdodin and Fedor Petrov, St. Petersburg State University)

Solution. Let $d = 1000$ and let $0 < p < 1$. Choose the set S randomly such that each people is selected with probability p, independently from the others.

The probability that a certain person is selected for S and knows exactly two members of S is

$$q = \binom{d}{2} p^3 (1-p)^{d-2}.$$

Choose $p = 3/(d+1)$ (this is the value of p for which q is maximal); then

$$q = \binom{d}{2} \left(\frac{3}{d+1}\right)^3 \left(\frac{d-2}{d+1}\right)^{d-2} =$$

$$= \frac{27d(d-1)}{2(d+1)^3} \left(1 + \frac{3}{d-2}\right)^{-(d-2)} > \frac{27d(d-1)}{2(d+1)^3} \cdot e^{-3} > \frac{1}{2017}.$$

Hence, $E\big(|S|\big) = nq > \frac{n}{2017}$, so there is a choice for S when $|S| > \frac{n}{2017}$.

Problem 5. Let k and n be positive integers with $n \ge k^2 - 3k + 4$, and let

$$f(z) = z^{n-1} + c_{n-2} z^{n-2} + \ldots + c_0$$

be a polynomial with complex coefficients such that

$$c_0 c_{n-2} = c_1 c_{n-3} = \ldots = c_{n-2} c_0 = 0.$$

Prove that $f(z)$ and $z^n - 1$ have at most $n - k$ common roots.

(Proposed by Vsevolod Lev and Fedor Petrov, St. Petersburg State University)

Solution. Let $M = \{z : z^n = 1\}$, $A = \{z \in M : f(z) \ne 0\}$ and $A^{-1} = \{z^{-1} : z \in A\}$. We have to prove $|A| \ge k$.

Claim.

$$A \cdot A^{-1} = M.$$

That is, for any $\eta \in M$, there exist some elements $a, b \in A$ such that $ab^{-1} = \eta$.

Proof. As is well-known, for every integer m,

$$\sum_{z \in M} z^m = \begin{cases} n & \text{if } n \mid m \\ 0 & \text{otherwise.} \end{cases}$$

Define $c_{n-1} = 1$ and consider

$$\sum_{z\in M} z^2 f(z) f(\eta z) = \sum_{z\in M} z^2 \sum_{j=0}^{n-1} c_j z^j \sum_{\ell=0}^{n-1} c_\ell (\eta z)^\ell = \sum_{j=0}^{n-1}\sum_{\ell=0}^{n-1} c_j c_\ell \eta^\ell \sum_{z\in M} z^{j+\ell+2} =$$

$$= \sum_{j=0}^{n-1}\sum_{\ell=0}^{n-1} c_j c_\ell \eta^\ell \sum_{z\in M} \begin{Bmatrix} n & \text{if } n|j+\ell+2 \\ 0 & \text{otherwise} \end{Bmatrix} = c_{n-1}^2 n + \sum_{j=0}^{n-2} c_j c_{n-2-j} \eta^{n-2-j} n = n \neq 0.$$

Therefore there exists some $b \in M$ such that $f(b) \neq 0$ and $f(\eta b) \neq 0$, i.e. $b \in A$, and $a = \eta b \in A$, satisfying $ab^{-1} = \eta$.

By double-counting the elements of M, from the Claim we conclude

$$|A|\big(|A|-1\big) \geq \big|M \setminus \{1\}\big| = n-1 \geq k^2 - 3k + 3 > (k-1)(k-2)$$

which shows $|A| > k-1$.

IMC 2017, Blagoevgrad, Bulgaria

Day 2, August 3, 2017

Problem 6. Let $f : [0; +\infty) \to \mathbb{R}$ be a continuous function such that $\lim\limits_{x\to+\infty} f(x) = L$ exists (it may be finite or infinite). Prove that

$$\lim_{n\to\infty} \int\limits_0^1 f(nx)\,\mathrm{d}x = L.$$

(Proposed by Alexandr Bolbot, Novosibirsk State University)

Solution 1. *Case 1: L is finite.* Take an arbitrary $\varepsilon > 0$. We construct a number $K \ge 0$ such that $\left|\int\limits_0^1 f(nx)\,\mathrm{d}x - L\right| < \varepsilon$.

Since $\lim\limits_{x\to+\infty} f(x) = L$, there exists a $K_1 \ge 0$ such that $\big|f(x) - L\big| < \frac{\varepsilon}{2}$ for every $x \ge K_1$. Hence, for $n \ge K_1$ we have

$$\left|\int_0^1 f(nx)\,\mathrm{d}x - L\right| = \left|\frac{1}{n}\int_0^n f(x)\,\mathrm{d}x - L\right| = \frac{1}{n}\left|\int_0^n \big(f - L\big)\right| \le$$
$$\le \frac{1}{n}\int_0^n |f-L| = \frac{1}{n}\left(\int_0^{K_1} |f-L| + \int_{K_1}^n |f-L|\right) < \frac{1}{n}\left(\int_0^{K_1} |f-L| + \int_{K_1}^n \frac{\varepsilon}{2}\right) =$$
$$= \frac{1}{n}\int_0^{K_1} |f-L| + \frac{n-K_1}{n}\cdot\frac{\varepsilon}{2} < \frac{1}{n}\int_0^{K_1} |f-L| + \frac{\varepsilon}{2}.$$

If $n \ge K_2 = \frac{2}{\varepsilon}\int_0^{K_1} |f-L|$ then the first term is at most $\frac{\varepsilon}{2}$. Then for $x \ge K := \max(K_1, K_2)$ we have

$$\left|\int_0^1 f(nx)\,\mathrm{d}x - L\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Case 2: $L = +\infty$. Take an arbitrary real M; we need a $K \ge 0$ such that $\int\limits_0^1 f(nx)\,\mathrm{d}x > M$ for every $x \ge K$.

Since $\lim\limits_{x\to+\infty} f(x) = \infty$, there exists a $K_1 \ge 0$ such that $f(x) > M + 1$ for every $x \ge K_1$. Hence, for $n \ge 2K_1$ we have

$$\int_0^1 f(nx)\,\mathrm{d}x = \frac{1}{n}\int_0^n f(x)\,\mathrm{d}x = \frac{1}{n}\int_0^n f = \frac{1}{n}\left(\int_0^{K_1} f + \int_{K_1}^n f\right) =$$
$$= \frac{1}{n}\left(\int_0^{K_1} f + \int_{K_1}^n (M+1)\right) = \frac{1}{n}\left(\int_0^{K_1} f - K_1(M+1)\right) + M + 1.$$

If $n \ge K_2 := \left|\int_0^{K_1} f - K_1(M+1)\right|$ then the first term is at least -1. For $x \ge K := \max(K_1, K_2)$ we have $\int_0^1 f(nx)\,\mathrm{d}x > M$.

Case 3: $L = -\infty$. We can repeat the steps in Case 2 for the function $-f$.

Solution 2. Let $F(x)=\int_0^x f$. For $t>0$ we have

$$\int_0^1 f(tx)\,\mathrm{d}x=\frac{F(t)}{t}.$$

Since $\lim\limits_{t\to\infty} t=\infty$ in the denominator and $\lim\limits_{t\to\infty} F'(t)=\lim\limits_{t\to\infty} f(t)=L$, L'Hospital's rule proves $\lim\limits_{t\to\infty}\frac{F(t)}{t}=\lim\limits_{t\to\infty}\frac{F'(t)}{1}=\lim\limits_{t\to\infty}\frac{f(t)}{1}=L$. Then it follows that $\lim\frac{F(n)}{n}=L$.

Problem 7. Let $p(x)$ be a nonconstant polynomial with real coefficients. For every positive integer n, let

$$q_n(x)=(x+1)^n p(x)+x^n p(x+1).$$

Prove that there are only finitely many numbers n such that all roots of $q_n(x)$ are real.

(Proposed by Alexandr Bolbot, Novosibirsk State University)

Solution.

Lemma. If $f(x)=a_mx^m+\ldots+a_1x+a_0$ is a polynomial with $a_m\neq 0$, and all roots of f are real, then

$$a_{m-1}^2-2a_ma_{m-2}\geq 0.$$

Proof. Let the roots of f be $w_1,\ldots,w_n$. By the Viéte-formulas,

$$\sum_{i=1}^m w_i=-\frac{a_{m-1}}{a_m},\qquad \sum_{i<j} w_iw_j=\frac{a_{m-2}}{a_m},$$

$$0\leq\sum_{i=1}^m w_i^2=\left(\sum_{i=1}^m w_i\right)^2-2\sum_{i<j}w_iw_j=\left(\frac{a_{m-1}}{a_m}\right)^2-2\frac{a_{m-2}}{a_m}=\frac{a_{m-1}^2-2a_ma_{m-2}}{a_m^2}.$$

In view of the Lemma we focus on the asymptotic behavior of the three terms in $q_n(x)$ with the highest degrees. Let $p(x)=ax^k+bx^{k-1}+cx^{k-2}+\ldots$ and $q_n(x)=A_nx^{n+k}+B_nx^{n+k-1}+C_nx^{n+k-2}+\ldots$; then

$$\begin{aligned}
q_n(x)&=(x+1)^np(x)+x^np(x+1)=\\
&=\left(x^n+nx^{n-1}+\frac{n(n-1)}{2}x^{n-2}+\ldots\right)(ax^k+bx^{k-1}+cx^{k-2}+\ldots)\\
&\quad+x^n\bigg(a\Big(x^k+kx^{k-1}+\frac{k(k-1)}{2}x^{k-2}+\ldots\Big)\\
&\qquad\quad+b\Big(x^{k-1}+(k-1)x^{k-2}+\ldots\Big)+c\Big(x^{k-2}\ldots\Big)+\ldots\bigg)\\
&=2a\cdot x^{n+k}+\big((n+k)a+2b\big)x^{n+k-1}\\
&\qquad+\left(\frac{n(n-1)+k(k-1)}{2}a+(n+k-1)b+2c\right)x^{n+k-2}+\ldots,
\end{aligned}$$

so

$$A_n=2a,\quad B_n=(n+k)a+2b=\quad C_n=\frac{n(n-1)+k(k-1)}{2}a+(n+k-1)b+2c.$$

If $n\to\infty$ then

$$B_n^2-2A_nC_n=\big(na+O(1)\big)^2-2\cdot 2a\left(\frac{n^2a}{2}+O(n)\right)=-an^2+O(n)\to-\infty,$$

so $B_n^2-2A_nC_n$ is eventually negative, indicating that q_n cannot have only real roots.

Problem 8. Define the sequence $A_1, A_2, \ldots$ of matrices by the following recurrence:

$$A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A_{n+1} = \begin{pmatrix} A_n & I_{2^n} \\ I_{2^n} & A_n \end{pmatrix} \quad (n = 1, 2, \ldots)$$

where I_m is the $m \times m$ identity matrix.

Prove that A_n has $n+1$ distinct integer eigenvalues $\lambda_0 < \lambda_1 < \ldots < \lambda_n$ with multiplicities $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$, respectively.

(Proposed by Snježana Majstorović, University of J. J. Strossmayer in Osijek, Croatia)

Solution. For each $n \in \mathbb{N}$, matrix A_n is symmetric $2^n \times 2^n$ matrix with elements from the set $\{0, 1\}$, so that all elements on the main diagonal are equal to zero. We can write

$$A_n = I_{2^{n-1}} \otimes A_1 + A_{n-1} \otimes I_2, \tag{1}$$

where $\otimes$ is binary operation over the space of matrices, defined for arbitrary $B \in \mathbb{R}^{n \times p}$ and $C \in \mathbb{R}^{m \times s}$ as

$$B \otimes C := \begin{bmatrix} b_{11}C & b_{12}C & \ldots & b_{1p}C \\ b_{21}C & b_{22}C & \ldots & b_{2p}C \\ \vdots & & & \\ b_{n1}C & b_{12}C & \ldots & b_{np}C \end{bmatrix}_{nm \times ps}.$$

Lemma 1. If $B \in \mathbb{R}^{n \times n}$ has eigenvalues λ_i, $i = 1, \ldots, n$ and $C \in \mathbb{R}^{m \times m}$ has eigenvalues μ_j, $j = 1, \ldots, m$, then $B \otimes C$ has eigenvalues $\lambda_i \mu_j$, $i = 1, \ldots, n$, $j = 1, \ldots, m$. If B and C are diagonalizable, then $A \otimes B$ has eigenvectors $y_i \otimes z_j$, with (λ_i, y_i) and (μ_j, z_j) being eigenpairs of B and C, respectively.

Proof 1. Let (λ, y) be an eigenpair of B and (μ, z) an eigenpar of C. Then

$$(B \otimes C)(y \otimes z) = By \otimes Cz = \lambda y \otimes \mu z = \lambda\mu(y \otimes z).$$

If we take (λ, y) to be an eigenpair of A_1 and (μ, z) to be an eigenpair of A_{n-1}, then from (1) and Lemma 1 we get

$$\begin{aligned} A_n(z \otimes y) &= (I_{2^{n-1}} \otimes A_1 + A_{n-1} \otimes I_2)(z \otimes y) \\ &= (I_{2^{n-1}} \otimes A_1)(z \otimes y) + (A_{n-1} \otimes I_2)(z \otimes y) \\ &= (\lambda + \mu)(z \otimes y). \end{aligned}$$

So the entire spectrum of A_n can be obtained from eigenvalues of A_{n-1} and A_1: just sum up each eigenvalue of A_{n-1} with each eigenvalue of A_1. Since the spectrum of A_1 is $\sigma(A_1) = \{-1, 1\}$, we get

$$\begin{aligned} \sigma(A_2) &= \{-1 + (-1), -1 + 1, 1 + (-1), 1 + 1\} = \{-2, 0^{(2)}, 2\} \\ \sigma(A_3) &= \{-1 + (-2), -1 + 0, -1 + 0, -1 + 2, 1 + (-2), 1 + 0, 1 + 0, 1 + 2\} = \{-3, (-1)^{(3)}, 1^{(3)}, 3\} \\ \sigma(A_4) &= \{-1 + (-3), -1 + (-1^{(3)}), -1 + 1^{(3)}, -1 + 3, 1 + (-3), 1 + (-1^{(3)}), 1 + 1^{(3)}, 1 + 3\} \\ &= \{-4, (-2)^{(4)}, 0^{(3)}, 2^{(4)}, 4\}. \end{aligned}$$

Inductively, A_n has $n+1$ distinct integer eigenvalues $-n, -n+2, -n+4, \ldots, n-4, n-2, n$ with multiplicities $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}$, respectively.

Problem 9. Define the sequence $f_1, f_2, \ldots : [0, 1) \to \mathbb{R}$ of continuously differentiable functions by the following recurrence:

$$f_1 = 1; \qquad f'_{n+1} = f_n f_{n+1} \quad \text{on } (0, 1), \quad \text{and} \quad f_{n+1}(0) = 1.$$

Show that $\lim_{n \to \infty} f_n(x)$ exists for every $x \in [0, 1)$ and determine the limit function.

(Proposed by Tomáš Bárta, Charles University, Prague)

Solution. First of all, the sequence f_n is well defined and it holds that

$$f_{n+1}(x) = e^{\int_0^x f_n(t)\mathrm{d}t}. \tag{2}$$

The mapping $\Phi : C([0,1)) \to C([0,1))$ given by

$$\Phi(g)(x) = e^{\int_0^x g(t)\mathrm{d}t}$$

is monotone, i.e. if $f < g$ on $(0,1)$ then

$$\Phi(f)(x) = e^{\int_0^x f(t)\mathrm{d}t} < e^{\int_0^x g(t)\mathrm{d}t} = \Phi(g)(x)$$

on $(0,1)$. Since $f_2(x) = e^{\int_0^x 1 mathrmdt} = e^x > 1 = f_1(x)$ on $(0,1)$, we have by induction $f_{n+1}(x) > f_n(x)$ for all $x \in (0,1)$, $n \in \mathbb{N}$. Moreover, function $f(x) = \frac{1}{1-x}$ is the unique solution to $f' = f^2$, $f(0) = 1$, i.e. it is the unique fixed point of Φ in $\{\varphi \in C([0,1)) : \ \varphi(0) = 1\}$. Since $f_1 < f$ on $(0,1)$, by induction we have $f_{n+1} = \Phi(f_n) < \Phi(f) = f$ for all $n \in \mathbb{N}$. Hence, for every $x \in (0,1)$ the sequence $f_n(x)$ is increasing and bounded, so a finite limit exists.

Let us denote the limit $g(x)$. We show that $g(x) = f(x) = \frac{1}{1-x}$. Obviously, $g(0) = \lim f_n(0) = 1$. By $f_1 \equiv 1$ and (2), we have $f_n > 0$ on $[0,1)$ for each $n \in \mathbb{N}$, and therefore (by (2) again) the function f_{n+1} is increasing. Since f_n, f_{n+1} are positive and increasing also f'_{n+1} is increasing (due to $f'_{n+1} = f_n f_{n+1}$), hence f_{n+1} is convex. A pointwise limit of a sequence of convex functions is convex, since we pass to a limit $n \to \infty$ in

$$f_n(\lambda x + (1-\lambda)y) \le \lambda f_n(x) + (1-\lambda) f_n(y)$$

and obtain

$$g(\lambda x + (1-\lambda)y) \le \lambda g(x) + (1-\lambda) g(y)$$

for any fixed x, $y \in [0,1)$ and $\lambda \in (0,1)$. Hence, g is convex, and therefore continuous on $(0,1)$. Moreover, g is continuous in 0, since $1 \equiv f_1 \le g \le f$ and $\lim_{x\to 0+} f(x) = 1$. By Dini's Theorem, convergence $f_n \to g$ is uniform on $[0, 1-\varepsilon]$ for each $\varepsilon \in (0,1)$ (a monotone sequence converging to a continuous function on a compact interval). We show that Φ is continuous and therefore f_n have to converge to a fixed point of Φ.

In fact, let us work on the space $C([0,1-\varepsilon])$ with any fixed $\varepsilon \in (0,1)$, $\|\cdot\|$ being the supremum norm on $[0,1-\varepsilon]$. Then for a fixed function h and $\|\varphi - h\| < \delta$ we have

$$\sup_{x\in[0,1-\varepsilon]} |\Phi(h)(x) - \Phi(\varphi)(x)| = \sup_{x\in[0,1-\varepsilon]} e^{\int_0^x h(t)\mathrm{d}t} \left|1 - e^{\int_0^x \varphi(t)-h(t)\mathrm{d}t}\right| \le C(e^\delta - 1) < 2C\delta$$

for $\delta > 0$ small enough. Hence, Φ is continuous on $C([0,1-\varepsilon])$. Let us assume for contradiction that $\Phi(g) \ne g$. Hence, there exists $\eta > 0$ and $x_0 \in [0, 1-\varepsilon]$ such that $|\Phi(g)(x_0) - g(x_0)| > \eta$. There exists $\delta > 0$ such that $\|\Phi(\varphi) - \Phi(g)\| < \frac{1}{3}\eta$ whenever $\|\varphi - g\| < \delta$. Take n_0 so large that $\|f_n - g\| < \min\{\delta, \frac{1}{3}\eta\}$ for all $n \ge n_0$. Hence, $\|f_{n+1} - \Phi(g)\| = \|\Phi(f_n) - \Phi(g)\| < \frac{1}{3}\eta$. On the other hand, we have $|f_{n+1}(x_0) - \Phi(g)(x_0)| > |\Phi(g)(x_0) - g(x_0)| - |g(x_0) - f_{n+1}(x_0)| > \eta - \frac{1}{3}\eta = \frac{2}{3}\eta$, contradiction. So, $\Phi(g) = g$.

Since f is the only fixed point of Φ in $\{\varphi \in C([0,1-\varepsilon]) : \ \varphi(0) = 1\}$, we have $g = f$ on $[0, 1-\varepsilon]$. Since $\varepsilon \in (0,1)$ was arbitrary, we have $\lim_{n\to\infty} f_n(x) = \frac{1}{1-x}$ for all $x \in [0,1)$.

Problem 10. Let K be an equilateral triangle in the plane. Prove that for every $p > 0$ there exists an $\varepsilon > 0$ with the following property: If n is a positive integer, and $T_1, \dots, T_n$ are non-overlapping triangles inside K such that each of them is homothetic to K with a negative ratio, and

$$\sum_{\ell=1}^{n} \operatorname{area}(T_\ell) > \operatorname{area}(K) - \varepsilon,$$

then

$$\sum_{\ell=1}^{n} \operatorname{perimeter}(T_\ell) > p.$$

(Proposed by Fedor Malyshev, Steklov Math. Inst. and Ilya Bogdanov, MIPT, Moscow)

Solution. For an arbitrary $\varepsilon > 0$ we will establish a lower bound for the sum of perimeters that would tend to $+\infty$ as $\varepsilon \to +0$; this solves the problem.

Rotate and scale the picture so that one of the sides of K is the segment from $(0,0)$ to $(0,1)$, and stretch the picture horizontally in such a way that the projection of K to the x axis is $[0,1]$. Evidently, we may work with the lengths of the projections to the x or y axis instead of the perimeters and consider their sum, that is why we may make any affine transformation.

Let $f_i(a)$ be the length of intersection of the straight line $\{x = a\}$ with T_i and put $f(a) = \sum_i f_i(a)$. Then f is piece-wise increasing with possible downward gaps, $f(a) \le 1 - a$, and

$$\int_0^1 f(x)\,\mathrm{d}x \ge \frac{1}{2} - \varepsilon.$$

Let $d_1, \dots, d_N$ be the values of the gaps of f. Every gap is a sum of side-lengths of some of T_i and every T_i contributes to one of d_j, we therefore estimate the sum of the gaps of f.

In the points of differentiability of f we have $f'(a) \ge f(a)/a$; this follows from $f_i'(a) \ge f_i(a)/a$ after summation. Indeed, if f_i is zero this inequality holds trivially, and if not then $f_i' = 1$ and the inequality reads $f_i(a) \le a$, which is clear from the definition.

Choose an integer $m = \lfloor 1/(8\varepsilon) \rfloor$ (considering ε sufficiently small). Then for all $k = 0, 1, \dots, [(m-1)/2]$ in the section of K by the strip $k/m \le x \le (k+1)/m$ the area, covered by the small triangles T_i is no smaller than $1/(2m) - \varepsilon \ge 1/(4m)$. Thus

$$\int_{k/m}^{(k+1)/m} f'(x)\,\mathrm{d}x \ge \int_{k/m}^{(k+1)/m} \frac{f(x)\,\mathrm{d}x}{x} \ge \frac{m}{k+1} \int_{k/m}^{(k+1)/m} f(x)\,\mathrm{d}x \ge \frac{m}{k+1} \cdot \frac{1}{4m} = \frac{1}{4(k+1)}.$$

Hence,

$$\int_0^{1/2} f'(x)\,\mathrm{d}x \ge \frac{1}{4}\left(\frac{1}{1} + \dots + \frac{1}{[(m-1)/2]}\right).$$

The right hand side tends to infinity as $\varepsilon \to +0$. On the other hand, the left hand side equals

$$f(1/2) + \sum_{x_i < 1/2} d_i;$$

hence $\sum_i d_i$ also tends to infinity.

IMC 2018, Blagoevgrad, Bulgaria

Day 1, July 24, 2018

Problem 1. Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be two sequences of positive numbers. Show that the following statements are equivalent:

(1) There is a sequence $(c_n)_{n=1}^{\infty}$ of positive numbers such that $\sum_{n=1}^{\infty} \frac{a_n}{c_n}$ and $\sum_{n=1}^{\infty} \frac{c_n}{b_n}$ both converge;

(2) $\sum_{n=1}^{\infty} \sqrt{\frac{a_n}{b_n}}$ converges.

(Proposed by Tomáš Bárta, Charles University, Prague)

Solution. Note that the sum of a series with positive terms can be either finite or $+\infty$, so for such a series, "converges" is equivalent to "is finite".

Proof for (1) $\implies$ (2)*:* By the AM-GM inequality,

$$\sqrt{\frac{a_n}{b_n}} = \sqrt{\frac{a_n}{c_n} \cdot \frac{c_n}{b_n}} \le \frac{1}{2}\left(\frac{a_n}{c_n} + \frac{c_n}{b_n}\right),$$

so

$$\sum_{n=1}^{\infty} \sqrt{\frac{a_n}{b_n}} \le \frac{1}{2}\sum_{n=1}^{\infty} \frac{a_n}{c_n} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{c_n}{b_n} < +\infty.$$

Hence, $\sum_{n=1}^{\infty} \sqrt{\frac{a_n}{b_n}}$ is finite and therefore convergent.

Proof for (2) $\implies$ (1)*:* Choose $c_n = \sqrt{a_n b_n}$. Then

$$\frac{a_n}{c_n} = \frac{c_n}{b_n} = \sqrt{\frac{a_n}{b_n}}.$$

By the condition $\sum_{n=1}^{\infty} \sqrt{\frac{a_n}{b_n}}$ converges, therefore $\sum_{n=1}^{\infty} \frac{a_n}{c_n}$ and $\sum_{n=1}^{\infty} \frac{c_n}{b_n}$ converge, too.

Problem 2. Does there exist a field such that its multiplicative group is isomorphic to its additive group?

(Proposed by Alexandre Chapovalov, New York University, Abu Dhabi)

Solution. There exist no such field.

Suppose that F is such a field and $g\colon F^* \to F^+$ is a group isomorphism. Then $g(1) = 0$.

Let $a = g(-1)$. Then $2a = 2 \cdot g(-1) = g((-1)^2) = g(1) = 0$; so either $a = 0$ or char $F = 2$. If $a = 0$ then $-1 = g^{-1}(a) = g^{-1}(0) = 1$; we have char $F = 2$ in any case.

For every $x \in F$, we have $g(x^2) = 2g(x) = 0 = g(1)$, so $x^2 = 1$. But this equation has only one or two solutions. Hence F is the 2-element field; but its additive and multiplicative groups have different numbers of elements and are not isomorphic.

Problem 3. Determine all rational numbers a for which the matrix

$$\begin{pmatrix} a & -a & -1 & 0 \\ a & -a & 0 & -1 \\ 1 & 0 & a & -a \\ 0 & 1 & a & -a \end{pmatrix}$$

is the square of a matrix with all rational entries.

(Proposed by Daniël Kroes, University of California, San Diego)

Solution. We will show that the only such number is $a = 0$.

Let $A = \begin{pmatrix} a & -a & -1 & 0 \\ a & -a & 0 & -1 \\ 1 & 0 & a & -a \\ 0 & 1 & a & -a \end{pmatrix}$ and suppose that $A = B^2$. It is easy to compute the characteristic polynomial of A, which is

$$p_A(x) = \det(A - xI) = (x^2 + 1)^2.$$

By the Cayley-Hamilton theorem we have $p_A(B^2) = p_A(A) = 0$.

Let $\mu_B(x)$ be the minimal polynomial of B. The minimal polynomial divides all polynomials that vanish at B; in particular $\mu_B(x)$ must be a divisor of the polynomial $p_A(x^2) = (x^4 + 1)^2$. The polynomial $\mu_B(x)$ has rational coefficients and degree at most 4. On the other hand, the polynomial $x^4 + 1$, being the 8th cyclotomic polynomial, is irreducible in $\mathbb{Q}[x]$. Hence the only possibility for μ_B is $\mu_B(x) = x^4 + 1$. Therefore,

$$A^2 + I = \mu_B(B) = 0. \tag{1}$$

Since we have

$$A^2 + I = \begin{pmatrix} 0 & 0 & -2a & 2a \\ 0 & 0 & -2a & 2a \\ 2a & -2a & 0 & 0 \\ 2a & -2a & 0 & 0 \end{pmatrix},$$

the relation (1) forces $a = 0$.

In case $a = 0$ we have

$$A = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^2,$$

hence $a = 0$ satisfies the condition.

Problem 4. Find all differentiable functions $f : (0, \infty) \to \mathbb{R}$ such that

$$f(b) - f(a) = (b - a) f'\left(\sqrt{ab}\right) \quad \text{for all} \quad a, b > 0. \tag{2}$$

(Proposed by Orif Ibrogimov, National University of Uzbekistan)

Solution. First we show that f is infinitely many times differentiable. By substituting $a = \frac{1}{2}t$ and $b = 2t$ in (2),

$$f'(t) = \frac{f(2t) - f(\frac{1}{2}t)}{\frac{3}{2}t}. \tag{3}$$

Inductively, if f is k times differentiable then the right-hand side of (3) is k times differentiable, so the $f'(t)$ on the left-hand-side is k times differentiable as well; hence f is $k+1$ times differentiable.

Now substitute $b = e^h t$ and $a = e^{-h} t$ in (2), differentiate three times with respect to h then take limits with $h \to 0$:

$$f(e^h t) - f(e^{-h} t) - (e^h t - e^{-h} t) f(t) = 0$$

$$\left(\frac{\partial}{\partial h}\right)^3 \Big(f(e^h t) - f(e^{-h} t) - (e^h t - e^{-h} t) f(t)\Big) = 0$$

$$e^{3h} t^3 f'''(e^h t) + 3e^{2h} t^2 f''(e^h t) + e^h t f'(e^h t) + e^{-3h} t^3 f'''(e^{-h} t) + 3e^{-2h} t^2 f''(e^{-h} t) + e^{-h} t f'(e^{-h} t) -$$
$$-(e^h t + e^{-h} t) f'(t) = 0$$

$$2t^3 f'''(t) + 6t^2 f''(t) = 0$$

$$t f'''(t) + 3f''(t) = 0$$

$$(t\, f(t))''' = 0.$$

Consequently, $tf(t)$ is an at most quadratic polynomial of t, and therefore

$$f(t) = C_1 t + \frac{C_2}{t} + C_3 \tag{4}$$

with some constants C_1, C_2 and C_3.

It is easy to verify that all functions of the form (4) satisfy the equation (1).

Problem 5. Let p and q be prime numbers with $p < q$. Suppose that in a convex polygon $P_1 P_2 \dots P_{pq}$ all angles are equal and the side lengths are distinct positive integers. Prove that

$$P_1P_2 + P_2P_3 + \cdots + P_kP_{k+1} \ge \frac{k^3 + k}{2}$$

holds for every integer k with $1 \le k \le p$.

(Proposed by Ander Lamaison Vidarte, Berlin Mathematical School, Berlin)

Solution. Place the polygon in the complex plane counterclockwise, so that $P_2 - P_1$ is a positive real number. Let $a_i = |P_{i+2} - P_{i+1}|$, which is an integer, and define the polynomial $f(x) = a_{pq-1} x^{pq-1} + \cdots + a_1 x + a_0$. Let $\omega = e^{\frac{2\pi i}{pq}}$; then $P_{i+1} - P_i = a_{i-1} \omega^{i-1}$, so $f(\omega) = 0$.

The minimal polynomial of ω over $\mathbb{Q}[x]$ is the cyclotomic polynomial $\Phi_{pq}(x) = \frac{(x^{pq}-1)(x-1)}{(x^p-1)(x^q-1)}$, so $\Phi_{pq}(x)$ divides $f(x)$. At the same time, $\Phi_{pq}(x)$ is the greatest common divisor of $s(x) = \frac{x^{pq}-1}{x^p-1} = \Phi_q(x^p)$ and $t(x) = \frac{x^{pq}-1}{x^q-1} = \Phi_p(x^q)$, so by Bézout's identity (for real polynomials), we can write $f(x) = s(x)u(x) + t(x)v(x)$, with some polynomials $u(x), v(x)$. These polynomials can be replaced by $u^*(x) = u(x) + w(x)\frac{x^p-1}{x-1}$ and $v^*(x) = v(x) - w(x)\frac{x^q-1}{x-1}$, so without loss of generality we may assume that $\deg u \le p-1$. Since $\deg a = pq-1$, this forces $\deg v \le q-1$.

Let $u(x) = u_{p-1}x^{p-1} + \cdots + u_1 x + u_0$ and $v(x) = v_{q-1}x^{q-1} + \cdots + v_1 x + v_0$. Denote by (i,j) the unique integer $n \in \{0, 1, \dots, pq-1\}$ with $n \equiv i \pmod p$ and $n \equiv j \pmod q$. By the choice of s and t, we have $a_{(i,j)} = u_i + v_j$. Then

$$P_1P_2 + \cdots + P_kP_{k+1} = \sum_{i=0}^{k-1} a_{(i,i)} = \sum_{i=0}^{k-1} u_i + v_i = \frac{1}{k} \sum_{i=0}^{k-1} \sum_{j=0}^{k-1} (u_i + v_j)$$
$$= \frac{1}{k} \sum_{i=0}^{k-1} \sum_{j=0}^{k-1} a_{(i,j)} \overset{(*)}{\ge} \frac{1}{k}\left(1 + 2 + \cdots + k^2\right) = \frac{k^3 + k}{2}$$

where $(*)$ uses the fact that the numbers (i,j) are pairwise different.

IMC 2018, Blagoevgrad, Bulgaria

Day 2, July 25, 2018

Problem 6. Let k be a positive integer. Find the smallest positive integer n for which there exist k nonzero vectors $v_1, \dots, v_k$ in $\mathbb{R}^n$ such that for every pair i, j of indices with $|i-j| > 1$ the vectors v_i and v_j are orthogonal.

(Proposed by Alexey Balitskiy, Moscow Institute of Physics and Technology and M.I.T.)

Solution. First we prove that if $2n+1 \le k$ then no sequence $v_1, \dots, v_k$ of vectors can satisfy the condition. Suppose to the contrary that $v_1, \dots, v_k$ are vectors with the required property and consider the vectors

$$v_1, v_3, v_5, \dots, v_{2n+1}.$$

By the condition these $n+1$ vectors should be pairwise orthogonal, but this is not possible in $\mathbb{R}^n$.

Next we show a possible construction for every pair k, n of positive integers with $2n \ge k$. Take an orthogonal basis $(e_1, \dots, e_n)$ of $\mathbb{R}^n$ and consider the vectors

$$v_1 = v_2 = e_1, \quad v_3 = v_4 = e_2, \quad \dots, \quad v_{2n-1} = v_{2n} = e_n.$$

For every pair (i, j) of indices with $1 \le i, j \le 2n$ and $|i-j| > 1$ the vectors v_i and v_j are distinct basis vectors, so they are orthogonal. Evidently the subsequence $v_1, v_2, \dots, v_k$ also satisfies the same property.

Hence, such a sequence of vectors exists if and only if $2n \ge k$; that is, for a fixed k, the smallest suitable n is $\left\lceil \dfrac{k}{2} \right\rceil$.

Problem 7. Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers such that $a_0 = 0$ and

$$a_{n+1}^3 = a_n^2 - 8 \quad \text{for} \quad n = 0, 1, 2, \dots$$

Prove that the following series is convergent:

$$\sum_{n=0}^{\infty} |a_{n+1} - a_n|. \tag{1}$$

(Proposed by Orif Ibrogimov, National University of Uzbekistan)

Solution. We will estimate the ratio between the terms $|a_{n+2} - a_{n+1}|$ and $|a_{n+1} - a_n|$.

Before doing that, we localize the numbers a_n; we prove that

$$-2 \le a_n \le -\sqrt[3]{4} \quad \text{for } n \ge 1. \tag{2}$$

The lower bound simply follows from the recurrence: $a_n = \sqrt[3]{a_{n-1}^2 - 8} \ge \sqrt[3]{-8} = -2$. The proof of the upper bound can be done by induction: we have $a_1 = -2 < -\sqrt[3]{4}$, and whenever $-2 \le a_n < 0$, it follows that $a_{n+1} = \sqrt[3]{a_n^2 - 8} \le \sqrt[3]{2^2 - 8} = -\sqrt[3]{4}$.

Now compare $|a_{n+2}-a_{n+1}|$ with $|a_{n+1}-a_n|$. By applying $x^3-y^3=(x-y)(x^2+xy+y^2)$, $x^2-y^2=(x-y)(x+y)$ and the recurrence,

$$(a_{n+2}^2+a_{n+2}a_{n+1}+a_{n+1}^2)\cdot|a_{n+2}-a_{n+1}| = \\ = |a_{n+2}^3-a_{n+1}^3| = \big|(a_{n+1}^2-8)-(a_n^2-8)\big| = \\ = |a_{n+1}+a_n|\cdot|a_{n+1}-a_n|.$$

On the left-hand side we have

$$a_{n+2}^2+a_{n+2}a_{n+1}+a_{n+1}^2 \geq 3\cdot 4^{2/3};$$

on the right-hand side

$$|a_{n+1}+a_n|\leq 4.$$

Hence,

$$|a_{n+2}-a_{n+1}| \leq \frac{4}{3\cdot 4^{2/3}}|a_{n+1}-a_n| = \frac{\sqrt[3]{4}}{3}|a_{n+1}-a_n|.$$

By a trivial induction it follows that

$$|a_{n+1}-a_n| < \left(\frac{\sqrt[3]{4}}{3}\right)^{n-1}|a_2-a_1|.$$

Hence the series $\sum_{n=0}^{\infty}|a_{n+1}-a_n|$ can be majorized by a geometric series with quotient $\dfrac{\sqrt[3]{4}}{3}<1$; that proves that the series converges.

Problem 8. Let $\Omega=\{(x,y,z)\in\mathbb{Z}^3 : y+1\geq x\geq y\geq z\geq 0\}$. A frog moves along the points of Ω by jumps of length 1. For every positive integer n, determine the number of paths the frog can take to reach (n,n,n) starting from $(0,0,0)$ in exactly $3n$ jumps.

(Proposed by Fedor Petrov and Anatoly Vershik, St. Petersburg State University)

Solution. Let $\Psi=\{(u,v)\in\mathbb{Z}^3 : v\geq 0, u\geq 2v\}$. Notice that the map $\pi:\Omega\to\Psi$, $\pi(x,y,z)=(x+y,z)$ is a bijection between the two sets; moreover π projects all allowed paths of the frogs to paths inside the set Ψ, using only unit jump vectors. Hence, we are interested in the number of paths from $\pi(0,0,0)=(0,0)$ to $\pi(n,n,n)=(2n,n)$ in the set Ψ, using only jumps $(1,0)$ and $(0,1)$.

For every lattice point $(u,v)\in\Psi$, let $f(u,v)$ be the number of paths from $(0,0)$ to (u,v) in Ψ with $u+v$ jumps. Evidently we have $f(0,0)=1$. Extend this definition to the points with $v=-1$ and $2v=u+1$ by setting

$$f(u,-1)=0,\quad f(2v-1,v)=0. \tag{3}$$

To any point (u,v) of Ψ other than the origin, the path can come either from $(u-1,v)$ or from $(u,v-1)$, so

$$f(u,v)=f(u-1,v)+f(u,v-1)\quad\text{for } (u,v)\in\Psi\setminus\{(0,0)\}. \tag{4}$$

If we ignore the boundary condition (3), there is a wide family of functions that satisfy (4); namely, for every integer c, $(u,v)\mapsto\binom{u+v}{v+c}$ is such a function, with defining this binomial coefficient to be 0 if $v+c$ is negative or greater than $u+v$.

Along the line $2v=u+1$ we have $\binom{u+v}{v}=\binom{3v-1}{v}=2\binom{3v-1}{v-1}=2\binom{u+v}{v-1}$. Hence, the function

$$f^*(u,v)=\binom{u+v}{v}-2\binom{u+v}{v-1}$$

satisfies (3), (4) and $f(0,0)=1$. These properties uniquely define the function f, so $f=f^*$.

In particular, the number of paths of the frog from $(0,0,0)$ to (n,n,n) is

$$f(\pi(n,n,n)) = f(2n,n) = \binom{3n}{n} - 2\binom{3n}{n-1} = \frac{\binom{3n}{n}}{2n+1}.$$

Remark. There exist direct proofs for the formula $\binom{3n}{n}/(2n+1)$. For instance, we can replicate the well-known proof of the formula for the Catalan numbers using the Cycle Lemma of Dvoretzky and Motzkin (related to the petrol station replenishment problem). See `https://en.wikipedia.org/wiki/Catalan_number#Sixth_proof`

Problem 9. Determine all pairs $P(x)$, $Q(x)$ of complex polynomials with leading coefficient 1 such that $P(x)$ divides $Q(x)^2+1$ and $Q(x)$ divides $P(x)^2+1$.

(Proposed by Rodrigo Angelo, Princeton University and Matheus Secco, PUC, Rio de Janeiro)

Solution. The answer is all pairs $(1,1)$ and $(P, P+i)$, $(P, P-i)$, where P is a non-constant monic polynomial in $\mathbb{C}[x]$ and i is the imaginary unit.

Notice that if $P|Q^2+1$ and $Q|P^2+1$ then P and Q are coprime and the condition is equivalent with $PQ|P^2+Q^2+1$.

Lemma. If $P, Q \in \mathbb{C}[x]$ are monic polynomials such that P^2+Q^2+1 is divisible by PQ, then $\deg P = \deg Q$.

Proof. Assume for the sake of contradiction that there is a pair (P,Q) with $\deg P \neq \deg Q$. Among all these pairs, take the one with smallest sum $\deg P + \deg Q$ and let (P,Q) be such pair. Without loss of generality, suppose that $\deg P > \deg Q$. Let S be the polynomial such that

$$\frac{P^2+Q^2+1}{PQ} = S.$$

Notice that P a solution of the polynomial equation $X^2 - QSX + Q^2 + 1 = 0$, in variable X. By Vieta's formulas, the other solution is $R = QS - P = \dfrac{Q^2+1}{P}$. By $R = QS - P$, the R is indeed a polynomial, and because P, Q are monic, $R = \dfrac{Q^2+1}{P}$ is also monic. Therefore the pair (R, Q) satisfies the conditions of the Lemma. Notice that $\deg R = 2\deg Q - \deg P < \deg P$, which contradicts the minimality of $\deg P + \deg Q$. This contradiction establishes the Lemma.

By the Lemma, we have that $\deg(PQ) = \deg(P^2+Q^2+1)$ and therefore $\dfrac{P^2+Q^2+1}{PQ}$ is a constant polynomial. If P and Q are constant polynomials, we have $P=Q=1$. Assuming that $\deg P = \deg Q \geq 1$, as P and Q are monic, the leading coefficient of P^2+Q^2+1 is 2 and the leading coefficient of PQ is 1, which give us $\dfrac{P^2+Q^2+1}{PQ} = 2$. Finally we have that $P^2+Q^2+1 = 2PQ$ and therefore $(P-Q)^2 = -1$, i.e $Q = P+i$ or $Q = P-i$. It's easy to check that these pairs are indeed solutions of the problem.

Problem 10. For $R > 1$ let $\mathcal{D}_R = \{(a,b) \in \mathbb{Z}^2 \colon 0 < a^2 + b^2 < R\}$. Compute

$$\lim_{R\to\infty} \sum_{(a,b)\in\mathcal{D}_R} \frac{(-1)^{a+b}}{a^2+b^2}.$$

(Proposed by Rodrigo Angelo, Princeton University and Matheus Secco, PUC, Rio de Janeiro)

Solution. Define $\mathcal{E}_R = \{(a,b) \in \mathbb{Z}^2 \setminus \{(0,0)\} : a^2+b^2 < R \text{ and } a+b \text{ is even}\}$. Then

$$\sum_{(a,b)\in\mathcal{D}_R} \frac{(-1)^{a+b}}{a^2+b^2} = 2\sum_{(a,b)\in\mathcal{E}_R} \frac{1}{a^2+b^2} - \sum_{(a,b)\in\mathcal{D}_R} \frac{1}{a^2+b^2}. \tag{5}$$

But $a+b$ is even if and only if one can write $(a,b) = (m-n, m+n)$, and such m,n are unique. Notice also that $a^2+b^2 = (m-n)^2 + (m+n)^2 = 2m^2+2n^2$, hence $a^2+b^2 < R$ if and only if $m^2+n^2 < R/2$. With that we get:

$$2\sum_{(a,b)\in\mathcal{E}_R} \frac{1}{a^2+b^2} = 2\sum_{(m,n)\in D_{R/2}} \frac{1}{(m-n)^2+(m+n)^2} = \sum_{(m,n)\in D_{R/2}} \frac{1}{m^2+n^2}. \tag{6}$$

Replacing (6) in (5), we obtain

$$\sum_{(a,b)\in\mathcal{D}_R} \frac{(-1)^{a+b}}{a^2+b^2} = -\sum_{R/2\le a^2+b^2<R} \frac{1}{a^2+b^2},$$

where the second sum is evaluated for a and b integers.

Denote by $N(r)$ the number of lattice points in the open disk $x^2+y^2<r^2$. Along the circle with radius r with $\sqrt{R/2} \le r < \sqrt{R}$, there are $N(r+0) - N(r-0)$ lattice points; each of them contribute $\frac{1}{r^2}$ in the sum (7). So we can re-write the sum as a Stieltjes integral:

$$\sum_{R/2\le a^2+b^2<R} \frac{1}{a^2+b^2} = \int_{\sqrt{R/2}}^{\sqrt{R}} \frac{1}{r^2}\,\mathrm{d}N(r).$$

It is well-known that $N(r) = \pi r^2 + O(r)$. (Putting a unit square around each lattice point, these squares cover the disk with radius $r-1$ and lie inside the disk with radius $r+1$, so there their total area is between $\pi(r-1)^2$ and $\pi(r+1)^2$). By integrating by parts,

$$\begin{aligned}
\int_{\sqrt{R/2}}^{\sqrt{R}} \frac{1}{r^2}\,\mathrm{d}N(r) &= \left[\frac{1}{r^2}N(r)\right]_{\sqrt{R/2}}^{\sqrt{R}} + \int_{\sqrt{R/2}}^{\sqrt{R}} \frac{2}{r^3}N(r)\,\mathrm{d}r \\
&= \left[\frac{\pi r^2 + O(r)}{r^2}\right]_{\sqrt{R/2}}^{\sqrt{R}} + 2\int_{\sqrt{R/2}}^{\sqrt{R}} \frac{\pi r^2 + O(r)}{r^3}\,\mathrm{d}r \\
&= 2\pi \int_{\sqrt{R/2}}^{\sqrt{R}} \frac{\mathrm{d}r}{r} + O\Big(1/\sqrt{R}\Big) = \pi\log 2 + O\Big(1/\sqrt{R}\Big).
\end{aligned}$$

Therefore,

$$\lim_{R\to\infty} \sum_{(a,b)\in\mathcal{D}_R} \frac{(-1)^{a+b}}{a^2+b^2} = -\lim_{R\to\infty} \sum_{R/2\le a^2+b^2<R} \frac{1}{a^2+b^2} = -\lim_{R\to\infty} \int_{\sqrt{R/2}}^{\sqrt{R}} \frac{1}{r^2}\,\mathrm{d}N(r) = -\pi\log 2.$$

IMC 2019, Blagoevgrad, Bulgaria

Day 1, July 30, 2019

Problem 1. Evaluate the product

$$\prod_{n=3}^{\infty} \frac{(n^3+3n)^2}{n^6-64}.$$

Proposed by Orif Ibrogimov, ETH Zurich and National University of Uzbekistan and Karen Keryan, Yerevan State University and American University of Armenia, Yerevan

Hint: Telescoping product.

Solution. Let

$$a_n = \frac{(n^3+3n)^2}{n^6-64}.$$

Notice that

$$a_n = \frac{(n^3+3n)^2}{(n^3-8)(n^3+8)} = \frac{n^2(n^2+3)^2}{(n-2)(n^2+2n+4)\cdot(n+2)(n^2-2n+4)}$$
$$= \frac{n}{n-2}\cdot\frac{n}{n+2}\cdot\frac{n^2+3}{(n-1)^2+3}\cdot\frac{n^2+3}{(n+1)^2+3}.$$

Hence, for $N \ge 3$ we have

$$\prod_{n=3}^{N} a_n = \left(\prod_{n=3}^{N}\frac{n}{n-2}\right)\left(\prod_{n=3}^{N}\frac{n}{n+2}\right)\left(\prod_{n=3}^{N}\frac{n^2+3}{(n-1)^2+3}\right)\left(\prod_{n=3}^{N}\frac{n^2+3}{(n+1)^2+3}\right)$$
$$= \frac{N(N-1)}{1\cdot 2}\cdot\frac{3\cdot 4}{(N+1)(N+2)}\cdot\frac{N^2+3}{2^2+3}\cdot\frac{3^2+3}{(N+1)^2+3}$$
$$= \frac{72}{7}\cdot\frac{N(N-1)(N^2+3)}{(N+1)(N+2)\big((N+1)^2+3\big)}$$
$$= \frac{72}{7}\cdot\frac{(1-\frac{1}{N})(1+\frac{3}{N^2})}{(1+\frac{1}{N})(1+\frac{2}{N})\big((1+\frac{1}{N})^2+\frac{3}{N^2}\big)},$$

so

$$\prod_{n=3}^{\infty} a_n = \lim_{N\to\infty}\prod_{n=3}^{N} a_n = \lim_{N\to\infty}\left(\frac{72}{7}\cdot\frac{(1-\frac{1}{N})(1+\frac{3}{N^2})}{(1+\frac{1}{N})(1+\frac{2}{N})\big((1+\frac{1}{N})^2+\frac{3}{N^2}\big)}\right) = \frac{72}{7}.$$

Problem 2. A four-digit number $YEAR$ is called *very good* if the system

$$\begin{aligned} Yx+Ey+Az+Rw &= Y \\ Rx+Yy+Ez+Aw &= E \\ Ax+Ry+Yz+Ew &= A \\ Ex+Ay+Rz+Yw &= R \end{aligned}$$

of linear equations in the variables x, y, z and w has at least two solutions. Find all very good YEARs in the 21st century.

(The 21st century starts in 2001 and ends in 2100.)

Proposed by Tomáš Bárta, Charles University, Prague

Hint: If the solution of the system is not unique then $\det\begin{pmatrix} Y & E & A & R \\ R & Y & E & A \\ A & R & Y & E \\ E & A & R & Y \end{pmatrix} = 0.$

Solution. Let us apply row transformations to the augmented matrix of the system to find its rank. First we add the second, third and fourth row to the first one and divide by $Y+E+A+R$ to get

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ R & Y & E & A & E \\ A & R & Y & E & A \\ E & A & R & Y & R \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & Y-R & E-R & A-R & E-R \\ 0 & R-A & Y-A & E-A & 0 \\ 0 & A-E & R-E & Y-E & R-E \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & Y-R & E-R & A-R & E-R \\ 0 & R-A & Y-A & E-A & 0 \\ 0 & A-E+Y-R & 0 & Y-E+A-R & 0 \end{pmatrix}$$

Let us first omit the last column and look at the remaining 4×4 matrix. If $E \neq R$, the first and second rows are linearly independent, so the rank of the matrix is at least 2 and rank of the augmented 4×5 matrix cannot be bigger than rank of the 4×4 matrix due to the zeros in the last column.

IF $E = R$, then we have three zeros in the last column, so rank of the 4×5 matrix cannot be bigger than rank of the 4×4 matrix. So, the original system has always at least one solution.

It follows that the system has more than one solution if and only if the 4×4 matrix (with the last column omitted) is singular. Let us first assume that $E \neq R$. We apply one more transform to get

$$\sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & Y-R & E-R & A-R \\ 0 & (R-A)(E-R)-(Y-R)(Y-A) & 0 & (E-A)(E-R)-(A-R)(Y-A) \\ 0 & A-E+Y-R & 0 & Y-E+A-R \end{pmatrix}$$

Obviously, this matrix is singular if and only if $A-E+Y-R = 0$ or the two expressions in the third row are equal, i.e.

$$RE - R^2 - AE + AR - Y^2 + RY + AY - AR = E^2 - AE - ER + AR - AY + RY + A^2 - AR$$

$$0 = (E-R)^2 + (A-Y)^2,$$

but this is impossible if $E \neq R$. If $E = R$, we have

$$\sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & Y-R & 0 & A-R \\ 0 & R-A & Y-A & R-A \\ 0 & A+Y-2R & 0 & Y+A-2R \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & Y-R & 0 & A-R \\ 0 & R-A & Y-A & R-A \\ 0 & A-R & 0 & Y-R \end{pmatrix}.$$

If $A = Y$, this matrix is singular. If $A \neq Y$, the matrix is regular if and only if $(Y-R)^2 \neq (A-R)^2$ and since $Y \neq A$, it means that $Y-R \neq -(A-R)$, i.e. $Y+A \neq 2R$. We conclude that YEAR is very good if and only if

1. $E \neq R$ and $A+Y = E+R$, or
2. $E = R$ and $Y = A$, or
3. $E = R$, $A \neq Y$ and $Y+A = 2R$.

We can see that if $Y = 2$, $E = 0$, then the very good years satisfying 1 are $A+2 = R \neq 0$, i.e. 2002, 2013, 2024, 2035, 2046, 2057, 2068, 2079, condition 2 is satisfied for 2020 and condition 3 never satisfied.

Problem 3. Let $f : (-1, 1) \to \mathbb{R}$ be a twice differentiable function such that

$$2f'(x) + xf''(x) \geq 1 \quad \text{for } x \in (-1, 1).$$

Prove that

$$\int_{-1}^{1} xf(x)\,\mathrm{d}x \geq \frac{1}{3}.$$

Proposed by Orif Ibrogimov, ETH Zurich and National University of Uzbekistan and Karim Rakhimov, Scuola Normale Superiore and National University of Uzbekistan

Hint: $2f'(x) + xf''(x)$ is the second derivative of a certain function.

Solution. Let

$$g(x) = xf(x) - \frac{x^2}{2}.$$

Notice that

$$g''(x) = 2f'(x) + xf''(x) - 1 \geq 0,$$

so g is convex. Estimate g by its tangent at 0: let $g'(0) = a$, then

$$g(x) = g(0) + g'(0)x = ax$$

and therefore

$$\int_{-1}^{1} xf(x)\,\mathrm{d}x = \int_{-1}^{1} \Big(g(x) + \frac{x^2}{2}\Big)\,\mathrm{d}x \geq \int_{-1}^{1} \Big(ax + \frac{x^2}{2}\Big)\,\mathrm{d}x = \frac{1}{3}.$$

Problem 4. Define the sequence $a_0, a_1, \ldots$ of numbers by the following recurrence:

$$a_0 = 1, \quad a_1 = 2, \quad (n+3)a_{n+2} = (6n+9)a_{n+1} - na_n \quad \text{for } n \geq 0.$$

Prove that all terms of this sequence are integers.

Proposed by Khakimboy Egamberganov, ICTP, Italy

Hint: Determine the generating function $\sum a_n x^n$.

Solution. Take the generating function of this sequence

$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

It is easy to see that the sequence is increasing and

$$\frac{a_{n+1}}{a_n} = \frac{(6n+3)a_n - (n-1)a_{n-1}}{(n+2)a_n} < \frac{6n+3}{n+2} \quad \Rightarrow \quad \limsup_{n\to\infty} \frac{a_{n+1}}{a_n} \leq 6.$$

So the generating function converges in some neighbourhood of 0. Then, we have

$$f(x) = 1+2x+\sum_{n=2}^{\infty} a_n x^n = 1+2x+\sum_{n=0}^{\infty} a_{n+2}x^{n+2} = 1+2x+\sum_{n=0}^{\infty} \frac{6n+9}{n+3}a_{n+1}x^{n+2} - \sum_{n=0}^{\infty} \frac{n}{n+3}a_n x^{n+2}.$$

Let $f_1(x)=\sum_{n=0}^{\infty}\frac{6n+9}{n+3}a_{n+1}x^{n+2}$ and $f_2(x)=\sum_{n=0}^{\infty}\frac{n}{n+3}a_nx^{n+2}$. Then

$$(xf_1(x))'=\sum_{n=0}^{\infty}(6n+9)a_{n+1}x^{n+2}=6x^2\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n+3x\sum_{n=0}^{\infty}a_{n+1}x^{n+1}=6x^2f'(x)+3x(f(x)-1)$$

and

$$(xf_2(x))'=\sum_{n=0}^{\infty}na_nx^{n+2}=x^2\sum_{n=0}^{\infty}(n+1)a_nx^n-x^2\sum_{n=0}^{\infty}a_nx^n=x^2(xf(x))'-x^2f(x)=x^3f'(x).$$

Using this relations, we arrive at the following differential equation for f:

$$(xf(x))'=1+4x+(xf_1(x))'-(xf_2(x))'=1+x+(6x^2-x^3)f'(x)+3xf(x)$$

or, equivalently,

$$(x^3-6x^2+x)f'(x)+(1-3x)f(x)-1-x=0.$$

So, we need solve this differential equation in some sufficiently smaller neighbourhood of 0. We know that $f(0)=1$ and we need a neighbourhood of 0 such that $x^2-6x+1>0$. Then

$$f'(x)+\frac{1-3x}{x(x^2-6x+1)}f(x)=\frac{1+x}{x(x^2-6x+1)}$$

for $x\neq 0$. So the integral multiplier is $\mu(x)=\frac{x}{\sqrt{x^2-6x+1}}$ and

$$(f(x)\mu(x))'=\frac{x+x^2}{(x^2-6x+1)^{\frac{3}{2}}},$$

so

$$f(x)=\left(\frac{1-x}{2\sqrt{x^2-6x+1}}-\frac{1}{2}\right)\frac{\sqrt{x^2-6x+1}}{x}=\frac{1-x-\sqrt{x^2-6x+1}}{2x}.$$

We found the generating function of (a_n) in some neighbourhood of 0, which $x^2-6x+1>0$. So our series uniformly converges to $f(x)=\frac{1-x-\sqrt{x^2-6x+1}}{2x}$ in $|x|<3-2\sqrt{2}$.

Instead of computing the coefficients of the Taylor series of $f(x)$ directly, we will find another recurrence relation for (a_n). It is easy to see that $f(x)$ satisfies the quadratic equation $xt^2-(1-x)t+1=0$. So

$$xf(x)^2-(1-x)f(x)+1=0.$$

Then

$$x\left(\sum_{n=0}^{\infty}a_nx^n\right)^2+1=\sum_{n=0}^{\infty}a_nx^n-\sum_{n=0}^{\infty}a_nx^{n+1}\quad\Rightarrow\quad\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}a_ka_{n-k}\right)x^{n+1}=\sum_{n=0}^{\infty}(a_{n+1}-a_n)x^{n+1}$$

and from here, we get

$$a_{n+1}=a_n+\sum_{k=0}^{n}a_ka_{n-k}.$$

If $a_0,a_1,...,a_n$ be integers, then a_{n+1} is also integer. We know that $a_0=1,a_1=2$ are integer numbers, so all terms of the sequence (a_n) are integers by induction.

Problem 5. Determine whether there exist an odd positive integer n and $n\times n$ matrices A and B with integer entries, that satisfy the following conditions:

(1) $\det(B)=1$;

(2) $AB=BA$;

(3) $A^4+4A^2B^2+16B^4=2019I$.

(Here I denotes the $n\times n$ identity matrix.)

Proposed by Orif Ibrogimov, ETH Zurich and National University of Uzbekistan

Hint: Consider the determinants modulo 4.

Remark. The proposed solution was more complicated and involved; during the contest it turned out that a signficantly simplified solution exists – which we now provide below.

Solution 1. We show that there are no such matrices.
Notice that $A^4 + 4A^2B^2 + 16B^4$ can factorized as

$$A^4 + 4A^2B^2 + 16B^4 = (A^2 + 2AB + 4B^2)(A^2 - 2AB + 4B^2).$$

Let $C = A^2 + 2AB + 4B^2$ and $D = A^2 - 2AB + 4B^2$ be the two factors above. Then

$$\det C \cdot \det D = \det(CD) = \det(A^4 + 4A^2B^2 + 16B^4) = \det(2019I) = 2019^n.$$

The matrices C, D have integer entries, so their determinants are integers. Moreover, from $C \equiv D \pmod 4$ we can see that

$$\det C \equiv \det D \pmod 4.$$

This implies that $\det C \cdot \det D \equiv (\det C)^2 \pmod 4$, but this is a contradiction because $2019^n \equiv 3 \pmod 4$ is a quadratic nonresidue modulo 4.

Solution 2. Notice that

$$A^4 \equiv A^4 + 4A^2B^2 + 16B^4 = 2019I \mod 4$$

so

$$(\det A)^4 = \det A^4 \equiv \det(2109I) = 2019^n \pmod 4.$$

But $2019^n \equiv 3$ is a quadratic nonresidue modulo 4, contradiction.

IMC 2019, Blagoevgrad, Bulgaria

Day 2, July 31, 2019

Problem 6. Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ be continuous functions such that g is differentiable. Assume that $\big(f(0) - g'(0)\big)\big(g'(1) - f(1)\big) > 0$. Show that there exists a point $c \in (0,1)$ such that $f(c) = g'(c)$.

Proposed by Fereshteh Malek, K. N. Toosi University of Technology

Solution. Define $F(x) = \int_0^x f(t)\mathrm{d}t$ and let $h(x) = F(x) - g(x)$. By the continuouity of f we have $F' = f$, so $h' = f - g'$.

The assumption can be re-written as $h'(0)\big(-h'(1)\big) > 0$, so $h'(0)$ and $h'(1)$ have opposite signs. Then, by the Mean Value Theorem For Derivatives (Darboux property of derivatives) it follows that there is a point c between 0 and 1 where $h'(c) = 0$, so $f(c) = g'(c)$.

Problem 7.

Let $C = \{4, 6, 8, 9, 10, \ldots\}$ be the set of composite positive integers. For each $n \in C$ let a_n be the smallest positive integer k such that $k!$ is divisible by n. Determine whether the following series converges:

$$\sum_{n \in C} \left(\frac{a_n}{n}\right)^n. \tag{1}$$

Proposed by Orif Ibrogimov, ETH Zurich and National University of Uzbekistan

Solution. The series (1) converges. We will show that $\frac{a_n}{n} \le \frac{2}{3}$ for $n > 4$; then the geometric series $\sum \left(\frac{2}{3}\right)^n$ majorizes (1).

Case 1: n has at least two distinct prime divisors. Then n can be factored as $n = qr$ with some co-prime positive integers $q, r \ge 2$; without loss of generality we can assume $q > r$. Notice that $q \,\big|\, q!$ and $r \,\big|\, r! \,\big|\, q!$, so $n = qr \,\big|\, q!$; this shows $a_n \le q$ and therefore

$$\frac{a_n}{n} \le \frac{q}{n} = \frac{1}{r} \le \frac{1}{2}.$$

Case 2: n is the square of a prime, $n = p^2$ with some prime $p \ge 3$. From $p^2 \,\big|\, p \cdot 2p \,\big|\, (2p)!$ we obtain $a_n = 2p$, so

$$\frac{a_n}{n} = \frac{2p}{p^2} = \frac{2}{p} \le \frac{2}{3}.$$

Case 3: n is a prime power, $n = p^k$ with some prime p and $k \ge 3$. Notice that $n = p^k \,\big|\, p \cdot p^2 \cdots p^{k-1}$, so $a_n \le p^{k-1}$ and therefore

$$\frac{a_n}{n} \le \frac{p^{k-1}}{p^k} = \frac{1}{p} \le \frac{1}{2}.$$

Problem 8. Let $x_1, \ldots, x_n$ be real numbers. For any set $I \subset \{1, 2, \ldots, n\}$ let $s(I) = \sum_{i \in I} x_i$. Assume that the function $I \mapsto s(I)$ takes on at least 1.8^n values where I runs over all 2^n subsets of $\{1, 2, \ldots, n\}$. Prove that the number of sets $I \subset \{1, 2, \ldots, n\}$ for which $s(I) = 2019$ does not exceed 1.7^n.

Proposed by Fedor Part and Fedor Petrov, St. Petersburg State University

Solution. Choose disctint sets $I_1, \ldots, I_A \subset \{1, 2, \ldots, n\}$ where $A \geq 1.8^n$, and let $J_1, \ldots, J_B \subset \{1, 2, \ldots, n\}$ be all sets so that $S(J_i) = 2019$; for the sake of contradiction, assume that $B \geq 1.7^n$.

Every set $I \subset \{1, 2, \ldots, n\}$ can be identified with a $0-1$ vector of length n: the kth coordinate in the vector is 1 if $k \in I$. Then $s(I) = \langle I, X \rangle$, where $X = (x_1, \ldots, x_n)$ and $\langle \cdot, \cdot \rangle$ stands for the usual scalar product.

For all ordered pairs $(a, b) \in \{1, \ldots, A\} \times \{1, \ldots, B\}$ consider the vector $I_a - J_b \in \{-1, 0, 1\}^n$. By the pigeonhole principle, since $AB \geq (1.8 \cdot 1.7)^n > 3^n$, there are two pairs (a, b) and (c, d) such that $I_a - J_b = I_c - J_d$. Multiplying this by X we get $s(I_a) - 2019 = s(I_c) - 2019$; that implies $a = c$. But then $J_b = J_d$, that is, $b = d$, and our pairs coincide. Contradiction.

Problem 9. Determine all positive integers n for which there exist $n \times n$ real invertible matrices A and B that satisfy $AB - BA = B^2 A$.

Proposed by Karen Keryan, Yerevan State University & American University of Armenia, Yerevan

Solution. We prove that there exist such matrices A and B inf and only if n is even.

I. Assume that n is odd and some invertible $n \times n$ matrices A, B satisfy $AB - BA = B^2 A$. Hence $B = A^{-1}(B^2 + B)A$, so the matrices B and $B^2 + B$ are similar and therefore have the same eigenvalues. Since n is odd, the matrix B has a real eigenvalue, denote it by λ_1. Therefore $\lambda_2 := \lambda_1^2 + \lambda_1$ is an eigenvalue of $B^2 + B$, hence an eigenvalue of B. Similarly, $\lambda_3 := \lambda_2^2 + \lambda_2$ is an eigenvalue of $B^2 + B$, hence an eigenvalue of B. Repeating this process and taking into account that the number of eigenvalues of B is finite we will get there exist numbers $k \leq l$ so that $\lambda_{l+1} = \lambda_k$. Hence

$$\begin{aligned}
\lambda_{k+1} &= \lambda_k^2 + \lambda_k \\
\lambda_{k+2} &= \lambda_{k+1}^2 + \lambda_{k+1} \\
&\cdots\cdots\cdot \\
\lambda_l &= \lambda_{l-1}^2 + \lambda_{l-1} \\
\lambda_k &= \lambda_l^2 + \lambda_l.
\end{aligned}$$

Adding this equations we get $\lambda_k^2 + \lambda_{k+1}^2 + \ldots + \lambda_l^2 = 0$. Taking into account that all λ_i's are real (as λ_1 is real), we have $\lambda_k = \ldots = \lambda_l = 0$, which implies that B is not invertible, contradiction.

II. Now we construct such matrices A, B for even n. Let $A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $B_2 = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$. It is easy to check that the matrices A_2, B_2 are invertible and satisfy the condition. For $n = 2k$ the $n \times n$ block matrices

$$A = \begin{bmatrix} A_2 & 0 & \ldots & 0 \\ 0 & A_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & A_2 \end{bmatrix}, \quad B = \begin{bmatrix} B_2 & 0 & \ldots & 0 \\ 0 & B_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & B_2 \end{bmatrix}$$

are also invertible and satisfy the condition.

Problem 10. 2019 points are chosen at random, independently, and distributed uniformly in the unit disc $\{(x,y)\in\mathbb{R}^2\colon x^2+y^2\le 1\}$. Let C be the convex hull of the chosen points. Which probability is larger: that C is a polygon with three vertices, or a polygon with four vertices?

Proposed by Fedor Petrov, St. Petersburg State University

Solution. We will show that the quadrilateral has larger probability.

Let $\mathcal{D}=\{(x,y)\in\mathbb{R}^2\colon x^2+y^2\le 1\}$. Denote the random points by $X_1,\dots,X_{2019}$ and let

$$p = P\big(C \text{ is a triangle with vertices } X_1,X_2,X_3\big),$$
$$q = P\big(C \text{ is a convex quadrilateral with vertices } X_1,X_2,X_3,X_4\big).$$

By symmetry we have $P\big(C \text{is a triangle}\big)=\binom{2019}{3}p$, $P\big(C \text{is a quadrilateral}\big)=\binom{2019}{4}q$ and we need to prove that $\binom{2019}{4}q>\binom{2019}{3}p$, or equivalently $p<\frac{2016}{4}q=504q$.

Note that p is the average over X_1,X_2,X_3 of the following expression:

$$u(X_1,X_2,X_3)=P\big(X_4\in\triangle X_1X_2X_3\big)\cdot P\big(X_5,X_6,\dots,X_{2019}\in\triangle X_1X_2X_3\big),$$

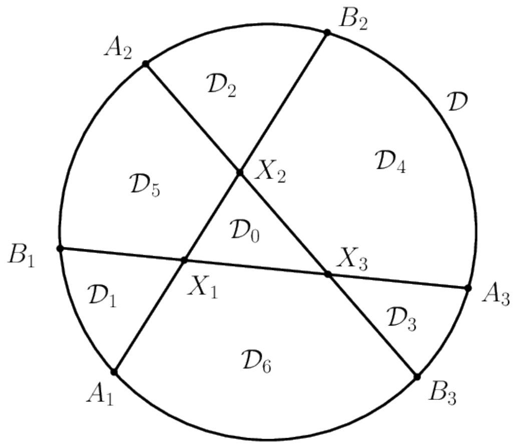
and q is not less than the average over X_1,X_2,X_3 of

$$v(X_1,X_2,X_3)=P\big(X_1,X_2,X_3,X_4 \text{ form a convex quad.}\big)\cdot P\big(X_5,X_6,\dots,X_{2019}\in\triangle X_1X_2X_3)\big).$$

Thus it suffices to prove that $u(X_1,X_2,X_3)\le 500v(X_1,X_2,X_3)$ for all X_1,X_2,X_3. It reads as $\mathrm{area}(\triangle X_1X_2X_3)\le 500\mathrm{area}(\Omega)$, where $\Omega=\{Y: X_1,X_2,X_3,Y \text{ form a convex quadrilateral}\}$.

Assume the contrary, i.e., $\mathrm{area}(\triangle X_1X_2X_3)>500\mathrm{area}(\Omega)$.

Let the lines X_1X_2, X_1X_3, X_2X_3 meet the boundary of $\mathcal{D}$ at A_1,A_2,A_3,B_1,B_2,B_3; these lines divide $\mathcal{D}$ into 7 regions as shown in the picture; $\Omega=\mathcal{D}_4\cup\mathcal{D}_5\cup\mathcal{D}_6$.



By our indirect assumption,

$$\mathrm{area}(\mathcal{D}_4)+\mathrm{area}(\mathcal{D}_5)+\mathrm{area}(\mathcal{D}_6)=\mathrm{area}(\Omega)<\frac{1}{500}\mathrm{area}(\mathcal{D}_0)<\frac{1}{500}\mathrm{area}(\mathcal{D})=\frac{\pi}{500}.$$

From $\triangle X_1X_3B_3\subset\Omega$ we get $X_3B_3/X_3X_2=\mathrm{area}(\triangle X_1X_3B_3)/\mathrm{area}(\triangle X_1X_2X_3)<1/500$, so $X_3B_3<\frac{1}{500}X_2X_3<\frac{1}{250}$. Similarly, the lengths segments $A_1X_1,B_1X_1,A_2X_2,B_2X_2,A_3X_2$ are less than $\frac{1}{250}$.

The regions $\mathcal{D}_1,\mathcal{D}_2,\mathcal{D}_3$ can be covered by disks with radius $\frac{1}{250}$, so

$$\mathrm{area}(\mathcal{D}_1)+\mathrm{area}(\mathcal{D}_2)+\mathrm{area}(\mathcal{D}_3)<3\cdot\frac{\pi}{250^2}.$$

Finally, it is well-known that the area of any triangle inside the unit disk is at most $\frac{3\sqrt{3}}{4}$, so

$$\mathrm{area}(\mathcal{D}_0)\le\frac{3\sqrt{3}}{4}.$$

But then

$$\sum_{i=0}^{6} \mathrm{area}(\mathcal{D}_i) < \frac{3\sqrt{3}}{4} + 3 \cdot \frac{\pi}{250^2} + \frac{\pi}{500} < \mathrm{area}(\mathcal{D}),$$

contradiction.

IMC 2020 Online

Day 1, July 26, 2020

Problem 1. Let n be a positive integer. Compute the number of words w (finite sequences of letters) that satisfy all the following three properties:

(1) w consists of n letters, all of them are from the alphabet $\{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\}$;

(2) w contains an even number of letters a;

(3) w contains an even number of letters b.

(For example, for $n = 2$ there are 6 such words: aa, bb, cc, dd, cd and dc.)

Armend Sh. Shabani, University of Prishtina

Solution 1. Let $N = \{1, 2, \ldots, n\}$. Consider a word w that satisfies the conditions and let $A, B, C, D \subset N$ be the sets of positions of letters a, b, c and d in w, respectively. By the definition of the words we have $A \sqcup B \sqcup C \sqcup D = N$. The sets A and B are constrained to have even sizes.

In order to construct all suitable words w, choose the set $S = A \cup B$ first; by the conditions, $|S| = |A| + |B|$ must be even. It is well-known that an n-element set (with $n \geq 1$) has 2^{n-1} even subsets, so there are 2^{n-1} possibilities for S.

If $S = \emptyset$ then we can choose $C \subset N$ arbitrarily, and then the set $D = S \setminus C$ is determined D uniquely. Since N has 2^n subsets, we have 2^n options for set C and therefore 2^n suitable words w with $S = \emptyset$.

Otherwise, if $k = |S| > 0$, we have to choose an arbitrary subset C of $N \setminus S$ and an even subset A of S; then $D = (N \setminus S) \setminus C$ and $B = S \setminus A$ are determined and $|B| = |S| - |A|$ will automatically be even. We have 2^{n-k} choices for C and 2^{k-1} independent choices for A; so for each nonempty even S we have $2^{n-k} \cdot 2^{k-1} = 2^{n-1}$ suitable words.

The number of nonempty even sets S is $2^{n-1} - 1$, so in total, the number of words satisfying the conditions is

$$1 \cdot 2^n + (2^{n-1} - 1) \cdot 2^{n-1} = 4^{n-1} + 2^{n-1}.$$

Solution 2. Let a_n denote the number of words of length n over $\mathcal{A} = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\}$ such that a and b appear even number of times. Further, we define the following sequences for the number of words of length n, all over $\mathcal{A}$.

- b_n - the number of words with an odd number of a's and even number of b's
- c_n - the number of words with even number of a's and an odd number of b's
- d_n - the number of words with an odd number of a's and an odd number of b's

We will call them A-words, B-words, C-words and D-words, respectively.
It is clear that $a_1 = 2$ and that

$$a_n + b_n + c_n + d_n = 4^n.$$

First, we find a recurrence relation for a_n. If an A-word of length n begins with c or d, it can be followed by any A-word of length $n-1$, contributing with $2a_{n-1}$. If an A-word of length n begins with a, it can be followed by any word of length $n-1$ that contains an odd number of a's and even number of b's, thus contributing with b_{n-1}. If an A-word of length n begins with b, it can be followed by any word of length $n-1$ that contains even number of a's and an odd number of b's, thus contributing with c_{n-1}. Therefore we have the following recurrence relation:

$$a_n = 2a_{n-1} + b_{n-1} + c_{n-1}. \tag{1}$$

Next, we find a recurrence relation for b_n.
If a B-word of length n begins with c or d, it can be followed by any B-word of length $n-1$, contributing with $2b_{n-1}$. If a B-word of length n begins with a, it can be followed by any word of length $n-1$ that contains even number of a's and even number of b's, contributing with a_{n-1}. If a B-word of length n begins with b, it can be followed by any word of length $n-1$ that contains an odd number of a's and an odd number of b's, contributing with $d_{n-1} = 4^{n-1} - a_{n-1} - b_{n-1} - c_{n-1}$. Therefore we have the following recurrence relation:

$$b_n = b_{n-1} + 4^{n-1} - c_{n-1}. \tag{2}$$

Now observe that $b_k = c_k$ for all k, since simultaneously replacing a's to b's and vice versa we get a C-word from a B-word. Therefore (2) yields $b_n = 4^{n-1}$. Now (1) yields

$$a_n = 2 \cdot a_{n-1} + 2 \cdot 4^{n-2}.$$

Solving the last recurrence relation (for example, diving by 2^n we get $x_n := a_n 2^{-n}$ satisfies $x_n - x_{n-1} = 2^{n-3}$, and it remains to sum up consecutive powers of 2) we get

$$a_n = 2^{n-1} + 4^{n-1}.$$

Solution 3. Consider the sum

$$\frac{(a+b+c+d)^n + (-a-b+c+d)^n + (-a+b+c+d)^n + (a-b+c+d)^n}{4}. \tag{$*$}$$

Expanding the parentheses as

$$(a+b+c+d)^n = (a+b+c+d)(a+b+c+d)\dots(a+b+c+d),$$

we get a sum of products $x_1 \dots x_n$, $x_i \in \{a, b, c, d\}$, naturally corresponding to the words of length n over the alphabet $\{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\}$. Consider the other terms in the numerator similarly.

If a word $x_1 \dots x_n$ contains A, B, C, D letters a,b,c and d respectively, we get $a^A b^B c^C d^D$ with the coefficient

$$\frac{1 + (-1)^{A+B} + (-1)^A + (-1)^B}{4} = \frac{(1 + (-1)^A)(1 + (-1)^B)}{4} = \begin{cases} 1, & \text{if } A \text{ and } B \text{ are even} \\ 0, & \text{otherwise.} \end{cases}$$

Hence, by substituting $a = b = c = d = 1$ in $(*)$ we get the answer $(4^n + 2^{n+1})/4 = 4^{n-1} + 2^{n-1}$.

Problem 2. Let A and B be $n \times n$ real matrices such that

$$\mathrm{rk}(AB - BA + I) = 1$$

where I is the $n \times n$ identity matrix.

Prove that

$$\mathrm{trace(ABAB)} - \mathrm{trace(A^2B^2)} = \frac{1}{2}\mathrm{n(n-1)}.$$

($\mathrm{rk}(M)$ denotes the rank of matrix M, i.e., the maximum number of linearly independent columns in M. trace(M) denotes the trace of M, that is the sum of diagonal elements in M.)

Rustam Turdibaev, V. I. Romanovskiy Institute of Mathematics

Solution. Let $X = AB - BA$. The first important observation is that

$$\mathrm{trace(X^2)} = \mathrm{trace(ABAB - ABBA - BAAB + BABA)} = \mathrm{2trace(ABAB)} - \mathrm{2trace(A^2B^2)}$$

using that the trace is cyclic. So we need to prove that $\mathrm{trace(X^2) = n(n-1)}$.

By assumption, $X + I$ has rank one, so we can write $X + I = v^t w$ for two vectors v, w. So

$$X^2 = (v^t w - I)^2 = I - 2v^t w + v^t w v^t w = I + (wv^t - 2)v^t w.$$

Now by definition of X we have $\mathrm{trace(X)} = 0$ and hence $wv^t = \mathrm{trace(wv^t)} = \mathrm{trace(v^t w)} = \mathrm{n}$ so that indeed

$$\mathrm{trace(X^2)} = \mathrm{n + (n-2)n = n(n-1)}.$$

An alternative way to use the rank one condition is via eigenvalues: Since $X + I$ has rank one, it has eigenvalue 0 with multiplicity $n-1$. So X has eigenvalue -1 with multiplicity $n-1$. Since $\mathrm{trace(X)} = 0$ the remaining eigenvalue of X must be $n-1$. Hence

$$\mathrm{trace(X^2)} = \mathrm{(n-1)^2 + (n-1) \cdot 1^2 = n(n-1)}.$$

Problem 3. Let $d \geq 2$ be an integer. Prove that there exists a constant $C(d)$ such that the following holds: For any convex polytope $K \subset \mathbb{R}^d$, which is symmetric about the origin, and any $\varepsilon \in (0, 1)$, there exists a convex polytope $L \subset \mathbb{R}^d$ with at most $C(d)\varepsilon^{1-d}$ vertices such that

$$(1 - \varepsilon)K \subseteq L \subseteq K.$$

(For a real α, a set $T \subset \mathbb{R}^d$ with nonempty interior is a *convex polytope with at most α vertices*, if T is a convex hull of a set $X \subset \mathbb{R}^d$ of at most α points, i.e., $T = \{\sum_{x\in X} t_x x \mid t_x \geq 0, \sum_{x\in X} t_x = 1\}$. For a real λ, put $\lambda K = \{\lambda x \mid x \in K\}$. A set $T \subset \mathbb{R}^d$ is *symmetric about the origin* if $(-1)T = T$.)

Fedor Petrov, St. Petersburg State University

Solution [in elementary terms] Let $\{p_1, \dots, p_m\}$ be an inclusion-maximal collection of points on the boundary ∂K of K such that the homothetic copies $K_i := p_i + \frac{\varepsilon}{2}K$ have disjoint interiors. We claim that the convex hull $L := \mathrm{conv}\{p_1, \dots, p_m\}$ satisfies all the conditions.

First, note that by convexity of K we have $aK + bK = (a+b)K$ for $a, b > 0$. It follows that $K_i \subset (1 + \frac{\varepsilon}{2})K$. On the other hand, if $k \in K$, $a > 0$ and and $ak \in K_i$, then

$$p_i \in ak - \frac{\varepsilon}{2}K = ak + \frac{\varepsilon}{2}K \subset (a + \frac{\varepsilon}{2})K,$$

and since p_i is a boundary point of K, we get $a + \frac{\varepsilon}{2} \geqslant 1$, $a \geqslant 1 - \frac{\varepsilon}{2}$. It means that all K_i lie between $(1 - \frac{\varepsilon}{2})K$ and $(1 + \frac{\varepsilon}{2})K$. Since their interiors are disjoint, by the volume counting we obtain

$$m\left(\frac{\varepsilon}{2}\right)^d \leqslant \left(1 + \frac{\varepsilon}{2}\right)^d - \left(1 - \frac{\varepsilon}{2}\right)^d \leqslant (3/2)^d\varepsilon$$

(since $F(\varepsilon) = (1 + \frac{\varepsilon}{2})^d - (1 - \frac{\varepsilon}{2})^d$ is a polynomial in ε without constant term with non-negative coefficients which sum up to $(3/2)^d - (1/2)^d$), therefore $m \leqslant 3^d\varepsilon^{1-d}$.

It is clear that $L \subseteq K$, so it remains to prove that $(1 - \varepsilon)K \subseteq L$. Assume the contrary: there exists a point $p \in (1 - \varepsilon)K \setminus L$. Separate p from L by a hyperplane: Choose a linear functional ℓ such that $\ell(p) > \max_{x\in L} \ell(x) = \max_i \ell(p_i)$. Choose $x \in K$ such that $\ell(x) =: a$ is maximal possible. Note that by our construction $x + \frac{\varepsilon}{2}K$ has a common point with some K_i: there exists a point $z \in (x + \frac{\varepsilon}{2}K) \cap (p_i + \frac{\varepsilon}{2}K)$. We have

$$\ell(p_i) + \frac{\varepsilon}{2}a \geqslant \ell(z) \geqslant \ell(x) - \frac{\varepsilon}{2}a,$$

and therefore $\ell(p_i) \geqslant a(1 - \varepsilon)$. Since $p \in (1 - \varepsilon)K$, we obtain $\ell(p) \leqslant a(1 - \varepsilon)$. A contradiction.

Solution [in the language of Banach spaces] Equip $\mathbb{R}^d$ with the norm $\|\cdot\|$, whose unit ball is K, call this Banach space V. Choose an inclusion maximal set $X \subset \partial K$ whose pairwise distances are $\geq \varepsilon$. Put $L = \mathrm{conv}X$.

The inclusion $L \subseteq K$ follows from the convexity of K. If the inclusion $(1-\varepsilon)K \subseteq L$ fails then the Hahn–Banach theorem provides a unit linear functional $\lambda \in V^*$ such that $\max\{\lambda(L)\} = \max\{\lambda X\} \leq 1 - \varepsilon$. Then the point $x \in K$, where the maximum $\max\{\lambda(K)\} = 1$ is attained (thanks to the finite dimension and compactness) is in ∂K and, as λ witnesses, at distance $\geq \varepsilon$ from all other points of L and X, contradicting the inclusion-maximality of X.

The upper bound for the cardinality $|X|$ is obtained by noting that the $\varepsilon/2$ balls centered at the points of X are pairwise disjoint and lie in the difference of balls $(1+\varepsilon/2)K \setminus (1-\varepsilon/2)K$, whose volume is $\left((1 + \varepsilon/2)^d - (1 - \varepsilon/2)^d\right) \mathrm{vol}K$, the volume of each of the small balls being $\varepsilon^d/2^d\mathrm{vol}K$. Hence

$$|X| \leq \frac{(2 + \varepsilon)^d - (2 - \varepsilon)^d}{\varepsilon^d} = O(\varepsilon^{1-d}).$$

Problem 4. A polynomial p with real coefficients satisfies the equation $p(x+1)-p(x)=x^{100}$ for all $x\in\mathbb{R}$. Prove that $p(1-t)\geqslant p(t)$ for $0\leqslant t\leqslant 1/2$.

Daniil Klyuev, St. Petersburg State University

Solution 1. Denote $h(z)=p(1-\bar{z})-p(z)$ for complex z. For $t\in\mathbb{R}$ we have $h(it)=p(1+it)-p(it)=t^{100}$, $h(1/2+it)=0$.

If $p(z)=c_nz^n+\ldots+c_0, c_n\neq 0$, we have

$$h(a+it)=p((1-a)+it)-p(a+it)=(1-2a)\left(nc_ni^{n-1}t^{n-1}+Q(t,a)\right)$$

for some polynomial Q having degree at most $n-2$ with respect to the variable t. Substituting $a=0$ we get $n=101$, $c_n=1/101$.

Next, for large $|t|$ we see that $\Re(h(a+it))>0$ for $0\leqslant a<1/2$.

Therefore by Maximum Principle for the harmonic function $\Re h$ and the rectangle $[0,1/2]\times[-N,N]$ for large enough N we conclude that $\Re h$ is non-negative in this rectangle, in particular on $[0,1/2]$, as we need.

Solution 2. Let $p(x)=\sum_{j=0}^{m}a_jx^j$. Then

$$p(x+1)-p(x)=\sum_{j=0}^{m}a_j\left((x+1)^j-x^j\right)=a_1+a_2(2x+1)+\cdots+a_m\left(mx^{m-1}+\binom{m}{2}x^{m-2}+\cdots+1\right).$$

This implies that $m=101$, $ma_m=1$ so $a_{101}=\frac{1}{101}$, $(m-1)a_{m-1}+a_m\binom{m}{2}=0$ so $a_{100}=-\frac{1}{2}$ etc. For $j\geqslant 1$ a_j is uniquely defined, a_0 may be chosen arbitrarily.

The equality $p_{2n}(\frac{1}{2})=0$ holds because $0=p_{2n}(\frac{1}{2})+p_{2n}(1-\frac{1}{2})=2p_{2n}(\frac{1}{2})$. Let $n\geqslant 1$ be an integer and let p_n be a polynomial such that $p_n(x+1)-p_n(x)=x^n$ for all x and $p_n(0)=0=p_n(1)$. The above considerations prove the uniqueness of p_n. We have $p_1(x)=\frac{1}{2}x^2-\frac{1}{2}x$. Also $p_n'(x+1)-p_n'(x)=nx^{n-1}=n\left(p_{n-1}(x+1)-p_{n-1}(x)\right)$. Therefore $p_n'(x)=np_{n-1}(x)+c_{n-1}$ for a properly chosen constant c_{n-1}. We shall prove that

(1) $p_{2n-1}(x)-p_{2n-1}(1-x)=0,\ p_{2n}(x)+p_{2n}(1-x)=0,\ c_{2n}=0,\ p_{2n}''(x)=2n(2n-1)p_{2n-2}(x)$

for $n=1,2,\ldots$ and for all x. Simple computation shows that $p_1(x)-p_1(1-x)=0$. We have $(p_2(x)+p_2(1-x))'=2p_1(x)+c_1-(2p_1(1-x)+c_1)=0$ so the map $x\mapsto p_2(x)+p_2(1-x)$ is constant thus $p_2(x)+p_2(1-x)=p_2(0)+p_2(1-0)=0$. If the first two equalities hold for some n then $(p_{2n+1}(x)-p_{2n+1}(1-x))'=(2n+1)p_{2n}(x)+c_{2n}+(p_{2n}(1-x)+c_{2n})=2c_{2n}$ so there exists $b\in R$ such that $p_{2n+1}(x)-p_{2n+1}(1-x)=2c_{2n}x+b$ for all x. $p_{2n+1}(0)-p_{2n+1}(1-0)=0$ and $p_{2n+1}(1)-p_{2n+1}(1-1)=0$ so $2c_{2n}=0=b$. This proves that $p_{2n+1}(x)-p_{2n+1}(1-x)=0$ for all x. In a similar way we shall prove the second equality: $(p_{2n+2}(x)+p_{2n+2}(1-x))'=(2n+2)p_{2n+1}'(x)+c_{2n+1}-(2n+2)\left(p_{2n+1}(1-x)+c_{2n+1}\right)=0$ so the map $x\mapsto p_{2n+2}(x)+p_{2n+2}(1-x)$ is constant hence $p_{2n+2}(x)+p_{2n+2}(1-x)=p_{2n+2}(0)+p_{2n+2}(1-0)=0$ for all x. Now $p_{2n+2}''(x)=((2n+2)p_{2n+1}(x)+c_{2n+1})'=(2n+2)p_{2n+1}'(x)=(2n+2)((2n+1)p_{2n}(x)+c_{2n})=(2n+2)(2n+1)p_{2n}(x)$. Since $p_2'(x)=2p_1(x)+c_1=x^2-x+c_1$ we obtain $p_2''(x)=2x-1<0$ for $x<\frac{1}{2}$.The function p_2 is strictly concave on $[0,\frac{1}{2}]$ and $p_2(0)=0=p_2(\frac{1}{2})$. Therefore $p_2(x)>0$ for $x\in(0,\frac{1}{2})$. This together with the equality $p_4(x)=12p_2(x)$ implies that p_4 is strictly convex on $[0,\frac{1}{2}]$ so in view of $p_4(0)=0=p_4(\frac{1}{2})$ we conclude that $p_4(x)<0$ for $x\in(0,\frac{1}{2})$. Easy induction shows that for $x\in(0,\frac{1}{2})$ one has $p_{2n}(x)>0$ for an odd n and $p_{2n}(x)<0$ for an even n. If $t\in(0,\frac{1}{2})$ then by (1) we get $p_{100}(1-t)-p_{100}(t)=-2p_{100}(t)>0$ as required.

IMC 2020 Online

Day 2, July 27, 2020

Problem 5. Find all twice continuously differentiable functions $f:\mathbb{R}\to(0,+\infty)$ satisfying

$$f''(x)f(x) \ge 2(f'(x))^2$$

for all $x\in\mathbb{R}$.

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Solution. We shall show that only positive constant functions satisfy the condition.

Let $g(x)=\dfrac{1}{f(x)}$. Notice that

$$g''=\left(\frac{1}{f}\right)''=\left(\frac{-f'}{f^2}\right)'=\left(\frac{2(f')^2-f''f}{f^3}\right)'\le 0,$$

so the positive function $g(x)$ is concave. We show that g must be constant.

Take two arbitrary real numbers $a<b$. By the concavity of g, for all $u<a$ and $v>b$ we have

$$\frac{g(a)-g(u)}{a-u}\ge\frac{g(b)-g(a)}{b-a}\ge\frac{g(v)-g(b)}{v-b}.$$

Combining this with $g(u),g(v)>0$ we get

$$\frac{g(a)}{a-u}>\frac{g(b)-g(a)}{b-a}>\frac{-g(b)}{v-b}$$

Now by taking limits $u\to-\infty$ and $v\to\infty$ we obtain

$$0\ge\frac{g(b)-g(a)}{b-a}\ge 0,$$

so $g(a)=g(b)$. This holds for any pair (a,b), so $g(x)$ is constant and $f(x)=1/g(x)$ also is constant.

If f is constant then $f'=f''=0$, so the condition is satisfied.

Remark. Instead of the function $1/f(x)$, the same idea works with $\arctan f(x)$:

$$(\arctan f(x))''=\frac{f''(1+f^2)-2(f')^2}{(1+f^2)^2}=\frac{f''(1+f^2)-2(f')^2(1+f^2)}{(1+f^2)^2}=\frac{f''-2(f')^2}{1+f^2}\ge 0.$$

As can be seen, $\arctan f(x)$ is a bounded convex function, therefore it must be constant.

Problem 6. Find all prime numbers p for which there exists a unique $a\in\{1,2,\dots,p\}$ such that a^3-3a+1 is divisible by p.

Géza Kós, Loránd Eötvös University, Budapest

Solution 1. We show that $p=3$ the only prime that satsfies the condition.

Let $f(x) = x^3 - 3x + 1$. As preparation, let's compute the roots of $f(x)$. By Cardano's formula, it can be seen that the roots are

$$2\mathrm{Re}\sqrt[3]{\frac{-1}{2} + \sqrt{\left(\frac{-1}{2}\right)^2 - \left(\frac{-3}{3}\right)^3}} = 2\mathrm{Re}\sqrt[3]{\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}} = \left\{2\cos\frac{2\pi}{9}, 2\cos\frac{4\pi}{9}, 2\cos\frac{8\pi}{9}\right\}$$

where all three values of the complex cubic root were taken.

Notice that, by the trigonometric identity $2\cos 2t = (2\cos t)^2 - 2$, the map $\varphi(x) = x^2 - 2$ cyclically permutes the three roots. We will use this map to find another root of f, when it is considered over $\mathbb{F}_p$.

Suppose that $f(a) = 0$ for some $a \in \mathbb{F}_p$ and consider

$$g(x) = \frac{f(x)}{x-a} = \frac{f(x) - f(a)}{x-a} = x^2 + ax + (a^2 - 3).$$

We claim that $b = a^2 - 2$ is a root of $g(x)$. Indeed,

$$g(b) = (a^2-2)^2 + a(a^2-2) + (a^2-3) = (a+1)\cdot f(a) = 0.$$

By Vieta's formulas, the other root of $g(x)$ is $c = -a - b = -a^2 - a + 2$.

If f has a single root then the three roots must coincide, so

$$a = a^2 - 2 = -a^2 - a + 2.$$

Here the quadratic equation $a = a^2 - 2$, or equivalently $(a+1)(a-2) = 0$, has two solutions, $a = -1$ and $a = 2$. By $f(-1) = f(2) = 3$, in both cases we have $0 = f(a) = 3$, so the only choice is $p = 3$.

Finally, for $p = 3$ we have $f(1) = -1$, $f(2) = 3$ and $f(3) = 19$, from these values only $f(2)$ is divisible by 3, so $p = 3$ satisfies the condition.

Solution 2 (outline) Define $f(x)$ and $g(x)$ like in Solution 1. The discriminant of $g(x)$ is

$$\Delta_g = a^2 - 4(a^2 - 3) = 12 - 3a^2.$$

We show that Δ_g has a square root in $\mathbb{F}_p$.

Take two integers k, m (to be determinated later) and consider

$$\Delta_g = \Delta_g + (ka + m)f(a) = ka^4 + ma^3 - (3k+1)a^2 + (k - 3m)a + (m + 12).$$

Our goal is to choose k, m in such a way that the last expression is a complete square. Either by direct calculations or guessing, we can find that $k = m = 4$ works:

$$\Delta_g = \Delta_g + (4a+4)f(a) = 4a^4 + 4a^3 - 15a^2 - 8a + 16 = (2a^2 + a - 4)^2.$$

If $p \neq 2$ then we can conclude that $f(x)$ has either no or three roots, therefore p is suitable if and only is $f(x)$ is a complete cube: $x^3 - 3x + 1 = (x-a)^3$. From Vieta's formulas $a^3 = 1$, so $a \neq 0$ and $3a = 0$, which is possible if $p = 3$.

For $p = 3$ we have $f(x) = (x+1)^3$, so $p = 3$ is suitable.

The case $p = 2$ must be checked separately because the quadratic formula contains a division by 2. $f(1) = -1$ and $f(2) = 3$, so $p = 2$ is not suitable.

Solution 3 (outline) Assume $p > 3$; the cases $p = 2$ and $p = 3$ will be checked separately.

Let $f(x) = x^3 - 3x + 1$ and suppose that $a \in \mathbb{F}_p$ is a root of $f(x)$, and let $b, c \in \mathbb{F}_{p^2}$ be the other two roots. The discriminant Δ_f of $f(x)$ can be expressed by the elementary symmetric polynomials of a, b, c; it can be calculated that

$$\Delta_f = (b-c)^2(a-b)^2(a-c)^2 = 81 = 9^2,$$

so

$$(b-c)(a-b)(a-c) = \pm 9 \in \mathbb{F}_p.$$

Notice that $\Delta_f \neq 0$, so the three roots are distinct.

Either $b, c \in \mathbb{F}_p$ or b, c are conjugate elements in $\mathbb{F}_{p^2} \setminus \mathbb{F}_p$, we have $(a-b)(a-c) \in \mathbb{F}_p$, so $b - c = \frac{(b-c)(a-b)(a-c)}{(a-b)(a-c)} \in \mathbb{F}_p$. From Vieta's formulas we have $b + c \in \mathbb{F}_p$ as well; since $p \neq 2$, it follows that $b, c \in \mathbb{F}_p$. Now $f(x)$ has three distinct roots in $\mathbb{F}_p$, so p cannot be suitable.

$p = 2$ does not satisfies the condition because both $f(1) = -1$ and $f(2) = 3$ are odd. $p = 3$ is suitable, because $f(2) = 3$ is divisible by 3 while $f(1) = -1$ and $f(3) = 19$ are not.

Problem 7. Let G be a group and $n \geq 2$ be an integer. Let H_1 and H_2 be two subgroups of G that satisfy

$$[G : H_1] = [G : H_2] = n \quad \text{and} \quad [G : (H_1 \cap H_2)] = n(n-1).$$

Prove that H_1 and H_2 are conjugate in G.

(Here $[G : H]$ denotes the *index* of the subgroup H, i.e. the number of distinct left cosets xH of H in G. The subgroups H_1 and H_2 are *conjugate* if there exists an element $g \in G$ such that $g^{-1}H_1g = H_2$.)

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Solution 1. Denote $K = H_1 \cap H_2$. Since

$$n(n-1) = [G : K] = [G : H_1][H_1 : K] = n[H_1 : K],$$

we obtain that $[H_1 : K] = n - 1$. Thus, the subgroup H_1 is partitioned into $n - 1$ left cosets of K, say $H_1 = \bigsqcup_{i=1}^{n-1} h_iK$. Therefore, the set $H_1H_2 = \{ab\colon a \in H_1,\ b \in H_2\}$ is partitioned as

$$H_1H_2 = \left(\bigsqcup_{i=1}^{n-1} h_iK\right) H_2 = \bigsqcup_{i=1}^{n-1} h_iKH_2 = \bigsqcup_{i=1}^{n-1} h_iH_2.$$

The last equality holds because $K \subseteq H_2$, so $KH_2 = H_2$. The last expression is a disjoint union since

$$h_iH_2 \cap h_jH_2 \neq \varnothing \iff h_i^{-1}h_j \in H_2 \iff h_i^{-1}h_j \in K \iff h_i = h_j.$$

Thus, H_1H_2 is a disjoint union of $n-1$ left cosets with respect to H_2; hence $L = G \setminus (H_1H_2)$ is the remaining such left coset. Similarly, L is a right coset with respect to H_1. Therefore, for each $g \in L$ we have $L = gH_2 = H_1g$, which yields $H_2 = g^{-1}H_1g$.

Solution 2. Put $G/H_1 = X$ and $G/H_2 = Y$, those are n-element sets acted on by G from the left. Let G act on $X \times Y$ from the left coordinate-wise, consider this product as a table, with rows being copies of X and columns being copies of Y.

The stabilizer of a point (x, y) in $X \times Y$ is $H_1 \cap H_2$. By the orbit-stabilizer theorem, we obtain that the orbit Z of (x, y) has size $[G : H_1 \cap H_2] = n(n-1)$.

If Z contains a whole column then there is a subgroup G_1 of G that stabilizes x and acts transitively on Y. If we conjugate G_1 to a group G_1', then its action remains transitive on Y, so by conjugation we obtain columns of the table. Since G acts transitively on X, we cover all the columns. It follows that $Z = X \times Y$, so

$$n(n-1) = |Z| = |X \times Y| = n^2,$$

which is a contradiction.

Hence every column of $X \times Y$ has an element not from Z. The same holds for the rows of $X \times Y$. There are n elements not from Z in total and they induce a bijection between X and Y which allows us to identify $X = Y$.

After this identification, every element (x, x) from the diagonal of $X \times X$ (i.e. from $(X \times X) \setminus Z$) is moved to a diagonal element by any $g \in G$, because $gx = gx$. In this formula the action of g in the left hand side and the action of g in the right hand side are the actions of g on X and Y respectively.

Therefore our bijection between X and Y is an isomorphism of sets with a left action of G. Since H_1 and H_2 are stabilizers of the points in the same transitive action of G, we conclude that they are conjugate.

Remark. The situation in the problem is possible for every $n \geq 2$: let $G = S_n$ and let H_1 an H_2 be the stabilizer subgroups of two elements.

Problem 8. Compute

$$\lim_{n\to\infty} \frac{1}{\log\log n} \sum_{k=1}^{n} (-1)^k \binom{n}{k} \log k.$$

(Here log denotes the natural logarithm.)

Fedor Petrov, St. Petersburg State University

Solution 1. Answer: 1.

The idea is that if $f(k) = \int g^k$, then

$$\sum (-1)^k \binom{n}{k} f(k) = \int (1-g)^n.$$

To relate this to logarithm, we may use the Frullani integrals

$$\int_0^\infty \frac{e^{-x} - e^{-kx}}{x} dx = \lim_{c\to+0} \int_c^\infty \frac{e^{-x}}{x} dx - \int_c^\infty \frac{e^{-kx}}{x} dx = \lim_{c\to+0} \int_c^\infty \frac{e^{-x}}{x} dx - \int_{kc}^\infty \frac{e^{-x}}{x} dx =$$

$$\lim_{c\to+0} \int_c^{kc} \frac{e^{-x}}{x} dx = \log k + \lim_{c\to+0} \int_c^{kc} \frac{e^{-x} - 1}{x} dx = \log k.$$

This gives the integral representation of our sum:

$$A := \sum_{k=1}^{n} (-1)^k \binom{n}{k} \log k = \int_0^\infty \frac{-e^{-x} + 1 - (1-e^{-x})^n}{x} dx.$$

Now the problem is reduced to a rather standard integral asymptotics.

We have $(1-e^{-x})^n \geqslant 1 - ne^{-x}$ by Bernoulli inequality, thus $0 \leqslant -e^{-x} + 1 - (1-e^{-x})^n \leqslant ne^{-x}$, and we get

$$0 \leqslant \int_M^\infty \frac{-e^{-x} + 1 - (1-e^{-x})^n}{x} dx \leqslant n \int_M^\infty \frac{e^{-x}}{x} dx \leqslant nM^{-1} \int_M^\infty e^{-x} dx = nM^{-1} e^{-M}.$$

So choosing M such that $Me^M = n$ (such M exists and goes to ∞ with n) we get

$$A = O(1) + \int_0^M \frac{-e^{-x} + 1 - (1-e^{-x})^n}{x} dx.$$

Note that for $0 \leqslant x \leqslant M$ we have $e^{-x} \geqslant e^{-M} = M/n$, and $(1-e^{-x})^{n-1} \leqslant e^{-e^{-x}(n-1)} \leqslant e^{-M(n-1)/n}$ tends to 0 uniformly in x. Therefore

$$\int_0^M \frac{(1-e^{-x})(1-(1-e^{-x})^{n-1})}{x} dx = (1+o(1)) \int_0^M \frac{1-e^{-x}}{x} dx.$$

Finally

$$\int_0^M \frac{1-e^{-x}}{x} dx = \int_0^1 \frac{1-e^{-x}}{x} dx + \int_1^M \frac{-e^{-x}}{x} dx + \int_1^M \frac{dx}{x} =$$
$$\log M + O(1) = \log(M + \log M) + O(1) = \log\log n + O(1),$$

and we get $A = (1+o(1)) \log\log n$.

Solution 2. We start with a known identity (a finite difference of $1/x$).

Expand the rational function

$$f(x) = \frac{m!}{x(x+1)\dots(x+m)}$$

as the linear combination of simple fractions $f(x) = \sum_{j=0}^m c_j/(x+j)$. To find c_j we use

$$c_j = ((x+j)f(x))\,|_{x=-j} = (-1)^j \binom{m}{j}.$$

So we get

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} \frac{1}{x+k} = \frac{m!}{x(x+1)\dots(x+m)}. \tag{1}$$

Another known identity we use is

$$\sum_{k=j+1}^{n} (-1)^k \binom{n}{j} = \sum_{k=j+1}^{n} (-1)^k \left(\binom{n-1}{k} + \binom{n-1}{k-1} \right) = (-1)^{j+1} \binom{n-1}{j}. \tag{2}$$

Finally we write $\log k = \int_1^k \frac{dx}{x} = \sum_{j=1}^{k-1} I_j$, where $I_j = \int_0^1 \frac{dx}{x+j}$.

Now we have

$$S := \sum_{k=1}^{n} (-1)^k \binom{n}{k} \log k = \sum_{k=1}^{n} (-1)^k \binom{n}{k} \sum_{j=1}^{k-1} I_j = \sum_{j=1}^{n-1} I_j \sum_{k=j+1}^{n} (-1)^k \binom{n}{k} \underset{(2)}{=} \sum_{j=1}^{n-1} I_j (-1)^{j+1} \binom{n-1}{j} =$$

$$\int_0^1 \sum_{j=1}^{n-1} (-1)^{j+1} \binom{n-1}{j} \frac{dx}{x+j} = \int_0^1 \left(\frac{1}{x} - \sum_{j=0}^{n-1} (-1)^j \binom{n-1}{j} \frac{dx}{x+j} \right) dx \underset{(1)}{=}$$

$$\int_0^1 \left(\frac{1}{x} - \frac{(n-1)!}{x(x+1)\dots(x+(n-1))} \right) dx = \int_0^1 \frac{dx}{x} \left(1 - \frac{1}{(1+x)(1+x/2)\dots(1+x/(n-1))} \right).$$

So S is again expressed as an integral, for which it is not hard to get an asymptotics.

Since $e^t \geqslant 1+t$ for all real t (by convexity or any other reason), we have $e^{y^2-y} \geqslant 1+y^2-y = \frac{1+y^3}{1+y} \geqslant \frac{1}{1+y}$ and $\frac{1}{1+y} \geqslant \frac{1}{e^y} = e^{-y}$ for $y > 0$. Therefore

$$e^{y^2-y} \geqslant \frac{1}{1+y} \geqslant e^{-y},\ y > 0.$$

Using this double inequality we get

$$e^{x^2\left(1+\frac{1}{2^2}+\dots+\frac{1}{(n-1)^2}\right) - x\left(1+\frac{1}{2}+\dots+\frac{1}{n-1}\right)} \geqslant \frac{1}{(1+x)(1+x/2)\dots(1+x/(n-1))} \geqslant e^{-x\left(1+\frac{1}{2}+\dots+\frac{1}{n-1}\right)}.$$

Since $x^2(1+1/2^2+\dots) \leqslant 2x^2 \leqslant 2x$, we conclude that

$$\frac{1}{(1+x)(1+x/2)\dots(1+x/(n-1))} = e^{-C_n x}, \text{where } -2 + \sum_{j=1}^{n-1} \frac{1}{j} \leqslant C_n \leqslant \sum_{j=1}^{n-1} \frac{1}{j},$$

i.e., $C_n = \log n + O(1)$. Thus

$$S = \int_0^1 \frac{dx}{x}(1-e^{-C_n x}) = \int_0^{C_n} \frac{dt}{t}(1-e^{-t}) = \int_1^{C_n} \frac{dt}{t} + \int_0^1 (1-e^{-t}) \frac{dt}{t} + \int_1^{C_n} e^{-t} \frac{dt}{t}$$

$$= \log C_n + O(1) = \log\log n + O(1).$$

IMC 2021 Online

First Day, August 3, 2021

Solutions

Problem 1. Let A be a real $n \times n$ matrix such that $A^3 = 0$.

(a) Prove that there is a unique real $n \times n$ matrix X that satisfies the equation

$$X + AX + XA^2 = A.$$

(b) Express X in terms of A.

(proposed by Bekhzod Kurbonboev, Institute of Mathematics, Tashkent)

Hint: (a) Multiply the equation by some power of A from left and another power of A from right.

(b) Substitute repeatedly $X = A - AX - XA^2$.

Solution 1. First suppose that some matrix X satisfies the equation. We can obtain new equations if we multiply the given equation by some power of A from left and another power of A from right. For example,

$$A^2(X + AX + XA^2)A^2 = A^2XA^2 + A^3 \cdot XA^2 + A^2XA \cdot A^3 = A^2XA^2.$$

The right-hand side is $A^2 \cdot A \cdot A^2 = A^3 \cdot A^2 = 0$, so

$$\begin{aligned} A^2XA^2 &= A^2(X + AX + XA^2)A^2 = A^5 = 0. \qquad \text{Similarly,} \\ A^2X &= A^2(X + AX + XA^2) = A^3 = 0 \\ AXA &= A(X + AX + XA^2)A = A^3 = 0 \\ XA^2 &= (X + AX + XA^2)A^2 = A^3 = 0 \\ AX &= A(X + AX + XA^2)A = A^2. \qquad \text{Finally} \\ X &= A - AX - XA^2 = A - A^2. \end{aligned}$$

Hence, no matrix other than $A - A^2$ can satisfy the equation.

Note that the argument above does not prove that the matrix $X = A - A^2$ satisfies the equation, because the steps cannot be done in reverse order. That must be verified separately. Indeed,

$$X + AX + XA^2 = (A - A^2) + A(A - A^2) + (A - A^2)A^2 = A - A^4 = A.$$

Hence, $X = A - A^2$ is the unique solution of the equation.

Remark. By multiplying the equation by A^n from left and by A^k from right we can get 9 different equations:

$$\begin{array}{ccc} X + AX + XA^2 = A & XA + AXA = A^2 & XA^2 + AXA^2 = 0 \\ AX + A^2X + AXA^2 = A^2 & AXA + A^2XA = 0 & AXA^2 + A^2XA^2 = 0 \\ A^2X + A^2XA^2 = 0 & A^2XA = 0 & A^2XA^2 = 0 \end{array}$$

These formulas provide a system of linear equations for the nine matrices X, AX, A^2X, XA, AXA, A^2XA, XA^2, AXA^2 and A^2XA^2.

Solution 2. We use a different approach to express X in terms of A. If some matrix X satisfies the equation then

$$X = A - AX - XA^2.$$

Let us substitute this identity in the right-hand side repeatedly until X cancels out everywhere. Notice that by the condition $A^3 = 0$ we have $A^3 = A^4 = A^5 = A^3X = XA^4 = AXA^4 = A^3XA^2 = 0$, so

$$\begin{aligned}
X &= A - AX - XA^2 \\
&= A - A(A - AX - XA^2) - (A - AX - XA^2)A^2 \\
&= A - (A^2 - A^2X - AXA^2) - (A^3 - AXA^2 - XA^4) \\
&= A - A^2 + A^2X + 2AXA^2 \\
&= A - A^2 + A^2(A - AX - XA^2) + 2A(A - AX - XA^2)A^2 \\
&= A - A^2 + (A^3 - A^3X - A^2XA^2) + 2(A^4 - A^2XA^2 - AXA^4) \\
&= A - A^2 - 3A^2XA^2 \\
&= A - A^2 - 3A^2(A - AX - XA^2)A^2 \\
&= A - A^2 - 3(A^5 - A^3XA^2 - A^2XA^4) \\
&= A - A^2.
\end{aligned}$$

To complete the solution, we have to verify that $X = A - A^2$ is indeed a solution. This step is the same as in Solution 1.

Solution 3. Let $B = I - A + A^2$ so that B is the inverse of $I + A$. Multiplying by B from the left, the equation is equivalent to

$$X + BXA^2 = BA. \tag{1}$$

Now assume X satisfies the equation. Multiplying by A^2 from the right and using $A^3 = 0$ we get $XA^2 = 0$. Hence the equation simplifies to $X = BA = A - A^2$.

On the other hand, $X = BA$ obviously satisfies (1).

Problem 2. Let n and k be fixed positive integers, and let a be an arbitrary non-negative integer. Choose a random k-element subset X of $\{1, 2, \ldots, k + a\}$ uniformly (i.e., all k-element subsets are chosen with the same probability) and, independently of X, choose a random n-element subset Y of $\{1, \ldots, k + n + a\}$ uniformly.

Prove that the probability

$$\mathsf{P}\Big(\min(Y) > \max(X)\Big)$$

does not depend on a.

(proposed by Fedor Petrov, St. Petersburg State University)

Hint: The sets X and Y with $\min(Y) > \max(X)$ are uniquely determined by $X \cup Y$.

Solution 1. The number of choices for (X, Y) is $\binom{k+a}{k} \cdot \binom{n+k+a}{n}$.

The number of such choices with $\min(Y) > \max(X)$ is equal to $\binom{n+k+a}{n+k}$ since this is the number of choices for the $n+k$-element set $X \cup Y$ and this union together with the condition $\min(Y) > \max(X)$ determines X and Y uniquely (note in particular that no elements of X will be larger than $k + a$). Hence the probability is

$$\frac{\binom{n+k+a}{n+k}}{\binom{k+a}{k} \cdot \binom{n+k+a}{n}} = \frac{1}{\binom{n+k}{k}}$$

where the identity can be seen by expanding the binomial coefficients on both sides into factorials and canceling.

Since the right hand side is independent of a, the claim follows.

Solution 2. Let f be the increasing bijection from $\{1,2,\dots,k+a\}$ to $\{1,\dots,k+a+n\} \setminus Y$. Note that $\min(Y) > \max(X)$ if and only if $\min(Y) > \max(f(X))$.

Note that

$$\{Z_n := Y, Z_k := f(X), Z_a := f(\{1,2,\dots,k+a\} \setminus X)\}$$

is a random partition of

$$\{1,\dots,n+k+a\} = Z_n \sqcup Z_k \sqcup Z_a$$

into an n-subset, k-subset, and a-subset.

If an a-subset Z_a is fixed, the conditional probability that $\min(Z_k) > \max(Z_n)$ always equals $1/\binom{n+k}{k}$. Therefore the total probability also equals $1/\binom{n+k}{k}$.

Problem 3. We say that a positive real number d is *good* if there exists an infinite sequence $a_1, a_2, a_3, \dots \in (0,d)$ such that for each n, the points $a_1,\dots,a_n$ partition the interval $[0,d]$ into segments of length at most $1/n$ each. Find

$$\sup\left\{d \;\middle|\; d \text{ is good}\right\}.$$

(proposed by Josef Tkadlec)

Hint: To get an upper bound, use that some of the gaps after n steps are still intact some steps later.

Solution. Let $d^\star = \sup\{d \mid d \text{ is good}\}$. We will show that $d^\star = \ln(2) \doteq 0.693$.

1. $d^\star \le \ln 2$:

   Assume that some d is good and let $a_1, a_2, \dots$ be the witness sequence.

   Fix an integer n. By assumption, the prefix $a_1,\dots,a_n$ of the sequence splits the interval $[0,d]$ into $n+1$ parts, each of length at most $1/n$.

   Let $0 \le \ell_1 \le \ell_2 \le \dots \le \ell_{n+1}$ be the lengths of these parts. Now for each $k = 1,\dots,n$ after placing the next k terms $a_{n+1},\dots,a_{n+k}$, at least $n+1-k$ of these initial parts remain intact. Hence $\ell_{n+1-k} \le \frac{1}{n+k}$. Hence

   $$d = \ell_1 + \dots + \ell_{n+1} \le \frac{1}{n} + \frac{1}{n+1} + \dots + \frac{1}{2n}. \tag{2}$$

   As $n \to \infty$, the RHS tends to $\ln(2)$ showing that $d \le \ln(2)$.

   Hence $d^\star \le \ln 2$ as desired.

2. $d^\star \ge \ln 2$:

   Observe that

   $$\ln 2 = \ln 2n - \ln n = \sum_{i=1}^{n} \ln(n+i) - \ln(n+i-1) = \sum_{i=1}^{n} \ln\left(1 + \frac{1}{n+i-1}\right).$$

   Interpreting the summands as lengths, we think of the sum as the lengths of a partition of the segment $[0, \ln 2]$ in n parts. Moreover, the maximal length of the parts is $\ln(1+1/n) < 1/n$.

   Changing n to $n+1$ in the sum keeps the values of the sum, removes the summand $\ln(1+1/n)$, and adds two summands

   $$\ln\left(1 + \frac{1}{2n}\right) + \ln\left(1 + \frac{1}{2n+1}\right) = \ln\left(1 + \frac{1}{n}\right).$$

This transformation may be realized by adding one partition point in the segment of length $\ln(1 + 1/n)$.

In total, we obtain a scheme to add partition points one by one, all the time keeping the assumption that once we have $n-1$ partition points and n partition segments, all the partition segments are smaller than $1/n$.

The first terms of the constructed sequence will be $a_1 = \ln\frac{3}{2}, a_2 = \ln\frac{5}{4}, a_3 = \ln\frac{7}{4}, a_4 = \ln\frac{9}{8}, \ldots$.

Remark. This remark describes in fact the same solution from a different view and some ideas behind it. It could be erased after marking is finished. Estimate (2) is quite natural. To prove that RHS tends to $\ln 2$ we use some integral estimates by

$$\int_n^{2n+1} \frac{1}{x} dx = \ln(2n+1) - \ln n.$$

Here we can observe that

$$\int_n^{2n} \frac{1}{x} dx = \ln 2$$

is independent of n. This can help us with the construction since the above equality means

$$I_1 = \int_n^{n+1} \frac{1}{x} dx = \int_{2n}^{2n+1} \frac{1}{x} dx + \int_{2n+1}^{2n+2} \frac{1}{x} dx = I_2 + I_3,$$

so, interval of length I_1 can be splitted into two intervals of lengths I_2 and I_3. In fact, after placing the point a_n in the construction for $d = \ln 2$, the lengths of the $n+1$ intervals are

$$\int_{n+1}^{n+2} \frac{1}{x}, \int_{n+2}^{n+3} \frac{1}{x}, \ldots, \int_{2n+1}^{2n+2} \frac{1}{x}$$

with total length

$$d = \int_{n+1}^{2n+2} \frac{1}{x} = \ln 2.$$

Problem 4. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Suppose that for every $\varepsilon > 0$, there exists a function $g : \mathbb{R} \to (0, \infty)$ such that for every pair (x, y) of real numbers,

$$\text{if} \quad |x - y| < \min\{g(x), g(y)\}, \quad \text{then} \quad \big|f(x) - f(y)\big| < \varepsilon.$$

Prove that f is the pointwise limit of a sequence of continuous $\mathbb{R} \to \mathbb{R}$ functions, i.e., there is a sequence $h_1, h_2, \ldots$ of continuous $\mathbb{R} \to \mathbb{R}$ functions such that $\lim_{n\to\infty} h_n(x) = f(x)$ for every $x \in \mathbb{R}$.

(proposed by Camille Mau, Nanyang Technological University, Singapore)

Hint: Start from a segment in place of $\mathbb{R}$ and use its compactness. Or recall the cool things called "the Lebesgue characterization theorem" and "the Baire characterization theorem".

Solution 1. Since g depends also on ε, let us use the notation $g(x, \varepsilon)$. Considering only $\varepsilon = 1/n$ for positive integer n will suffice to reach our conclusions, hence we may use $\min\{g(x, 1/m) \mid m \le n\}$ in place of $g(x, 1/n)$ and thus assume $g(x, \varepsilon)$ decreasing in ε.

For any $x \in \mathbb{R}$, choose $\delta_n(x) = \min\{1/n, g(x, 1/n)\}$. Of the $\delta_n(x)$-neighborhoods of all x select (using local compactness of the reals) an inclusion-minimal locally finite covering $\{U_i\}$. From its inclusion-minimality it follows that we may enumerate U_i with $i \in \mathbb{Z}$ so that $U_i \cap U_j \ne \varnothing$ only when $|i - j| \le 1$ and the enumeration goes from left to right on the real line. For an assumed n, let x_i be the center of U_i and $\delta_i = \delta_n(x_i)$, so that $U_i = (x_i - \delta_i, x_i + \delta_i)$ and $\delta_i < 1/n$ for all i.

Now define a continuous $f_n : \mathbb{R} \to \mathbb{R}$ so that it equals $f(x_i)$ in $U_i \setminus (U_{i-1} \cup U_{i+1})$, and so that f_n changes continuously between $f(x_{i-1})$ and $f(x_i)$ in the intersection $U_{i-1} \cap U_i$.

Now we show that $f_n \to f$ pointwise. Fix a point x and $\varepsilon = 1/m > 0$, and choose

$$n > \max\{1/g(x, \varepsilon), m\}.$$

Examine the construction of f_n for any such n. Observe that $g(x, \varepsilon) > 1/n > \delta_i$ and $1/n < 1/m$. There are two cases:

- x belongs to the unique U_i. Then using the monotonicity of $g(x,\varepsilon)$ in ε we have

$$|x_i - x| < \delta_i \leq \min\left\{g\left(x_i, \frac{1}{n}\right), g(x,\varepsilon)\right\} \leq \min\left\{g(x_i,\varepsilon), g(x,\varepsilon)\right\}.$$

Hence

$$|f(x) - f_n(x)| = |f(x) - f(x_i)| < \varepsilon.$$

- x belongs to $U_{i-1} \cap U_i$. Similar to the previous case,

$$|f(x) - f(x_{i-1})|, |f(x) - f(x_i)| < \varepsilon.$$

Since $f_n(x)$ is between $f_n(x_{i-1}) = f(x_{i-1})$ and $f_n(x_i) = f(x_i)$ by construction, we have

$$|f(x) - f_n(x)| < \varepsilon.$$

We have that $|f(x) - f_n(x)| < \varepsilon$ holds for sufficiently large n, which proves the pointwise convergence.

Solution 2. This solution uses the Baire characterization theorem: *A function $f : \mathbb{R} \to \mathbb{R}$ is a pointwise limit of continuous functions if and only if its restriction to every non-empty closed subset of $\mathbb{R}$ has a point of continuity.*

Assume the contrary in view of the above theorem: $A \subseteq \mathbb{R}$ is a non-empty closed set and f has no point of continuity in A. Let's think that f is defined only on A.

Then for all $x \in A$ there exist rationals $p < q$ for which $\limsup_x f > q$, $\liminf_x f < p$. Apply the Baire category theorem: *If a complete metric space A is a countable union of sets then some of the sets is dense in a positive radius metric ball of A.* It follows that there exist p and q, which serve for a subset $B \subset A$ which is dense on a certain ball (in the induced metric of the real line) $A_1 \subset A$. It yields that both sets $Q = f^{-1}(q,\infty)$ and $P = f^{-1}(-\infty,p)$ are dense in A_1.

Choose $\varepsilon = (q-p)/10$ and find k for which the set $S = \{x : g(x) > 1/k\}$ is also dense on a certain ball $A_2 \subset A_1$. Partition S into subsets where $f(x) > (p+q)/2$ and $f(x) \leqslant (p+q)/2$, one of them is again dense somewhere in A_3, say the latter.

Now take any point $y \in A_3 \cap Q$ and a very close (within distance $\min(1/k, g(y))$) to y point x with $g(x) > 1/k$ but $f(x) \leqslant (p+q)/2$. This pair x, y contradicts the property of f from the problem statement.

Solution 3. This solution uses the Lebesgue characterization theorem: *If $f : \mathbb{R} \to \mathbb{R}$ is a function and, for all real c, the sublevel and superlevel sets $\{x \mid f(x) \geqslant c\}$, $\{x \mid f(x) \leqslant c\}$ are countable intersections of open sets then f is a pointwise limit of continuous functions.*

Now the solution follows from the formula with a countable intersection of the unions of intervals:

$$\{x \mid f(x) \geqslant c\} = \bigcap_{n,k=1}^{\infty} \bigcup_{\substack{y \in \mathbb{R} \\ f(y) \geqslant c}} \left(y - \min\left\{\frac{1}{k}, g\left(y, \frac{1}{n}\right)\right\}, y + \min\left\{\frac{1}{k}, g\left(y, \frac{1}{n}\right)\right\}\right) \qquad (*)$$

and the similar formula for $\{x : f(x) \leqslant c\}$. It remains to prove $(*)$.

The left hand side is obviously contained in the right hand side, just put $y = x$.

To prove the opposite inclusion assume the contrary, that $f(x) < c$, but x is contained in the right hand side. Choose a positive integer n such that $f(x) < c - 1/n$ and k such that $g(x, 1/n) > 1/k$. Then, since x belongs to the right hand side, we see that there exists y such that $f(y) \geqslant c$ and

$$|x - y| < \min\left\{g\left(y, \frac{1}{n}\right), \frac{1}{k}\right\} \leqslant \min\left\{g\left(y, \frac{1}{n}\right), g\left(x, \frac{1}{n}\right)\right\},$$

which yields $f(x) \geqslant f(y) - 1/n \geqslant c - 1/n$, a contradiction.

IMC 2021 Online

Second Day, August 4, 2021

Solutions

Problem 5. Let A be a real $n \times n$ matrix and suppose that for every positive integer m there exists a real symmetric matrix B such that

$$2021B = A^m + B^2.$$

Prove that $|\det A| \le 1$.

(proposed by Rafael Filipe dos Santos, Instituto Militar de Engenharia, Rio de Janeiro)

Hint: The determinant is the product of the eigenvalues.

Solution. Let B_m be the corresponding matrix B depending on m:

$$2021B_m = A^m + B_m^2.$$

For $m = 1$, we obtain $A = 2021B_1 - B_1^2$. Since B_1 is real and symmetric, so is A. Thus A is diagonalizable and all eigenvalues of A are real.

Now fix a positive integer m and let λ be any real eigenvalue of A. Considering the diagonal form of both A and B_m, we know that there exists a real eigenvalue μ of B_m such that

$$2021\mu = \lambda^m + \mu^2 \Rightarrow \mu^2 - 2021\mu + \lambda^m = 0.$$

The last equation is a second degree equation with a real root. Therefore, the discriminant is non-negative:

$$2021^2 - 4\lambda^m \ge 0 \Rightarrow \lambda^m \le \frac{2021^2}{4}.$$

If $|\lambda| > 1$, letting m even sufficiently large we reach a contradiction. Thus $|\lambda| \le 1$.

Finally, since $\det A$ is the product of the eigenvalues of A and each of them has absolute value less then or equal to 1, we get $|\det A| \le 1$ as desired.

Solution. Different solution can be found in paper s2002

Problem 6. For a prime number p, let $\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$ be the group of invertible 2×2 matrices of residues modulo p, and let S_p be the symmetric group (the group of all permutations) on p elements. Show that there is no injective group homomorphism $\varphi : \mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z}) \to S_p$.

(proposed by Thiago Landim, Sorbonne University, Paris)

Hint: First find what the monomorphism must do with elements of order p.

Solution. For $p = 2$, just note that $\mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z})$ has more than $2 = |S_2|$ elements.

From now on, let p be an odd prime and suppose that there exists such a homomorphism.

The matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

has order p and commutes with the matrix

$$B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

of order 2, hence AB has order $2p$. But there is no permutation in S_p of order $2p$ since only p-cycles have order divisible by p, and their order is exactly p.

Problem 7. Let $D \subseteq \mathbb{C}$ be an open set containing the closed unit disk $\{z : |z| \le 1\}$. Let $f : D \to \mathbb{C}$ be a holomorphic function, and let $p(z)$ be a monic polynomial. Prove that

$$\left|f(0)\right| \le \max_{|z|=1} \left|f(z)p(z)\right|.$$

(proposed by Lars Hörmander)

Hint: Apply the maximum principle or the Cauchy formula to a suitable function $f(z)q(z)$.

Solution.

Let $q(z) = z^n \cdot \overline{p(1/\overline{z})}$, or more explicitly, if

$$p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0,$$

let

$$q(z) = 1 + \overline{a_{n-1}}z + \cdots + \overline{a_0}z^n.$$

Note that for $|z| = 1$ we have $1/\overline{z} = z$ and hence $|q(z)| = |p(z)|$. Hence by the maximum principle or the Cauchy formula for the product of f and q, it follows that

$$|f(0)| = |f(0)q(0)| \le \max_{|z|=1} |f(z)q(z)| = \max_{|z|=1} |f(z)p(z)|.$$

Problem 8. Let n be a positive integer. At most how many distinct unit vectors can be selected in $\mathbb{R}^n$ such that from any three of them, at least two are orthogonal?

(proposed by Alexander Polyanskii, Moscow Institute of Physics and Technology; based on results of Paul Erdős and Moshe Rosenfeld)

Hint: Play with the Gram matrix of these vectors.

Solution 1. $2n$ is the maximal number.

An example of $2n$ vectors in the set is given by a basis and its opposite vectors. In the rest of the text we prove that it is impossible to have $2n+1$ vectors in the set.

Consider the Gram matrix A with entries $a_{ij} = e_i \cdot e_j$. Its rank is at most n, its eigenvalues are real and non-negative. Put $B = A - I_{2n+1}$, this is the same matrix, but with zeros on the diagonal. The eigenvalues of B are real, greater or equal to -1, and the multiplicity of -1 is at least $n+1$.

The matrix $C = B^3$ has the following diagonal entries

$$c_{ii} = \sum_{i \ne j \ne k \ne i} a_{ij}a_{jk}a_{ki}.$$

The problem statement implies that in every summand of this expression at least one factor is zero. Hence $\operatorname{tr} C = 0$. Let $x_1, \ldots, x_m$ be the positive eigenvalues of B, their number is $m \le n$ as noted above. From $\operatorname{tr} B = \operatorname{tr} C$ we deduce (taking into account that the eigenvalues between -1 and 0 satisfy $\lambda^3 \ge \lambda$):

$$x_1 + \cdots + x_m \ge x_1^3 + \cdots + x_m^3$$

Applying $\operatorname{tr} C = 0$ once again and noting that C has eigenvalue -1 of multiplicity at least $n+1$, we obtain

$$x_1^3 + \cdots + x_m^3 \geq n + 1.$$

It also follows that

$$\left(x_1 + \cdots + x_m\right)^3 \geq \left(x_1^3 + \cdots + x_m^3\right)(n+1)^2.$$

By Hölder's inequality, we obtain

$$\left(x_1^3 + \cdots + x_m^3\right) m^2 \geq \left(x_1 + \cdots + x_m\right)^3,$$

which is a contradiction with $m \leq n$.

Solution 2. Let P_i denote the projection onto i-th vector, $i = 1, \ldots, N$. Then our relation reads as $\operatorname{tr}(P_i P_j P_k) = 0$ for distinct i, j, k. Consider the operator $Q = \sum_{i=1}^{N} P_i$, it is non-negative definite, let $t_1, \ldots, t_n$ be its eigenvalues, $\sum t_i = \operatorname{tr} Q = N$. We get

$$\sum t_i^3 = \operatorname{tr} Q^3 = N + 6 \sum_{i<j} \operatorname{tr} P_i P_j = N + 3(\operatorname{tr} Q^2 - N) = 3 \sum t_i^2 - 2N$$

(we used the obvious identities like $\operatorname{tr} P_i P_j P_i = \operatorname{tr} P_i^2 P_j = \operatorname{tr} P_i P_j$). But $(t_i - 2)^2 (t_i + 1) = t_i^3 - 3t_i^2 + 4 \geqslant 0$, thus $-2N = \sum t_i^3 - 3t_i^2 \geqslant -4n$ and $N \leqslant 2n$.

IMC 2022

First Day, August 3, 2022

Solutions

Problem 1. Let $f : [0, 1] \to (0, \infty)$ be an integrable function such that $f(x) \cdot f(1 - x) = 1$ for all $x \in [0, 1]$. Prove that

$$\int_0^1 f(x)\ \mathrm{d}x \geq 1.$$

(proposed by Mike Daas, Universiteit Leiden)

Hint: Apply the AM–GM inequality.

Solution 1. By the AM–GM inequlity we have

$$f(x) + f(1 - x) \geq 2\sqrt{f(x)f(1 - x)} = 2.$$

By integrating in the interval $[0, \frac{1}{2}]$ we get

$$\int_0^1 f(x)\mathrm{d}x = \int_0^{\frac{1}{2}} f(x)\mathrm{d}x + \int_0^{\frac{1}{2}} f(1 - x)\mathrm{d}x = \int_0^{\frac{1}{2}} \big(f(x) + f(1 - x)\big)\mathrm{d}x \geq \int_0^{\frac{1}{2}} 2\mathrm{d}x = 1.$$

Solution 2. From the condition, we have

$$\int_0^1 f(x)dx = \int_0^1 f(1 - x)dx = \int_0^1 \frac{1}{f(x)}dx$$

and hence, using the positivity of f, the claim follows since

$$\left(\int_0^1 f(x)dx\right)^2 = \int_0^1 f(x)dx \cdot \int_0^1 \frac{1}{f(x)}dx \geq \left(\int_0^1 1dx\right)^2 \geq 1$$

by the Cauchy-Schwarz inequality.

Problem 2. Let n be a positive integer. Find all $n \times n$ real matrices A with only real eigenvalues satisfying

$$A + A^k = A^T$$

for some integer $k \geq n$.

(A^T denotes the transpose of A.)

(proposed by Camille Mau, Nanyang Technological University)

Hint: Consider the eigenvalues of A.

Solution 1. Taking the transpose of the matrix equation and substituting we have

$$A^T + (A^T)^k = A \implies A + A^k + (A + A^k)^k = A \implies A^k(I + (I + A^{k-1})^k) = 0.$$

Hence $p(x) = x^k(1 + (1 + x^{k-1})^k)$ is an annihilating polynomial for A. It follows that all eigenvalues of A must occur as roots of p (possibly with different multiplicities). Note that for all $x \in \mathbb{R}$ (this can be seen by considering even/odd cases on k),

$$(1 + x^{k-1})^k \geq 0,$$

and we conclude that the only eigenvalue of A is 0 with multiplicity n.

Thus A is nilpotent, and since A is $n \times n$, $A^n = 0$. It follows $A^k = 0$, and $A = A^T$. Hence A can only be the zero matrix: A is real symmetric and so is orthogonally diagonalizable, and all its eigenvalues are 0.

Remark. It's fairly easy to prove that eigenvalues must occur as roots of any annihilating polynomial. If λ is an eigenvalue and v an associated eigenvector, then $f(A)v = f(\lambda)v$. If f annihilates A, then $f(\lambda)v = 0$, and since $v \neq 0$, $f(\lambda) = 0$.

Solution 2. If λ is an eigenvalue of A, then $\lambda + \lambda^k$ is an eigenvalue of $A^T = A + A^k$, thus of A too. Now, if k is odd, then taking λ with maximal absolute value we get a contradiction unless all eigenvalues are 0. If k is even, the same contradiction is obtained by comparing the traces of A^T and $A + A^k$.

Hence all eigenvalues are zero and A is nilpotent. The hypothesis that $k \geq n$ ensures $A = A^T$. A nilpotent self-adjoint operator is diagonalizable and is necessarily zero.

Problem 3. Let p be a prime number. A flea is staying at point 0 of the real line. At each minute, the flea has three possibilities: to stay at its position, or to move by 1 to the left or to the right. After $p-1$ minutes, it wants to be at 0 again. Denote by $f(p)$ the number of its strategies to do this (for example, $f(3)=3$: it may either stay at 0 for the entire time, or go to the left and then to the right, or go to the right and then to the left). Find $f(p)$ modulo p.

(proposed by Fedor Petrov, St. Petersburg)

Hint: Find a recurrence for $f(p)$ or use generating functions.

Solution 1. The answer is $f(p) \equiv 0 \mod 3$ for $p=3$, $f(p) \equiv 1 \mod 3$ for $p=3k+1$, and $f(p) \equiv -1 \mod 3$ for $p=3k-1$.

The case $p=3$ is already considered, let further $p \neq 3$. For a residue i modulo p denote by $a_i(k)$ the number of Flea strategies for which she is at position i modulo p after k minutes. Then $f(p)=a_0(p-1)$. The natural recurrence is $a_i(k+1)=a_{i-1}(k)+a_i(k)+a_{i+1}(k)$, where the indices are taken modulo p. The idea is that modulo p we have $a_0(p) \equiv 3$ and $a_i(p) \equiv 0$. Indeed, for all strategies for p minutes for which not all p actions are the same, we may cyclically shift the actions, and so we partition such strategies onto groups by p strategies which result with the same i. Remaining three strategies correspond to $i=0$. Thus, if we denote $x_i=a_i(p-1)$, we get a system of equations $x_{-1}+x_0+x_1=3$, $x_{i-1}+x_i+x_{i+1}=0$ for all $i=1,\ldots,p-1$. It is not hard to solve this system (using the 3-periodicity, for example). For $p=3k+1$ we get $(x_0,x_1,\ldots,x_{p-1})=(1,1,-2,1,1,-2,\ldots,1)$, and $(x_0,x_1,\ldots,x_{p-1})=(-1,2,-1,-1,2,\ldots,2)$ for $p=3k+2$.

Solution 2. Note that $f(p)$ is the constant term of the Laurent polynomial $(x+1+1/x)^{p-1}$ (the moves to right, to left and staying are in natural correspondence with x, $1/x$ and 1.) Thus, working with power series over $\mathbb{F}_p$ we get (using the notation $[x^k]P(x)$ for the coefficient of x^k in P)

$$\begin{aligned} f(p) &= [x^{p-1}](1+x+x^2)^{p-1} = [x^{p-1}](1-x^3)^{p-1}(1-x)^{1-p} = [x^{p-1}](1-x^3)^p(1-x)^{-p}(1-x^3)^{-1}(1-x) \\ &= [x^{p-1}](1-x^{3p})(1-x^p)^{-1}(1-x^3)^{-1}(1-x) = [x^{p-1}](1-x^3)^{-1}(1-x), \end{aligned}$$

and expanding $(1-x^3)^{-1}=\sum x^{3k}$ we get the answer.

Problem 4. Let $n > 3$ be an integer. Let Ω be the set of all triples of distinct elements of $\{1, 2, \ldots, n\}$. Let m denote the minimal number of colours which suffice to colour Ω so that whenever $1 \le a < b < c < d \le n$, the triples $\{a, b, c\}$ and $\{b, c, d\}$ have different colours. Prove that

$$\frac{1}{100} \log \log n \leqslant m \leqslant 100 \log \log n.$$

(proposed by Danila Cherkashin, St. Petersburg)

Hint: Define two graphs, one on Ω and another graph on pairs (2-element sets).

Solution. For $k = 1, 2, \ldots, n$ denote by Ω_k the set of all $\binom{n}{k}$ k-subsets of $[n]$. For each $k = 1, 2, \ldots, n-1$ define a directed graph G_k whose vertices are elements of Ω_k, and edges correspond to elements of Ω_{k+1} as follows: if $1 \leqslant a_1 < a_2 < \ldots < a_{k+1} \leqslant n$, then the edge of G_k corresponding to $(a_1, \ldots, a_{k+1})$ goes from $(a_1, \ldots, a_k)$ to $(a_2, \ldots, a_{k+1})$.

For a directed graph $G = (V, E)$ we call a subset $E_1 \subset E$ *admissible*, if E_1 does not contain a directed path $a - b - c$ of length 2. Define *b-index* $b(G)$ of the G as the minimal number of admissible sets which cover E. As usual, a subset $V_1 \subset V$ is called *independent*, if there are no edges with both endpoints in V_1; a *chromatic number* of G is defined as the minimal number of independent sets which cover V.

A straightforward but crucial observation is the following

Lemma. For all $k = 2, 3, \ldots, n$ a subset $A_k \subset \Omega_k$ is independent in G_k if and only if it is admissible as a set of edges of G_{k-1}.

Corollary. $\chi(G_k) = b(G_{k-1})$ for all $k = 2, 3, \ldots, n$.

Now the bounds for numbers $\chi(G_k)$ follow by induction using the following general

Lemma. For a directed graph $G = (V, E)$ we have

$$\log_2 \chi(G) \leqslant b(G) \leqslant 2\lceil \log_2 \chi(G) \rceil.$$

Proof. 1) Denote $b(G) = m$ and prove that $\log_2 \chi(G) \leqslant m$. For this we take a covering of E by m admissible subsets $E_1, \ldots, E_m$ and define a color $c(v)$ of a vertex $v \in V$ as the following subset of $[m]$: $c(v) := \{i \in [m] : \exists vw \in E_i\}$. Note that for any edge $vw \in E$ there exists i such that $vw \in E_i$ which yields $i \in c(v)$ and $i \notin c(w)$, therefore $c(v) \neq c(w)$. So, each color class is an independent set and we get $\chi(G) \leqslant 2^m$ as needed.

2) Denote $\chi(G) = k$ and prove that $b(G) \leqslant 2\lceil \log_2 k \rceil$. Take a proper coloring $\tau \colon V \to \{0, 1, \ldots, k-1\}$ (that means that $\tau(u) \neq \tau(v)$ for all edges $vu \in E$). For an integer $x \in \{0, 1, \ldots, k-1\}$ take a binary representation $x = \sum_{i=0}^{r-1} \varepsilon_i(x) 2^i$, $\varepsilon_i(x) \in \{0, 1\}$, where $r = \lceil \log_2 k \rceil$. Consider the following $2r$ subsets of E, two subsets $E_{i,+}$ and $E_{i,-}$ for each $i \in \{0, 1, \ldots, k-1\}$:

$$E_{i,+} = \{vu \in E \colon \varepsilon_i(\tau(v)) = 0, \varepsilon_i(\tau(u)) = 1\},$$
$$E_{i,-} = \{vu \in E \colon \varepsilon_i(\tau(v)) = 1, \varepsilon_i(\tau(u)) = 0\}.$$

Each of them is admissible, and they cover E, thus $b(G) \leqslant 2r$.

Note that $\chi(G_1) = n$, thus $b(G_1) \geqslant \log_2 n$. Actually we have $b(G_1) = \lceil \log_2 n \rceil$: indeed, if we define $\tau(v) = v - 1$ for all $v \in [n] = \Omega_1$, then the above sets $E_{i,+}$ cover all edges of G_1.

The Lemma above now yields for our number $m = \chi(G_3) = b(G_2)$ the following bounds, which are better than required:

$$b(G_2) \geqslant \log_2 \chi(G_2) = \log_2 b(G_1) = \log_2 \lceil \log_2 n \rceil$$
$$b(G_2) \leqslant 2\lceil \log_2 \chi(G_2) \rceil = 2\lceil \log_2 b(G_1) \rceil = 2\lceil \log_2 \lceil \log_2 n \rceil \rceil.$$

Remark. Actually the upper bound in the Lemma may be improved to $(1 + o(1)) \log_2 \chi(G)$ that yields $m = (1 + o(1)) \log_2 \log_2 n$.

IMC 2022

Second Day, August 4, 2022

Solutions

Problem 5. We colour all the sides and diagonals of a regular polygon P with 43 vertices either red or blue in such a way that every vertex is an endpoint of 20 red segments and 22 blue segments. A triangle formed by vertices of P is called monochromatic if all of its sides have the same colour. Suppose that there are 2022 blue monochromatic triangles. How many red monochromatic triangles are there?

(proposed by Mike Daas, Universiteit Leiden)

Hint: Call two connecting edges a *cherry*. Double-count cherries.

Solution. 1 Define a *cherry* to be a set of two distinct edges from K_{43} that have a vertex in common. We observe that a monochromatic triangle always contains three monochromatic cherries, and that a polychromatic triangle always contains one monochromatic cherry and two polychromatic cherries. Therefore we study the quantity $2M - P$, where M is the number of monochromatic cherries and P is the number of polychromatic cherries. By observing that every cherry is part of a unique triangle, we can split this quantity up into all the distinct triangles in K_{43}. By construction the contribution of a polychromatic triangle will vanish, whereas a monochromatic triangle will contribute 6. We conclude that

$$2M - P = 6 \cdot \{\text{number of monochromatic triangles}\}.$$

Consider any vertex v. Let M_v be the number of monochromatic cherries with central vertex v and P_v the number such polychromatic cherries. It then follows that

$$M_v = \frac{20 \cdot 19}{2} + \frac{22 \cdot 21}{2} = 421 \quad \text{and} \quad P_v = 20 \cdot 22 = 440.$$

In other words, for any vertex v it holds that $2M_v - P_v = 402$. Adding up all these contributions, we find that

$$2M - P = 43 \cdot 402.$$

We conclude that there are $43 \cdot 402/6 = 43 \cdot 67 = 2881$ monochromatic triangles in total. Since 2022 of these were blue, 859 must be red.

Problem 6. Let $p > 2$ be a prime number. Prove that there is a permutation $(x_1, x_2, ..., x_{p-1})$ of the numbers $(1, 2, ..., p-1)$ such that
$$x_1x_2 + x_2x_3 + ... + x_{p-2}x_{p-1} \equiv 2 \pmod{p}.$$
(proposed by Giorgi Arabidze, Tbilisi Free University, Georgia)

Hint:

Solution 1. We show such a permutation.

Let $x_i \equiv i^{-1} \pmod{p}$ for $i = 1, 2, \cdots, p-1$. Then

$$\sum_{i=1}^{p-2} x_i x_{i+1} \equiv \sum_{i=1}^{p-2} \frac{1}{i} \cdot \frac{1}{i+1} \equiv \sum_{i=1}^{p-2} \left(\frac{1}{i} - \frac{1}{i+1} \right) \equiv 1 - \frac{1}{p-1} \equiv \frac{p-2}{p-1} \equiv 2 \pmod{p}$$

Solution 2. We begin by noting that the identity permutation yields the value

$$1 \cdot 2 + 2 \cdot 3 + \cdots + (p-2)(p-1) = 2 \cdot \binom{p}{3} \equiv 0 \pmod{p}$$

as soon as $p > 3$. The idea now is to perturb that permutation to obtain the desired value 2.

One thing we can do is to replace $(i, i+1, i+2, i+3)$ by $(i, i+2, i+1, i+3)$. Indeed, this will decrease the sum by 3. So if $p \equiv 2 \pmod{3}$, we can just take the permutation $(1, 3, 2, 4, 6, 5, 7, \dots, p-4, p-2, p-3, p-1)$ i.e. exchanging $3k-1$ and $3k$ whenever $k = 1, 2, \dots, \frac{p-2}{3}$. This means we decrease the sum $\frac{p-2}{3}$ times by 3, leading to a remaining sum of $-(p-2) \equiv 2 \pmod{p}$.

If $p \equiv 1 \pmod{3}$, this strategy does not work immediately. Instead, we can change $(1, 2, 3, 4, 5)$ to $(1, 4, 3, 2, 5)$ resulting in a decrement of the sum by 8. If we then exchange $3k$ and $3k+1$ for $k = 2, 3, \dots, \frac{p-7}{3}$ as before, we get another $\frac{p-10}{3}$ times a decrement by 3, leading to a remaining sum of $-8 - \frac{p-10}{3} \cdot 3 \equiv 2 \pmod{p}$.

Of course this only works if $p \geq 13$. It thus remains to consider the cases $p = 3$ and $p = 7$ by hand. For $p = 3$, we just take $(1, 2)$ and for $p = 7$ we can take $(1, 4, 5, 2, 3, 6)$.

Problem 7. Let $A_1, A_2, \ldots, A_k$ be $n \times n$ idempotent complex matrices such that

$$A_iA_j = -A_jA_i \quad \text{for all } i \neq j.$$

Prove that at least one of the given matrices has rank $\leq \frac{n}{k}$.

(A matrix A is called idempotent if $A^2 = A$.)

(proposed by Danila Belousov, Novosibirsk)

Hint: Consider the trace and the rank of A.

Solution 1.

Lemma. For any idempotent matrix B

$$\mathrm{tr}(B) = \mathrm{rank}(B)$$

Proof. Observe that an idempotent matrix satisfies the equation $\lambda(1-\lambda) = 0$. Hence the minimal polynomial is a product of linear factors and the matrix is diagonalizable. Therefore, the rank of the matrix equals the number of non-zero eigenvalues. Since the matrix has eigenvalues 0 or 1, this provides that the trace is equal to the number of unity eigenvalues, or non-zero eigenvalues.

It can be shown that $\sum\limits_{i=1}^{k} A_i$ is also an idempotent. Indeed,

$$\left(\sum_{i=1}^{k} A_i\right)^2 = \sum_{i=1}^{k} A_i^2 + \sum_{i \neq j}(A_iA_j + A_jA_i) = \sum_{i=1}^{k} A_i$$

Applying the lemma one can obtain

$$\sum_{i=1}^{k} rank(A_i) = \sum_{i=1}^{k} tr(A_i) = tr\left(\sum_{i=1}^{k} A_i\right) = rank\left(\sum_{i=1}^{k} A_i\right) \leqslant n$$

The required inequality follows.

Solution 2. We first prove that for idempotents A, B with $AB = -BA$ we already must have $AB = BA = 0$. Indeed, it is clear that $ABx = BAx = 0$ for $x \in \ker(A)$ so it suffices to prove the same for $x \in \mathrm{im}(A)$, i.e. when $Ax = x$. But then writing $Bx = y$ we have $Ay = -y$ i.e. $y = -Ay = -A^2y = Ay = -y$ and hence $y = 0$ so that again $ABx = BAx = 0$.

Henceforth, we can assume the stronger condition $A_iA_j = 0$ for all $i \neq j$. We next claim that all the image spaces V_i of A_i are linearly independent. This will imply the claim, since then the sum of their dimensions can be at most n, and so one of them has to be $\leq \frac{n}{k}$. Now, for the sake of contradiction, suppose that $\sum_i v_i = 0$ with $v_i \in V_i$ and w.l.o.g. $v_1 \neq 0$. But then

$$0 = A_1(v_1 + \cdots + v_k) = v_1 + A_1v_2 + \cdots + A_1v_k = v_1 + A_1A_2v_2 + \cdots + A_1A_kv_k = v_1$$

since $A_1A_i = 0$ for all i.

Remark. Here is a different argument for $AB = BA = 0$, without eigenvectors: multiplying by A and using its idempotence and the super-commutativity , we have

$$-BA = AB = A^2B = AAB = -ABA = BAA = BA^2 = BA$$

thus $BA = 0$.

Problem 8. Let $n, k \geq 3$ be integers, and let S be a circle. Let n blue points and k red points be chosen uniformly and independently at random on the circle S. Denote by F the intersection of the convex hull of the red points and the convex hull of the blue points. Let m be the number of vertices of the convex polygon F (in particular, $m = 0$ when F is empty). Find the expected value of m.

(proposed by Fedor Petrov, St. Petersburg)

Hint:

Solution 1. We prove that

$$E(m) = \frac{2kn}{n+k-1} - 2\frac{k!n!}{(k+n-1)!}.$$

Let $A_1, \ldots, A_n$ be blue points. Fix $i \in \{1, \ldots, n\}$. Enumerate our $n+k$ points starting from a blue point A_i counterclockwise as $A_i, X_{1,i}, X_{2,i}, \ldots, X_{(n+k-1),i}$. Denote the minimal index j for which the point $X_{j,i}$ is blue as $m(i)$. So, $A_i X_{m(i),i}$ is a side of the convex hull of blue points. Denote by b_i the following random variable:

$$b_i = \begin{cases} 1, & \text{if the chord } A_i X_{m(i),i} \text{ contains a side of } F \\ 0, & \text{otherwise.} \end{cases}$$

Define analogously k random variables $r_1, \ldots, r_k$ for the red points. Clearly,

$$m = b_1 + \ldots + b_n + r_1 + \ldots + r_k. \qquad (\heartsuit)$$

We proceed with computing the expectation of each b_i and r_j. Note that $b_i = 0$ if and only if all red points lie on the side of the line $A_i X_{m(i),i}$. This happens either if $m(i) = 1$, *i.e., the point* $X_{i,1}$ *is blue* (which happens with probability $\frac{n-1}{k+n-1}$), or if $i = k+1$, *points* $X_{1,i}, \ldots, X_{k,i}$ *are red, and points* $X_{k+1,i}, \ldots, X_{k+n-1,i}$ *are blue* (which happens with probability $1/\binom{k+n-1}{k}$, since all subsets of size k of $\{1, 2, \ldots, n+k-1\}$ have equal probabilities to correspond to the indices of red points between $X_{1,i}, \ldots, X_{n+k-1,i}$). Thus the expectation of b_i equals $1 - \frac{n-1}{k+n-1} - 1/\binom{k+n-1}{k} = \frac{k}{n+k-1} - \frac{k!(n-1)!}{(k+n-1)!}$. Analogously, the expectation of r_j equals $\frac{n}{n+k-1} - \frac{n!(k-1)!}{(k+n-1)!}$. It remains to use $(\heartsuit)$ and linearity of expectation.

Solution 2. Let $C_1, \ldots, C_{n+k}$ be the colours of the points, scanned counterclockwise from a fixed point on the circle. We consider the sequence as cyclic (so C_{n+k} is also adjacent to C_1). There are two cases: Either (i) all red points appear contiguously, followed by all blue points contiguously, or (ii) the red and blue points alternate at least twice. It can be seen that in the second case, m is exactly equal to the number of colour changes in the C_i sequence: For example, if C_i is red and C_{i+1} is blue, then the intersection of the red chord from C_i to the next red point with the blue chord from C_{i+1} to the previous blue point is a vertex of F, and every vertex is of this form. Case (i) is exceptional, as we have two colour changes, but $m = 0$, so it is 2 less than the number of changes in that case.

Now observe that the distribution of C_i is purely combinatorial: Each of the $\binom{n+k}{n,k}$ distributions of colours is equally likely (for example, because we can generate the distribution by first choosing all $n+k$ points on the circle, and then assigning colours uniformly). In particular the probability that $C_i C_{i+1}$ is a colour change is exactly $\frac{2nk}{(n+k)(n+k-1)}$, and by linearity of expectation, the total expected number of color changes (including $i = n+k$) is $n+k$ times this, i.e. $\frac{2nk}{n+k-1}$.

To get the expected value of m, we must subtract from the above 2 times the probability of case (i). Exactly $n+k$ of the $\binom{n+k}{n,k}$ distributions belong to case (i), so we must subtract $2(n+k)\binom{n+k}{n,k}^{-1} = 2\frac{n!k!}{(n+k-1)!}$, as claimed.

Solution 3. Let $A_1, \ldots, A_n$ be the blue points and $B_1, \ldots, B_k$ be the red points. For every pair of blue points A_i, A_j, $1 \leqslant i < j \leqslant n$, we evaluate the probability p that A_iA_j contains a side of F (it obviously does not depend on the choice of i and j). By q denote the analogous probability for the red points. Then by linearity of expectation we have $\mathbb{E}m = \binom{n}{2}p + \binom{k}{2}q$.

We proceed with finding p. Without loss of generality $i = 1$, $j = 2$. Let the length of the circle be 1, and the length of arc A_1A_2 (counterclockwise from A_1 to A_2) be x. Then x is uniformly distributed on $[0, 1]$. Then A_1A_2 contains a side of F if

(i) all blue points are on the same side of A_1A_2, but

(ii) the red points are not on the same side of A_1A_2.

The probability of (i) is $x^{n-2} + (1-x)^{n-2}$. The probability of (ii) is $1 - (x^k + (1-x)^{n-k})$.

Thus, using Beta function value $B(a, b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx = B(a, b) = \frac{(a-1)!(b-1)!}{(a+b-1)!}$ for positive integers a, b

$$p = \int_0^1 (x^{n-2} + (1-x)^{n-2})(1 - (x^k + (1-x)^{n-k}))dx = \frac{2}{n-1} - \frac{2}{n+k-1} - 2B(n-1, k+1)$$
$$= \frac{2}{n-1} - \frac{2}{n+k-1} - 2\frac{(n-2)!k!}{(n+k-1)!}.$$

Next,

$$\binom{n}{2}p = n - \frac{n(n-1)}{n+k-1} - \frac{n!k!}{(n+k-1)!} = \frac{nk}{n+k-1} - \frac{n!k!}{(n+k-1)!},$$

and by symmetry $\binom{k}{2}q$ takes the same value (that is in agreement with the observation that red and blue sides of F alternate).

IMC 2023

First Day, August 2, 2023

Solutions

Problem 1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ that have a continuous second derivative and for which the equality $f(7x+1) = 49f(x)$ holds for all $x \in \mathbb{R}$.

(proposed by Alex Avdiushenko, Neapolis University Paphos, Cyprus)

Hint:

- The fixed point of $7x + 1$ is $-1/6$.
- Differentiating twice cancels out the coefficient 49.

Solution. Differentiating the equation twice, we get

$$f''(7x+1) = f''(x) \quad \text{or} \quad f''(x) = f''\left(\frac{x-1}{7}\right). \tag{1}$$

Take an arbitrary $x \in \mathbb{R}$, and construct a sequence by the recurrence

$$x_0 = x, \quad x_{k+1} = \frac{x_k - 1}{7}.$$

By (1), the values of f'' at all points of this sequence are equal. The limit of this sequence is $-\frac{1}{6}$, since $\left|x_{k+1} + \frac{1}{6}\right| = \frac{1}{7}\left|x_k + \frac{1}{6}\right|$.

Due to the continuity of f'', the values of f'' at all points of this sequence are equal to $f''\left(-\frac{1}{6}\right)$, which means that $f''(x)$ is a constant.

Then f is an at most quadratic polynomial, $f(x) = ax^2 + bx + c$. Substituting this expression into the original equation, we get a system of equations, from which we find $a = 36c$, $b = 12c$, and hence

$$f(x) = c(6x+1)^2.$$

Problem 2. Let A, B and C be $n \times n$ matrices with complex entries satisfying

$$A^2 = B^2 = C^2 \quad \text{and} \quad B^3 = ABC + 2I.$$

Prove that $A^6 = I$.

(proposed by Mike Daas, Universiteit Leiden)

Hint: Factorize $B^3 - ABC$.

Solution. Note that $B^3 = A^2B$, from which it follows that

$$A^2B - ABC = 2I \implies A(AB - BC) = 2I.$$

Similarly, using that $B^3 = BC^2$, we find that

$$BC^2 - ABC = 2I \implies (BC - AB)C = 2I.$$

It follows that A is a left-inverse of $(AB - BC)/2$, whereas $-C$ is a right inverse. Hence $A = -C$ and as such, it must hold that $ABA = 2I - B^3$. It follows that ABA must commute with B, and so it follows that $(AB)^2 = (BA)^2$. Now we compute that

$$(AB - BA)(AB + BA) = (AB)^2 + AB^2A - BA^2B - (BA)^2 = (AB)^2 + A^4 - B^4 - (AB)^2 = 0.$$

However, we noted before that the matrix $AB - BC = AB + BA$ must be invertible. As such, it must follow that $AB = BA$. We conclude that $ABA = A^2B = B^3$ and so it readily follows that $B^3 = I$. Finally, $A^6 = B^6 = (B^3)^2 = I^2 = I$, completing the proof.

Problem 3. Find all polynomials P in two variables with real coefficients satisfying the identity

$$P(x, y)P(z, t) = P(xz - yt, xt + yz).$$

(proposed by Giorgi Arabidze, Free University of Tbilisi, Georgia)

Hint: The polynomials $(x+iy)^n$ and $(x-iy)^m$ are trivial complex solutions. Suppose that $P(x, y) = (x+iy)^n(x-iy)^m Q(x, y)$, where $Q(x, y)$ is divisible neither by $x+iy$ nor $x = iy$ and consider $Q(x, y)$.

Solution. First we find all polynomials $P(x, y)$ with complex coefficients which satisfies the condition of the problem statement. The identically zero polynomial clearly satisfies the condition. Let consider other polynomials.

Let $i^2 = -1$ and $P(x, y) = (x + iy)^n(x - iy)^m Q(x, y)$, where n and m are non-negative integers and $Q(x, y)$ is a polynomial with complex coefficients such that it is not divisible neither by $x + iy$ nor by $x - iy$. By the problem statement we have $Q(x, y)Q(z, t) = Q(xz - yt, xt + yz)$. Note that $z = t = 0$ gives $Q(x, y)Q(0, 0) = Q(0, 0)$. If $Q(0, 0) \neq 0$, then $Q(x, y) = 1$ for all x and y. Thus $P(x, y) = (x + iy)^n(x - iy)^m$. Now consider the case when $Q(0, 0) = 0$.

Let $x = iy$ and $z = -it$. We have $Q(iy, y)Q(-it, t) = Q(0, 0) = 0$ for all y and t. Since $Q(x, y)$ is not divisible by $x - iy$, $Q(iy, y)$ is not identically zero and since $Q(x, y)$ is not divisible by $x + iy$, $Q(-it, t)$ is not identically zero. Thus there exist y and t such that $Q(iy, y) \neq 0$ and $Q(-it, t) \neq 0$ which is impossible because $Q(iy, y)Q(-it, t) = 0$ for all y and t.

Finally, $P(x, y)$ polynomials with complex coefficients which satisfies the condition of the problem statement are $P(x, y) = 0$ and $P(x, y) = (x + iy)^n(x - iy)^m$. It is clear that if $n \neq m$, then $P(x, y) = (x + iy)^n(x - iy)^m$ cannot be polynomial with real coefficients. So we need to require $n = m$, and for this case $P(x, y) = (x + iy)^n(x - iy)^n = (x^2 + y^2)^n$.

So, the answer of the problem is $P(x, y) = 0$ and $P(x, y) = (x^2 + y^2)^n$ where n is any non-negative integer.

Problem 4. Let p be a prime number and let k be a positive integer. Suppose that the numbers $a_i = i^k + i$ for $i = 0, 1, \ldots, p-1$ form a complete residue system modulo p. What is the set of possible remainders of a_2 upon division by p?

(proposed by Tigran Hakobyan, Yerevan State University, Armenia)

Hint: Consider $\prod_{i=0}^{p-1}(i^k + i)$.

Solution. First observe that $p = 2$ does not satisfy the condtion, so p must be an odd prime.

Lemma. If $p > 2$ is a prime and $\mathbb{F}_p$ is the field containing p elements, then for any integer $1 \le n < p$ one has the following equality in the field $\mathbb{F}_p$

$$\prod_{\alpha\in\mathbb{F}_p^*}(1+\alpha^n) = \begin{cases} 0, & \text{if } \dfrac{p-1}{\gcd(p-1,n)} \text{ is even} \\ 2^n, & \text{otherwise} \end{cases}$$

Proof. We may safely assume that $n|p-1$ since it can be easily proved that the set of n-th powers of the elements of $\mathbb{F}_p^*$ coincides with the set of $\gcd(p-1,n)$-th powers of the same elements. Assume that $t_1, t_2, ..., t_n$ are the roots of the polynomial $t^n + 1 \in \mathbb{F}_p[x]$ in some extension of the field $\mathbb{F}_p$. It follows that

$$\prod_{\alpha\in\mathbb{F}_p^*}(1+\alpha^n) = \prod_{\alpha\in\mathbb{F}_p^*}\prod_{i=1}^{n}(\alpha - t_i) = \prod_{i=1}^{n}\prod_{\alpha\in\mathbb{F}_p^*}(\alpha - t_i) = \prod_{i=1}^{n}\prod_{\alpha\in\mathbb{F}_p^*}(t_i - \alpha) = \prod_{i=1}^{n}\Phi(t_i),$$

where we define $\Phi(t) = \prod_{\alpha\in\mathbb{F}_p^*}(t-\alpha) = t^{p-1} - 1$. Therefore

$$\prod_{\alpha\in\mathbb{F}_p^*}(1+\alpha^n) = \prod_{i=1}^{n}(t_i^{p-1} - 1) = \prod_{i=1}^{n}((t_i^n)^{\frac{p-1}{n}} - 1) = \prod_{i=1}^{n}((-1)^{\frac{p-1}{n}} - 1) = \begin{cases} 0, & \text{if } \frac{p-1}{n} \text{ is even} \\ 2^n, & \text{otherwise} \end{cases}$$

Let us now get back to our problem. Suppose the numbers $i^k + i, 0 \le i \le p-1$ form a complete residue system modulo p. It follows that

$$\prod_{\alpha\in\mathbb{F}_p^*}(\alpha^k + \alpha) = \prod_{\alpha\in\mathbb{F}_p^*}\alpha$$

so that $\prod_{\alpha\in\mathbb{F}_p^*}(\alpha^{k-1} + 1) = 1$ in $\mathbb{F}_p$. According to the Lemma, this means that $2^{k-1} = 1$ in $\mathbb{F}_p$, or equivalently, that $2^{k-1} \equiv 1 (\text{mod } p)$. Therefore $a_2 = 2^k + 2 \equiv 4 (\text{mod } p)$ so that the remainder of a_2 upon division by p is either 4 when $p > 3$ or is 1, when $p = 3$.

Problem 5. Fix positive integers n and k such that $2 \le k \le n$ and a set M consisting of n fruits. A *permutation* is a sequence $x = (x_1, x_2, \dots, x_n)$ such that $\{x_1, \dots, x_n\} = M$. Ivan *prefers* some (at least one) of these permutations. He realized that for every preferred permutation x, there exist k indices $i_1 < i_2 < \dots < i_k$ with the following property: for every $1 \le j < k$, if he swaps x_{i_j} and $x_{i_{j+1}}$, he obtains another preferred permutation.

Prove that he prefers at least $k!$ permutations.

(proposed by Ivan Mitrofanov, École Normale Superieur Paris)

Hint: For every permutation z of M, choose a preferred permutation x such that $\sum\limits_{m\in M} x^{-1}(m)z^{-1}(m)$ is maximal.

Solution. Let S be the set of all $n!$ permutations of M, and let P be the set of preferred permutations. For every permutation $x \in S$ and $m \in M$, let $x^{-1}(m)$ denote the unique number $i \in \{1, 2, \dots, n\}$ with $x_i = m$.

For every $x \in P$, define

$$A(x) = \left\{ z \in S \ : \ \forall y \in P \ \sum_{m\in M} x^{-1}(m)z^{-1}(m) \ge \sum_{m\in M} y^{-1}(m)z^{-1}(m) \right\}.$$

For every permutation $z \in S$, we can choose a permutation $x \in P$ for which $\sum\limits_{m\in M} x^{-1}(m)z^{-1}(m)$ is maximal, and then we have $z \in A(x)$; hence, all $z \in S$ is contained in at least one set $A(x)$.

So, it suffices to prove that $\left|A(x)\right| \le \dfrac{n!}{k!}$ for every preferred permutation x. Fix $x \in P$, and consider an arbitrary $z \in A(x)$. Let the indices $i_1 < \dots < i_k$ be as in the statement of the problem, and let $m_j = x_{i_j}$ for $j = 1, 2, \dots, k$.

For $s = 1, 2, \dots, k-1$ consider the permutation y obtained from x by swapping m_s and m_{s+1}. Since $y \in P$, the definition of $A(x)$ provides

$$i_s z^{-1}(m_s) + i_{s+1} z^{-1}(m_{s+1}) \ge i_{s+1} z^{-1}(m_s) + i_s z^{-1}(m_{s+1}),$$

$$z^{-1}(m_{s+1}) \ge z^{-1}(m_s).$$

Therefore, the elements $m_1, m_2, \dots, m_k$ appear in z in this order. There are exactly $n!/k!$ permutations with this property, so $\left|A(x)\right| \le \dfrac{n!}{k!}$.

IMC 2023

Second Day, August 3, 2023

Solutions

Problem 6. Ivan writes the matrix $\begin{pmatrix} 2 & 3 \\ 2 & 4 \end{pmatrix}$ on the board. Then he performs the following operation on the matrix several times:

- he chooses a row or a column of the matrix, and
- he multiplies or divides the chosen row or column entry-wise by the other row or column, respectively.

Can Ivan end up with the matrix $\begin{pmatrix} 2 & 4 \\ 2 & 3 \end{pmatrix}$ after finitely many steps?

(proposed by Alex Avdiushenko, Neapolis University Paphos, Cyprus)

Hint: Construct an invariant quantity that does not change during Ivan's prcedure.

Solution. We show that starting from $A = \begin{pmatrix} 2 & 3 \\ 2 & 4 \end{pmatrix}$, Ivan cannot reach the matrix $B = \begin{pmatrix} 2 & 4 \\ 2 & 3 \end{pmatrix}$.

Notice first that the allowed operations preserve the positivity of entries; all matrices Ivan can reach have only positive entries.

For every matrix $X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$ with positive entries, let $L(X) = \begin{pmatrix} \log_2 x_{11} & \log_2 x_{12} \\ \log_2 x_{21} & \log_2 x_{22} \end{pmatrix}$. By taking logarithms of the entries, the steps in Ivan game will be replaced by adding or subtracting a row or column to the other row. Such standard row and column operations preserve the determinant.

Hence, if the matrices in the game are $A = X_0, X_1, X_2, \ldots$, then we have $\det L(A) = \det L(X_1) = \det L(X_2) = \ldots$, and it suffices to verify that $\det L(A) \neq \det L(B)$.

Indeed,

$$\det L(A) = \log_2 2 \cdot \log_2 4 - \log_2 4 \cdot \log_2 3 = \log_2(4/3) > 0$$

and similarly $\det L(B) < 0$, so $\det L(A) \neq \det L(B)$.

Problem 7. Let V be the set of all continuous functions $f\colon [0,1] \to \mathbb{R}$, differentiable on $(0,1)$, with the property that $f(0)=0$ and $f(1)=1$. Determine all $\alpha \in \mathbb{R}$ such that for every $f \in V$, there exists some $\xi \in (0,1)$ such that
$$f(\xi)+\alpha = f'(\xi).$$

(proposed by Mike Daas, Leiden University)

Hint: Find a function $h \in V$ such that $h' - h$ is constant, then apply Rolle's theorem to $f - h$.
Alternatively, you can apply Cauchys's mean value theorem with some auxiliary functions.

Solution 1. First consider the function
$$h(x) = \frac{e^x - 1}{e-1}, \quad \text{which has the property that} \quad h'(x) = \frac{e^x}{e-1}.$$
Note that $h \in V$ and that $h'(x) - h(x) = 1/(e-1)$ is constant. As such, $\alpha = 1/(e-1)$ is the only possible value that could possibly satisfy the condition from the problem. For $f \in V$ arbitrary, let
$$g(x) = f(x)e^{-x} + h(-x), \quad \text{with} \quad g(0)=0 \quad \text{and also} \quad g(1) = e^{-1} + \frac{e^{-1}-1}{e-1} = 0.$$
We compute that
$$g'(x) = f'(x)e^{-x} - f(x)e^{-x} - h'(-x).$$
Now apply Rolle's Theorem to g on the interval $[0,1]$; it yields some $\xi \in (0,1)$ with the property that
$$g'(\xi) = 0 \implies f'(\xi)e^{-\xi} - f(\xi)e^{-\xi} - \frac{e^{-\xi}}{e-1} = 0 \implies f'(\xi) = f(\xi) + \frac{1}{e-1},$$
showing that $\alpha = 1/(e-1)$ indeed satisfies the condition from the problem.

Solution 2. Notice that the expression $f'(x) - f(x)$ appears in the derivative of the function $F(x) = f(x)\cdot e^{-x}$: $F'(x) = \big(f'(x) - f(x)\big)e^{-x}$.

Apply Cauchy's mean value theorem to $F(x)$ and the function $G(x) = -e^{-x}$. By the theorem, there is some $\xi \in (0,1)$ such that
$$\frac{F'(\xi)}{G'(\xi)} = \frac{F(1)-F(0)}{G(1)-G(0)}$$
$$f'(\xi) - f(\xi) = \frac{e^{-1}-0}{-e^{-1}+1} = \frac{1}{e-1}.$$
This proves the required property for $a = \dfrac{1}{e-1}$.

Now we show that no other α is possible. Choose f and F in such a way that $\dfrac{F'(x)}{G'(x)} = f'(x) - f(x) = \dfrac{1}{e-1}$ is constant. That means
$$F'(x) = \frac{G'(x)}{e-1} = \frac{e^{-x}}{e-1},$$
$$F(x) = \frac{1-e^{-x}}{e-1},$$
$$f(x) = F(x)\cdot e^x = \frac{e^x - 1}{e-1}.$$
With this choice we have $f(0)=0$ and $f(1)=1$, so $f' \in V$, and $f'(x) - f(x) \equiv \dfrac{1}{e-1}$ for all x, so for this function the only possible value for α is $\dfrac{1}{e-1}$.

Problem 8. Let T be a tree with n vertices; that is, a connected simple graph on n vertices that contains no cycle. For every pair u, v of vertices, let $d(u, v)$ denote the distance between u and v, that is, the number of edges in the shortest path in T that connects u with v.

Consider the sums

$$W(T) = \sum_{\substack{\{u,v\} \subseteq V(T) \\ u \neq v}} d(u, v) \quad \text{and} \quad H(T) = \sum_{\substack{\{u,v\} \subseteq V(T) \\ u \neq v}} \frac{1}{d(u, v)}.$$

Prove that

$$W(T) \cdot H(T) \geq \frac{(n-1)^3(n+2)}{4}.$$

(proposed by Slobodan Filipovski, University of Primorska, Koper)

Hint: There are $n-1$ pairs u, v with $d(u, v) = 1$; in all other cases $d(u, v) \geq 2$.

Solution. Let $k = \binom{n}{2}$ and let $x_1 \leq x_2 \leq \ldots \leq x_k$ be the distances between the pairs of vertices in the tree T. Thus

$$W(T) \cdot H(T) = (x_1 + x_2 + \ldots + x_k) \cdot \left(\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_k}\right).$$

Since the tree has exactly $n-1$ edges, there are exactly $n-1$ pairs of vertices at distance one, that is, $x_1 = x_2 = \ldots = x_{n-1} = 1$. Thus

$$W(T) \cdot H(T) = (n - 1 + x_n + x_{n+1} + \ldots + x_k) \cdot \left(n - 1 + \frac{1}{x_n} + \frac{1}{x_{n+1}} + \ldots + \frac{1}{x_k}\right) =$$

$$= (n-1)^2 + (n-1)\left(\left(x_n + \frac{1}{x_n}\right) + \ldots + \left(x_k + \frac{1}{x_k}\right)\right) +$$

$$+ (x_n + \ldots + x_k)\left(\frac{1}{x_n} + \ldots + \frac{1}{x_k}\right).$$

From Cauchy inequality we have

$$(x_n + \ldots + x_k)\left(\frac{1}{x_n} + \ldots + \frac{1}{x_k}\right) \geq (1 + 1 + \ldots + 1)^2 = (k - n + 1)^2 = \frac{(n-1)^2(n-2)^2}{4}.$$

The equality holds if and only if $x_n = x_{n+1} = \ldots = x_k$.
Now we minimize the expression $\left(x_n + \frac{1}{x_n}\right) + \ldots + \left(x_k + \frac{1}{x_k}\right)$, where $x_i \in [2, n-1]$.
It is clear that the minimal value is achieved for $x_n = x_{n+1} = \ldots = x_k = 2$. Therefore we get

$$W(T) \cdot H(T) \geq (n-1)^2 + (n-1)\left(\left(2 + \frac{1}{2}\right)(k - n + 1)\right) + \frac{(n-1)^2(n-2)^2}{4} = \frac{(n-1)^3(n+2)}{4}.$$

The equality holds for $x_1 = \ldots = x_{n-1} = 1$ and $x_n = x_{n+1} = \ldots = x_k = 2$, that is, the smallest value is achieved for the tree where $n-1$ pairs are at distance one, and the remaining $k-(n-1) = \frac{(n-1)(n-2)}{2}$ pairs are at distance two. The unique tree which satisfies these conditions is the star graph S_n. In this case it holds

$$W(S_n) \cdot H(S_n) = (n-1)^2 \cdot \frac{(n-1)(n+2)}{4} = \frac{(n-1)^3(n+2)}{4}.$$

Problem 9. We say that a real number V is *good* if there exist two closed convex subsets X, Y of the unit cube in $\mathbb{R}^3$, with volume V each, such that for each of the three coordinate planes (that is, the planes spanned by any two of the three coordinate axes), the projections of X and Y onto that plane are disjoint.

Find $\sup\{V \mid V \text{ is good}\}$.

(proposed by Josef Tkadlec and Arseniy Akopyan)

Hint: The two bodies can be replaced by a pair symmetric to the midpoint of the cube.

Solution. We prove that $\sup\{V \mid V \text{ is good}\} = 1/4$.

We will use the unit cube $U = [-1/2, 1/2]^3$.

For $\varepsilon \to 0$, the axis-parallel boxes $X = [-1/2, -\varepsilon] \times [-1/2, -\varepsilon] \times [-1/2, 1/2]$ and $Y = [\varepsilon, 1/2] \times [\varepsilon, 1/2] \times [-1/2, 1/2]$ show that $\sup\{V\} \geq 1/4$.

To prove the other bound, consider two admissible convex bodies X, Y. For any point $P = [x, y, z] \in U$ with $xyz \neq 0$, let $\overline{P} = \{[\pm x, \pm y, \pm z]\}$ be the set consisting of 8 points (the original P and its 7 "symmetric" points). If for each such P we have $|\overline{P} \cap (X \cup Y)| \leq 4$, then the conclusion follows by integrating. Suppose otherwise and let P be a point with $|\overline{P} \cap (X \cup Y)| \geq 5$. Below we will complete the proof by arguing that:

(1) we can replace one of the two bodies (the "thick" one) with the reflection of the other body about the origin, and

(2) for such symmetric pairs of bodies we in fact have $|\overline{P} \cap (X \cup Y)| \leq 4$, for all P.

To prove Claim (1), we say that a convex body is *thick* if each of its three projections contains the origin. We claim that one of the two bodies X, Y is thick. This is a short casework on the 8 points of $\overline{P}$. Since $|\overline{P} \cap (X \cup Y)| \geq 5$, by pigeonhole principle, we find a pair of points in $\overline{P} \cap (X \cup Y)$ symmetric about the origin. If both points belong to one body (say to X), then by convexity of X the origin belongs to X, thus X is thick. Otherwise, label $\overline{P}$ as $ABCDA'B'C'D'$. Wlog $A \in X$, $C' \in Y$ is the pair of points in $\overline{P}$ symmetric about the origin. Wlog at least 3 points of $\overline{P}$ belong to X. Since X, Y have disjoint projections, we have $C, B', D' \notin X$, so wlog $B, D \in X$. Then Y can contain no other point of $\overline{P}$ (apart from C'), so X must contain at least 4 points of $\overline{P}$ and thus $A' \in X$. But then each projection of X contains the origin, so X is indeed thick.

Note that if X is thick then none of the three projections of Y contains the origin. Consider the reflection $Y' = -Y$ of Y about the origin. Then (Y, Y') is an admissible pair with the same volume as (X, Y): the two bodies Y and Y' clearly have equal volumes V and they have disjoint projections (by convexity, since the projections of Y miss the origin). This proves Claim (1).

Claim (2) follows from a similar small casework on the 8-tuple $\overline{P}$: For contradiction, suppose $|\overline{P} \cap Y'| = |\overline{P} \cap Y| \geq 3$. Wlog $A \in Y'$. Then $C' \in Y$, so $C, B', D' \notin Y'$, so wlog $B, D \in Y'$. Then $B', D' \in Y$, a contradiction with (Y, Y') being admissible.

Remark. There are more examples with $V \to 1/4$, e.g. X a union of two triangular pyramids with base ACD' – one with apex D, one with apex at the origin (and Y symmetric with X about the origin).

Remark. The word "convex" matters. E.g., in a $3 \times 3 \times 3$ cube, one can set X to be a $2 \times 2 \times 2$ sub-cube, and Y to be the (non-convex) 3D L-shape consisting of 7 unit cubes. This shows that without convexity we have $V \geq 7/27 > 1/4$.

Problem 10. For every positive integer n, let $f(n), g(n)$ be the minimal positive integers such that

$$1+\frac{1}{1!}+\frac{1}{2!}+\ldots+\frac{1}{n!}=\frac{f(n)}{g(n)}.$$

Determine whether there exists a positive integer n for which $g(n)>n^{0.999\,n}$.

(proposed by Fedor Petrov, St. Petersburg State University)

Solution. We show that there does exist such a number n.

Let $\varepsilon=10^{-10}$. Call a prime p *special*, if for certain $k\in\{1,2,\ldots,p-1\}$ there exist at least $\varepsilon\cdot k$ positive integers $j\le k$ for which p divides $f(j)$.

Lemma. There exist only finitely many special primes.

Proof. Let p be a special prime number, and p divides $f(j)$ for at least $\varepsilon\cdot k$ values of $j\in\{1,2,\ldots,k\}$. Note that if p divides $f(j)$ and $f(j+r)$, then p divides

$$(j+r)!\left(\frac{f(j+r)}{g(j+r)}-\frac{f(j)}{g(j)}\right)=1+(j+r)+(j+r)(j+r-1)+\ldots+(j+r)\ldots(j+2)$$

that is a polynomial of degree $r-1$ with respect to j. Thus, for fixed j it equals to 0 modulo p for at most $r-1$ values of j. Look at our $\ge\varepsilon\cdot k$ values of $j\in\{1,2,\ldots,k\}$ and consider the gaps between consecutive j's. The number of such gaps which are greater than $2/\varepsilon$ does not exceed $\varepsilon\cdot k/2$ (since the total sum of gaps is less than k). Therefore, at least $\varepsilon\cdot k/2-1$ gaps are at most $2/\varepsilon$. But the number of such small gaps is bounded from above by a constant (not depending on k) by the above observation. Therefore, k is bounded, and, since p divides $f(1)f(2)\ldots f(k)$, p is bounded too.

Now we want to bound the product $g(1)g(2)\ldots g(n)$ (for a large integer n) from below. Let $p\le n$ be a non-special prime. Our nearest goal is to prove that

$$\nu_p\big(g(1)g(2)\ldots g(n)\big)\ge(1-\varepsilon)\nu_p(1!\cdot 2!\cdot\ldots\cdot n!) \qquad (1)$$

Partition the numbers $p,p+1,\ldots,n$ onto the intervals of length p (except possibly the last interval which may be shorter): $\{p,p+1,\ldots,2p-1\}$, $\ldots$, $\{p\lfloor n/p\rfloor,\ldots,n\}$. Note that in every interval $\Delta=[a\cdot p,a\cdot p+k]$, all factorials $x!$ with $x\in\Delta$ have the same p-adic valuation, denote it $T=\nu_p\big((ap)!\big)$. We claim that at least $(1-\varepsilon)(k+1)$ valuations of $g(x)$, $x\in\Delta$, are equal to the same number T. Indeed, if $j=0$ or $1\le j\le k$ and $f(j)$ is not divisible by p, then

$$\frac{1}{(ap)!}+\frac{1}{(ap+1)!}+\ldots+\frac{1}{(ap+j)!}=\frac{1}{(ap)!}\cdot\frac{A}{B}$$

where $A\equiv f(j) \pmod p$, $B\equiv g(j) \pmod p$, so, this sum has the same p-adic valuation as $1/(ap)!$, which is strictly less than that of the sum $\sum_{i=0}^{ap-1}1/i!$, that yields $\nu_p\big(g(ap+j)\big)=\nu_p\big((ap)!\big)$. Using this for every segment Δ, we get (1).

Now, using (1) for all non-special primes, we get

$$A\cdot g(1)g(2)\ldots g(n)\ge(1!\cdot 2!\cdot\ldots\cdot n!)^{1-\varepsilon},$$

where $A=\prod_{p,k}p^{\nu_p(g(k))}$, p runs over non-special primes, k from 1 to n. Since $\nu_p(g(k))\le\nu_p(k!)=\sum_{i=1}^{\infty}\lfloor k/p^i\rfloor\le k$, we get

$$A\le\Big(\prod_p p\Big)^{1+2+\ldots+n}\le C^{n^2}$$

for some constant C. But if we had $g(n)\le n^{0.999n}\le e^n n!^{0.999}$ for all n, then

$$\log\big(A\cdot g(1)g(2)\ldots g(n)\big)\le O(n^2)+0.999\log(1!\cdot 2!\cdot\ldots\cdot n!)<(1-\varepsilon)\log(1!\cdot 2!\cdot\ldots\cdot n!)$$

for large n, a contradiction.

IMC 2024

First Day, August 7, 2024

Solutions

Problem 1. Determine all pairs $(a,b) \in \mathbb{C} \times \mathbb{C}$ satisfying

$$|a| = |b| = 1 \quad \text{and} \quad a + b + a\overline{b} \in \mathbb{R}.$$

(proposed by Mike Daas, Universiteit Leiden)

Hint: Write $a = e^{ix}$ and $b = e^{iy}$, and transform the RHS to a product.

Solution 1. Write $a = e^{ix}$ and $b = e^{iy}$ for some $x, y \in [0, 2\pi)$. Using Euler's formula, and the well-known identities

$$\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2} \quad \text{and} \quad \sin x = 2\sin\frac{x}{2}\cos\frac{x}{2},$$

we get a product form of the left-hand side as

$$\begin{aligned}
\operatorname{Im}\left(a + b + a\overline{b}\right) &= \left(\sin x + \sin y\right) + \sin(x-y)\\
&= 2\sin\frac{x+y}{2}\cos\frac{x-y}{2} + 2\sin\frac{x-y}{2}\cos\frac{x-y}{2}\\
&= 2\left(\sin\frac{x+y}{2} + \sin\frac{x-y}{2}\right)\cos\frac{x-y}{2}\\
&= 4\sin\frac{x}{2}\cdot\cos\frac{y}{2}\cdot\cos\frac{x-y}{2}.
\end{aligned}$$

Hence, $a + b + a\overline{b}$ is real if and only if either $\sin\frac{x}{2} = 0$, $\cos\frac{y}{2} = 0$ or $\cos\frac{x-y}{2} = 0$, which respectivelly correspond to $x = 2k\pi$, $y = (2k+1)\pi$ and $x = y + (2k+1)\pi$.

Therefore, the solutions are

$$(1,b), \quad (a,-1) \quad \text{and} \quad (a,-a) \quad \text{with } |a| = 1,\ |b| = 1.$$

Solution 2. Notice that

$$a + b + a\overline{b} \in \mathbb{R} \iff 1 + a + b + a\overline{b} \in \mathbb{R}.$$

Let $c \in \mathbb{C}$ be such that $a = c^2$. Now observe that

$$\begin{aligned}
\overline{c}(1 + a + b + a\overline{b}) &= \overline{c} + \overline{c}c^2 + \overline{c}b + \overline{c}c^2\overline{b}\\
&= \overline{c} + c + \overline{c}b + c\overline{b} \in \mathbb{R},
\end{aligned}$$

where we used that $\overline{c}c = 1$ and $z + \overline{z} \in \mathbb{R}$ for any $z \in \mathbb{C}$. W e conclude that either $c \in \mathbb{R}$, or $1 + a + b + a\overline{b} = 0$. In the first case, $c = \pm 1$ and so $a = 1$. In the second case, we factor the equation as

$$(a+b)(1+\overline{b}) = 1 + a + 1b + a\overline{b} = 0, \quad \text{and as such,} \quad a = -b \quad \text{or} \quad b = -1.$$

We find precisely three families of pairs (a,b): the pairs $(1,b)$ for b on the unit circle; the pairs $(a,-1)$ for a on the unit circle; and the pairs $(a,-a)$ for a on the unit circle.

Problem 2. For $n = 1, 2, \ldots$ let

$$S_n = \log\left(\sqrt[n^2]{1^1 \cdot 2^2 \cdot \ldots \cdot n^n}\right) - \log(\sqrt{n}),$$

where log denotes the natural logarithm. Find $\lim\limits_{n\to\infty} S_n$.

(proposed by Sergey Chernov, Belarusian State University, Minsk)

Hint: S_n is (close to) a Riemann sum of a certain integral.

Solution. Transform S_n as

$$\begin{aligned} S_n &= \frac{1}{n^2}\sum_{k=1}^{n} k\log k - \frac{1}{2}\log n \\ &= \frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\left(\log\frac{k}{n} + \log n\right)\right) - \frac{1}{2}\log n \\ &= \frac{1}{n}\sum_{k=1}^{n}\frac{k}{n}\log\frac{k}{n} + \frac{\log n}{n^2}\sum_{k=1}^{n} k - \frac{1}{2}\log n \\ &= \frac{1}{n}\sum_{k=1}^{n}\frac{k}{n}\log\frac{k}{n} + \frac{\log n}{2n}. \end{aligned}$$

Here the last term $\frac{\log n}{2n}$ converges to 0. The sum $\frac{1}{n}\sum_{k=1}^{n}\frac{k}{n}\log\frac{k}{n}$ is a Riemann sum for the integrable function $f(x) = x\log x$ on the segment $[0,1]$ with the uniform grid $\left\{\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\right\}$. Therefore

$$\lim \frac{1}{n}\sum_{k=1}^{n}\frac{k}{n}\log\frac{k}{n} = \lim \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) = \int_0^1 x\log x\,\mathrm{d}x = \left[\frac{x^2}{2}\log x - \frac{x^2}{4}\right]_0^1 = -\frac{1}{4}.$$

Hence, $\lim S_n$ exists, and $\lim S_n = -\dfrac{1}{4}$.

Problem 3. For which positive integers n does there exist an $n \times n$ matrix A whose entries are all in $\{0, 1\}$, such that A^2 is the matrix of all ones?

(proposed by Alex Avdiushenko, Neapolis University Paphos, Cyprus)

Hint: Let J be the $n \times n$ matrix with all ones. Consider $A^3 = AJ = JA$.

Solution. **Answer:** Such a matrix A exists if and only if n is a complete square.

Let J_n be the $n \times n$ matrix with all ones, so $A^2 = J_n$. Consider the equality

$$A^3 = AJ_n = J_nA.$$

In the matrix AJ_n, all columns are equal to the sum of colums in A, that is, the (i, j)th entry in AJ_n is the number of ones in the ith row of A. Similarly, the (i, j)th entry in J_nA is the number of ones in the jth column of A. These numbers must be equal, so A contains the same number of ones in every row and every column. Let this common number be k; then $AJ_n = J_nA = kJ_n$.

Now from

$$nJ_n = J_n^2 = (A^2)^2 = A(AJ_n) = A(kJ_n) = k(AJ_n) = k^2J_n$$

we can read $n = k^2$, so n must be a complete square.

It remains to show an example for a matrix A of order $n = k^2$. For $l = 0, 1, \ldots, k-1$, let B_l be the $k \times k$ matrix whose (i, j)th entry is 1 if $j - i \equiv l \pmod{k}$ and 0 otherwise, i.e., B_l can be obtained from the identity matrix by cyclically shifting the colums l times, and let

$$A = \begin{pmatrix} B_0 & B_1 & B_2 & \ldots & B_{k-1} \\ B_0 & B_1 & B_2 & \ldots & B_{k-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B_0 & B_1 & B_2 & \ldots & B_{k-1} \end{pmatrix};$$

The (i, j)th block in A^2 is

$$\begin{pmatrix} B_0 & B_1 & \ldots & B_{k-1} \end{pmatrix} \begin{pmatrix} B_{j-1} \\ \vdots \\ B_{j-1} \end{pmatrix} = (B_0 + B_1 + \ldots + B_{k-1})B_{j-1} = J_kB_{j-1} = J_k,$$

so this matrix indeed satisfies $A^2 = J_{k^2}$.

Problem 4. Let g and h be two distinct elements of a group G, and let n be a positive integer. Consider a sequence $w = (w_1, w_2, \ldots)$ which is not eventually periodic and where each w_i is either g or h. Denote by H the subgroup of G generated by all elements of the form $w_k w_{k+1} \ldots w_{k+n-1}$ with $k \geq 1$. Prove that H does not depend on the choice of the sequence w (but may depend on n).

(proposed by Ivan Mitrofanov, Saarland University)

Solution. Let X_m denote the subset of G of products of the form $g_1 \ldots g_m$, where each g_i is either g or h.

Lemma. For all $j = 1, 2, \ldots, n$ and for all $a, b \in X_j$ the ratio $a^{-1}b$ is contained in H.

Proof. Induction in j.

We start with the base case $j = 1$. By the pigeonhole principle, there exist $k < \ell$ for which the sequences $(w_{k+1}, \ldots, w_{k+n-1})$ and $(w_{\ell+1}, \ldots, w_{\ell+n-1})$ coincide. If $w_{k+m} = w_{\ell+m}$ for all positive integer m, then the sequence w is eventually periodic with period $\ell - k$. Thus, there exists $m > 0$ for which $w_{k+m} \neq w_{\ell+m}$. We have $m \geqslant n$, so $w_{k+m-i} = w_{\ell+m-i}$ for $i = 1, 2, \ldots, n-1$. Therefore, since the products $x = w_{k+m-n+1} \ldots w_{k+m}$ and $y = w_{\ell+m-n+1} \ldots w_{\ell+m}$ both are elements of H, the subgroup H contains their ratios $x^{-1}y$ and $y^{-1}x$. These ratios are equal to $g^{-1}h$ and $h^{-1}g$ (in some order), that finishes the proof for $j = 1$.

Induction step from $j - 1$ to j, $2 \leqslant j \leqslant n$. We say that an element $a \in X_j$ is a g-element, correspondingly an h-element, if it can be represented as $a = ga_1$, correspondingly $a = ha_1$, where $a_1 \in X_{j-1}$. The ratio of two g-elements, or of two h-elements, is a ratio of two elements of X_{j-1}, thus, it is in H by the induction hypothesis. Since the property $a^{-1}b \in H$ is an equivalence relation on pairs (a, b), it suffices to find a g-element and h-element whose ratio is in H.

Define k, ℓ, m, as in the base case. The subgroup H contains the products

$$v = w_{k+m-n+j} \ldots w_{k+m} w_{k+m+1} \ldots w_{k+m+j-1},$$
$$u = w_{\ell+m-n+j} \ldots w_{\ell+m} w_{\ell+m+1} \ldots w_{\ell+m+j-1}.$$

Their ratio $u^{-1}v$ is a ratio of g-element and an h-element in X_j, since $\{w_{k+m}, w_{\ell+m}\} = \{g, h\}$ and $w_{k+m-i} = w_{\ell+m-i}$ for all $i = 1, 2, \ldots, n-j$.

The Lemma for $j = n$ yields that H is the subgroup of G generated by X_n, and this description does not depend on w.

Problem 5. Let $n > d$ be positive integers. Choose n independent, uniformly distributed random points $x_1, \dots, x_n$ in the unit ball $B \subset \mathbb{R}^d$ centered at the origin. For a point $p \in B$ denote by $f(p)$ the probability that the convex hull of $x_1, \dots, x_n$ contains p. Prove that if $p, q \in B$ and the distance of p from the origin is smaller than the distance of q from the origin, then $f(p) \geq f(q)$.

(proposed by Fedor Petrov, St Petersburg State University)

Solution. By radial symmetry of the distribution, $f(p)$ depends only on $|op|$ (the distance between o and p), so, we may assume that p lies on the segment between o and q. For points $x_1, \dots, x_n$ and $x \in B$ denote by $f_x(x_1, \dots, x_n)$ the indicator function of the event "x is in the convex hull of $x_1, \dots, x_n$". The claim follows from the following deterministic inequality

$$\sum f_p(\pm x_1, \dots, \pm x_n) \geqslant \sum f_q(\pm x_1, \dots, \pm x_n), \tag{1}$$

where $x_1, \dots, x_n \in B$ are arbitrary points in general position and the summations are over all 2^n choices of signs (here o is identified with the origin, that is, x and $-x$ are symmetric with respect to o). Indeed, taking the expectation in (1) over independent random uniform $x_1, \dots, x_n$, we get $2^n f(p) \geqslant 2^n f(q)$. (To be specific, here "general position" means that for any point set $A \subset \{\pm x_1, \dots, \pm x_n, p, q\}$, which does not contain simultaneosuly x_i and $-x_i$, is not contained in an (affine) $(|A| - 2)$-dimensional plane. This holds with probability 1.)

To prove (1), we use the following formula for the characteristic function χ_P of the convex polyhedron $P \subset \mathbb{R}^d$: if $P_1, \dots, P_k$ are all facets of P, and Q_i is the convex hull of o and P_i, then $\chi_P = \sum \pm \chi_{Q_i}$, where the sign is plus if o and P are on the same side of P_i, and minus otherwise. Indeed, for every point p in general position look how the ray op intersects the boundary of P and realize that for at most two summands the contribution of the RHS at point p is non-zero, and the total contribution equals 1 when p is inside P and 0 (possibly as $0 = 1 - 1$) otherwise. Use this formula for every polyhedron P with n vertices $y_1, \dots, y_n$, where each y_i is $\pm x_i$. These polyhedrons are simplicial (all facets are simplices) because of the general position condition. Sum up over all 2^n such P, we get the expression of $\sum_P \chi_P$ as a linear combination of χ_S, where S are simplices formed by o and some d points in $\{\pm x_1, \dots, \pm x_n\}$ (not containing x_i and $-x_i$ simultaneously).

For proving (1), it suffices to verify that all coefficients of χ_S in this linear combination are positive (since two sides of (1) are the values of the sum $\sum_P \chi_P$ at p and q). Let's find a coefficient of χ_S, where, say, S is a simplex with vertices $o, x_1, \dots, x_d$. The plane α through $x_1, \dots, x_d$ partitions $\mathbb{R}^d$ onto two parts H^+ (containing o) and H^- (not containing o). For every pair $\{x_i, -x_i\}$ with $i > d$, either both points belong to H^+, or one belongs to H^- and another to H^+. χ_S goes with the plus sign for P with vertices $x_1, \dots, x_d$ and other vertices from H^+, and with the minus sign for P with vertices $x_1, \dots, x_d$ and other vertices from H^-. It is immediate that there are at least as many pluses as minuses.

IMC 2024

Second Day, August 8, 2024

Solutions

Problem 6. Prove that for any function $f\colon \mathbb{Q} \to \mathbb{Z}$, there exist $a, b, c \in \mathbb{Q}$ such that $a < b < c$, $f(b) \geq f(a)$, and $f(b) \geq f(c)$.

(proposed by Mehdi Golafshan & Markus A. Whiteland, University of Liège, Liège)

Solution 1. We can replace $f(x)$ by the function $g(x) = f(1-x)$, so without loss of generality we can assume $f(0) \leq f(1)$.

If $f(1) \geq f(2)$ then we can choose $(a, b, c) = (0, 1, 2)$. Otherwise we have $f(0) \leq f(1) < f(2)$.

If there is some $x \in (1, 2)$ such that $f(x) \geq f(2)$ then we can chose $(a, b, c) = (1, x, 2)$; similarly, if there is some $x \in (1, 2)$ with $f(x) \leq f(1)$ then choose $(a, b, c) = (0, 1, x)$. Hence, in the remaining cases we have $f(1) \leq f(x) \leq f(2)$ for all $x \in (1, 2)$.

Now f is bonded on the interval $[1, 2]$, so it has only finitely many values on this interval. Since there are infintely many rational numbers in $[0, 1]$, there is a value y that is attained infinitely many times. The we can choose $1 \leq a < b < c \leq 2$ such that $f(a) = f(b) = f(c) = y$.

Solution 2. Assume towards a contradiction that there is a function f which does not satisfy the claim: for all rationals a, b, c with $a < b < c$ we have $f(b) < f(a)$ or $f(b) < f(c)$.

Let x and y be arbitrary rationals with $x < y$. Let $I(x, y) = [x, y] \cap \mathbb{Q}$. We first observe that $\inf f(I(x, y)) = -\infty$. Indeed, if the infimum was finite, then, as the set $f(I(x, y))$ is bounded ($\sup f(I(x, y)) = \max\{f(x), f(y)\}$) and thus finite, there are three points having the same value under f, which leads to a contradiction regarding our assumption on f.

So, going back to the question at hand, let x, b, y be arbitrary rationals with $x < b < y$. Applying the above observation to the set $I(x, b)$, there exists a point $a \in I(x, b)$ such that $f(a) < f(b)$. Similarly, there exists a point $c \in I(b, y)$ such that $f(c) < f(b)$. Hence we have the points a, b, c with $a < b < c$ and $f(b) > \max\{f(a), f(c)\}$, which contradicts our assumption on f.

Problem 7. Let n be a positive integer. Suppose that A and B are invertible $n\times n$ matrices with complex entries such that $A+B=I$ (where I is the identity matrix) and

$$(A^2+B^2)(A^4+B^4)=A^5+B^5.$$

Find all possible values of $\det(AB)$ for the given n.

(proposed by Sergey Bondarev, Sergey Chernov, Belarusian State University, Minsk)

Hint: Find a polynomial $p(x)$ such that $p(AB)=0$.

Solution 1. Notice first that $AB=A(I-A)=A-A^2=(I-A)A=BA$, so A and B commute. Let $C=AB=BA$; then

$$\begin{aligned}
A^2+B^2 &= (A+B)^2-2AB = I-2C,\\
A^4+B^4 &= (A+B)^4-4AB(A+B)^2+2A^2B^2 = I-4C+2C^2,\\
A^5+B^5 &= (A+B)^5-5AB(A+B)^3+5A^2B^2(A+B) = I-5C+5C^2,
\end{aligned}$$

so

$$\begin{aligned}
0 &= (A^5+B^5)-(A^2+B^2)(A^4+B^4) = (I-5C+5C^2)-(I-2C)(I-4C+2C^2)\\
&= 4C^3-5C^2+C = 4C(C-I)(C-\tfrac{1}{4}I);
\end{aligned}$$

since C is invertible, we have

$$(C-I)(C-\tfrac{1}{4}I)=0.$$

Hence, the polynomial $p(x)=(x-1)(x-\frac{1}{4})$ annihilates the matrix $C=AB$ and therefore all eigenvalues of C are roots of $p(x)$, so the possible eigenvalues are 1 and $\frac{1}{4}$. The determinant is the product of the n eigenvalues, so

$$\det(AB)=\det C\in\left\{1,\tfrac{1}{4},\tfrac{1}{4^2},\dots,\tfrac{1}{4^n}\right\}.$$

Now show that these values are indeed possible.
If

$$A=\operatorname{diag}\Big(\underbrace{\tfrac12,\dots,\tfrac12}_{k},\underbrace{e^{i\pi/3},\dots,e^{i\pi/3}}_{n-k}\Big) \quad\text{and}\quad B=\operatorname{diag}\Big(\underbrace{\tfrac12,\dots,\tfrac12}_{k},\underbrace{e^{-i\pi/3},\dots,e^{-i\pi/3}}_{n-k}\Big),$$

then $A+B=I$, $AB=\operatorname{diag}\Big(\underbrace{\tfrac14,\dots,\tfrac14}_{k},\underbrace{1,\dots,1}_{n-k}\Big)$ and $\det(AB)=\frac{1}{4^k}$.

Problem 8. Define the sequence $x_1, x_2, \ldots$ by the initial terms $x_1 = 2$, $x_2 = 4$, and the recurrence relation

$$x_{n+2} = 3x_{n+1} - 2x_n + \frac{2^n}{x_n} \quad \text{for } n \geq 1.$$

Prove that $\lim\limits_{n\to\infty} \frac{x_n}{2^n}$ exists and satisfies

$$\frac{1+\sqrt{3}}{2} \leq \lim_{n\to\infty} \frac{x_n}{2^n} \leq \frac{3}{2}.$$

(proposed by Karen Keryan, Yerevan State University & American University of Armenia, Armenia)

Hint: Prove that $2x_n \leq x_{n+1} \leq 2x_n + n$.

Solution. Let's prove by induction that $x_{n+1} \geq 2x_n$. It holds for $n = 1$. Assume it holds for n. Then by the induction hypothesis we have that $x_n \geq 2x_{n-1} \geq \ldots \geq 2^{n-1}x_1 > 0$ and

$$x_{n+2} = 2x_{n+1} + (x_{n+1} - 2x_n) + \frac{2^n}{x_n} > 2x_{n+1}.$$

Similarly we prove that $x_{n+1} \leq 2x_n + n$. Again it holds for $n = 1$. Assume that the inequality holds for n. Then using that $x_n \geq 2^n$ and the induction hypothesis we obtain

$$x_{n+2} \leq 3x_{n+1} - 2x_n + 1 \leq 2x_{n+1} + (2x_n + n) - 2x_n + 1 = 2x_{n+1} + n + 1.$$

Using the previous inequalities we obtain that the sequence $y_n = \frac{x_n}{2^n}$ is increasing and $y_{n+1} \leq y_n + \frac{n}{2^n} \leq \ldots \leq y_1 + \sum_{k=1}^{n} \frac{k}{2^k} < \infty$, thus $\lim\limits_{n\to\infty} y_n = \frac{x_n}{2^n} = c$ exists.

The recurrence relation has the following form for y_n:

$$4y_{n+2} - 2y_{n+1} = 4y_{n+1} - 2y_n + \frac{1}{2^n \cdot y_n}.$$

By summing up the above equality for $n = 1, \ldots, m$ we obtain

$$4y_{m+2} - 2y_{m+1} = 4y_2 - 2y_1 + \sum_{n=1}^{m} \frac{1}{2^n \cdot y_n} = 2 + \sum_{n=1}^{m} \frac{1}{2^n \cdot y_n}. \tag{1}$$

Now using the facts that $y_1 = 1$, y_n increases and $\lim_{n\to\infty} y_n = c$ we obtain $1 \leq y_n \leq c$. Hence

$$\frac{1}{c} \leq \sum_{n=1}^{\infty} \frac{1}{2^n \cdot y_n} \leq 1.$$

Thus we get from (1)

$$2c = \lim_{m\to\infty} (4y_{m+2} - 2y_{m+1}) = 2 + \sum_{n=1}^{\infty} \frac{1}{2^n \cdot y_n} \in \left[2 + \frac{1}{c}, 3\right].$$

So we have $2c^2 \geq 2c + 1$ and $2c \leq 3$. Recall that $c \geq 1$. Therefore $1 + \sqrt{3} \leq 2c \leq 3$, which finishes the proof.

Problem 9. A matrix $A = (a_{ij})$ is called *nice*, if it has the following properties:

(i) the set of all entries of A is $\{1, 2, \ldots, 2t\}$ for some integer t;

(ii) the entries are non-decreasing in every row and in every column: $a_{i,j} \leq a_{i,j+1}$ and $a_{i,j} \leq a_{i+1,j}$;

(iii) equal entries can appear only in the same row or the same column: if $a_{i,j} = a_{k,\ell}$, then either $i = k$ or $j = \ell$;

(iv) for each $s = 1, 2, \ldots, 2t - 1$, there exist $i \neq k$ and $j \neq \ell$ such that $a_{i,j} = s$ and $a_{k,\ell} = s + 1$.

Prove that for any positive integers m and n, the number of nice $m \times n$ matrices is even.

For example, the only two nice 2×3 matrices are $\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 & 3 \\ 2 & 4 & 4 \end{pmatrix}$.

(proposed by Fedor Petrov, St Petersburg State University)

Solution. Define a *standard Young tableaux* of shape $m \times n$ as an $m \times n$ matrix with the set of entries $\{1, 2, \ldots, mn\}$, increasing in every row and in every column as in (ii).

Call two standard Young tableaux Y_1, Y_2 *friends*, if they differ by a switch of two consecutive numbers $x, x + 1$ (the places of x and $x + 1$ must be not neighbouring, for such a switch preserving the monotonicity in rows and columns).

For a nice $m \times n$ matrix A we construct a standard Young tableaux Y_A of shape $m \times n$ as follows: if A has n_i entries equal to i ($i = 1, 2, \ldots, 2t$), we replace them by the numbers from $n_1 + \ldots + n_{i-1} + 1$ to $n_1 + \ldots + n_i$ preserving monotonicity.

Note that our Y_A has exactly $2t - 1$ friends, where $2t$ is the number of distinct entries in A, and moreover, every standard Young tableaux with odd number of friends corresponds to a unique nice matrix. It remains to apply the handshaking lemma (i.e., the sum of the degrees equals twice the number of edges in this graph).

Problem 10. We say that a square-free positive integer n is *almost prime* if

$$n \mid x^{d_1} + x^{d_2} + \ldots + x^{d_k} - kx$$

for all integers x, where $1 = d_1 < d_2 < \ldots < d_k = n$ are all the positive divisors of n. Suppose that r is a Fermat prime (i.e. it is a prime of the form $2^{2^m} + 1$ for an integer $m \geq 0$), p is a prime divisor of an almost prime integer n, and $p \equiv 1 \pmod{r}$. Show that, with the above notation, $d_i \equiv 1 \pmod{r}$ for all $1 \leq i \leq k$.

(An integer n is called *square-free* if it is not divisible by d^2 for any integer $d > 1$.)

(proposed by Tigran Hakobyan, Yerevan State University, Vanadzor, Armenia)

Solution. We first prove the following claims.

Lemma 1. If n is almost prime then $\gcd(n, \varphi(n)) = 1$.

Proof. Assume to the contrary that $\gcd(n, \varphi(n)) > 1$ so that there are primes p and q dividing n such that $p \equiv 1 (\text{mod } q)$. For $0 \leq i \leq p-2$ let h_i be the number of positive divisors of n congruent to i modulo $p-1$ and similarly for $0 \leq j \leq q-1$ let ν_j denote the number of positive divisors of n congruent to j modulo q. Observe that the polynomial $F_n(x) = x^{d_1} + x^{d_2} + \ldots + x^{d_k} - kx$ defines the zero function on $\mathbb{F}_p$ due to the condition of the problem. On the other hand, $F_n(x) = (h_1 - k)x + \sum_{i \neq 1} h_i x^i$ in $\mathbb{F}_p[x]$, so that $p | h_i$ for all $0 \leq i \leq p-2, i \neq 1$. It follows that $2^{\omega(n)-1} = \nu_0 = h_0 + h_q + h_{2q} + \ldots \equiv 0 (\text{mod } p)$ which is a contradiction (here $\omega(n)$ means the number of distinct prime divisors of n). Therefore our assumption was wrong and the lemma is proved. □

Lemma 2. Let q be a prime number and let h be a positive integer coprime to $q-1$. If l is the order of h modulo $q-1$, then there exists $a \in \mathbb{F}_q$ such that $a^{h^l} = a$ and

$$a - a^h + a^{h^2} - \ldots + (-1)^{l-1} a^{h^{l-1}} \neq 0$$

Proof. Observe that $a^{h^l} = a$ for any $a \in \mathbb{F}_q$ since $q - 1 | h^l - 1$. On the other hand, the numbers $h^0, h^1, \ldots, h^{l-1}$ leave different remainders upon division by $q-1$ and therefore the polynomial

$$f(x) = x - x^h + x^{h^2} - \ldots + (-1)^{l-1} x^{h^{l-1}}$$

defines a function on $\mathbb{F}_q$, which is not identically zero. Hence the existence of an element with the required properties is proved. □

Lemma 3. If n is almost prime then for any primes p and q dividing n, the order of p modulo $q-1$ is an odd number.

Proof. Observe that due to Lemma 1 the order l of p modulo $q-1$ is well defined and assume to the contrary that l is an even number. According to Lemma 2 there exists $a \in \mathbb{F}_q$ such that $a^{p^l} = a$ and $f(a) \neq 0$, where $f(x) = x - x^p + x^{p^2} - \ldots + (-1)^{l-1} x^{p^{l-1}}$. Let us consider the sequence $(a_i)_{i=0}^{l} \subset \mathbb{F}_q$ defined by $a_0 = a$ and $a_{i+1} = -a_i^p$ for $0 \leq i \leq l-1$. Notice that since l is even by the assumption, we have $a_l = a_0^{p^l} = a_0$. It follows that

$$\sum_{i=0}^{l-1} \sum_{d|n} a_i^d = \sum_{i=0}^{l-1} \left(\sum_{d|\frac{n}{p}} a_i^d + \sum_{d|\frac{n}{p}} a_i^{pd} \right) = \sum_{i=0}^{l-1} \sum_{d|\frac{n}{p}} \left(a_{i+1}^d + a_i^{pd} \right) = 0,$$

since d is always odd being a divisor of n (Recall that $\gcd(n, \varphi(n)) = 1$ due to Lemma 1, so that n is odd, except the trivial case $n = 2$), and $a_{i+1} = -a_i^p$ for all $0 \leq i \leq l-1$. On the other hand, according to the condition of the problem, $\sum_{d|n} a_i^d = k a_i$ in $\mathbb{F}_q$ for all i, which shows that

$$kf(a) = k \sum_{i=0}^{l-1} a_i = \sum_{i=0}^{l-1} k a_i = \sum_{i=0}^{l-1} \sum_{d|n} a_i^d = 0$$

in $\mathbb{F}_q$ which is impossible, since $f(a) \neq 0$ by construction and $k = 2^{\omega(n)-1}$ is coprime to q. The attained contradiction shows that our assumption was wrong and concludes the proof of the lemma. $\square$

Let us get back to the problem. Suppose that $p|n$ is prime and $r = 2^{2^m} + 1$ is a Fermat's prime such that $p \equiv 1 (\text{mod } r)$. If q is any prime divisor of n, then by Lemma 3 we have that $q^l \equiv 1(\text{mod } p-1)$ for some odd l, so that $q^l \equiv 1(\text{mod } r)$ and therefore $q = q^{\gcd(l,r-1)} \equiv 1(\text{mod } r)$. Hence $d \equiv 1(\text{mod } r)$ for any divisor d of n. $\square$