SEEMOUS 2007
South Eastern European Mathematical Olympiad for University Students
Agros, Cyprus
7-12 March 2007

Mathematical Society of South Eastern Europe
Cyprus Mathematical Society

COMPETITION PROBLEMS

9 March 2007

Do all problems 1-4. Each problem is worth 10 points. All answers should be answered in the booklet provided, based on the rules written in the Olympiad programme. Time duration: 9.00 – 14.00

PROBLEM 1

Given $a \in (0,1) \cap \mathbb{Q}$ let $a = 0,a_1\, a_2\, a_3 \ldots$ be its decimal representation. Define

$$f_a(x) = \sum_{n=1}^{\infty} a_n x^n, \; x \in (0\,,1).$$

Prove that f_a is a rational function of the form $f_a(x) = \dfrac{P(x)}{Q(x)}$, where P and Q are polynomials with integer coefficients.

Conversely, if $a_k \in \{0, 1, 2, \ldots, 9\}$ for all $k \in \mathbb{N}$, and $f_a(x) = \sum_{n=1}^{\infty} a_n x^n$ for $x \in (0\,,1)$ is a rational function of the form $f_a(x) = \dfrac{P(x)}{Q(x)}$, where P and Q are polynomials with integer coefficients, prove that the number $a = 0,a_1\, a_2\, a_3 \ldots$ is rational.

PROBLEM 2

Let $f(x) = \underbrace{\max}_{i} |x_i|$ for $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$ and let A be an nxn matrix such that $f(Ax) = f(x)$ for all $x \in \mathbb{R}^n$. Prove that there exists a positive integer m such that A^m is the identity matrix I_n.

PROBLEM 3

Let F be a field and let P: F x F $\to$ F be a function such that for every $x_0 \in F$ the function $P(x_0, y)$ is a polynomial in y and for every $y_0 \in F$ the function $P(x, y_0)$ is a polynomial in x.
Is it true that P is necessarily a polynomial in x and y, when

a) $F = \mathbb{Q}$, the field of rational numbers?

b) F is a finite field?

Prove your claims.

PROBLEM 4

For $x \in \mathbb{R}$, $y \geq 0$ and $n \in \mathbb{Z}$ denote by $w_n(x, y) \in [0, \pi)$ the angle in radians with which the segment joining the point (n , 0) to the point (n + y , 0) is seen from the point $(x, 1) \in \mathbb{R}^2$.

a) Show that for every $x \in \mathbb{R}$ and $y \geq 0$, the series $\sum_{n=-\infty}^{\infty} w_n(x, y)$ converges.

If we now set $w(x,y) = \sum_{n=-\infty}^{\infty} w_n(x, y)$, show that $w(x,y) \leq ([y] + 1)\pi$.
([y] is the integer part of y)

b) Prove that for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every y with $0 < y < \delta$ and every $x \in \mathbb{R}$ we have $w(x,y) < \varepsilon$.

c) Prove that the function $w : \mathbb{R} \times [0, +\infty) \to [0, +\infty)$ defined in (a) is continuous.

SEEMOUS 2008
South Eastern European Mathematical Olympiad for University Students

Athens – March 7, 2008

Problem 1

Let $f : [1, \infty) \to (0, \infty)$ be a continuous function. Assume that for every $a > 0$, the equation $f(x) = ax$ has at least one solution in the interval $[1, \infty)$.

(a) Prove that for every $a > 0$, the equation $f(x) = ax$ has infinitely many solutions.

(b) Give an example of a strictly increasing continuous function f with these properties.

Problem 2

Let $P_0, P_1, P_2, \dots$ be a sequence of convex polygons such that, for each $k \geq 0$, the vertices of P_{k+1} are the midpoints of all sides of P_k. Prove that there exists a unique point lying inside all these polygons.

Problem 3

Let $\mathcal{M}_n(\mathbb{R})$ denote the set of all real $n \times n$ matrices. Find all surjective functions $f : \mathcal{M}_n(\mathbb{R}) \to \{0, 1, \dots, n\}$ which satisfy

$$f(XY) \leq \min\{f(X), f(Y)\}$$

for all $X, Y \in \mathcal{M}_n(\mathbb{R})$.

Problem 4

Let n be a positive integer and $f : [0, 1] \to \mathbb{R}$ be a continuous function such that

$$\int_0^1 x^k f(x)\, dx = 1$$

for every $k \in \{0, 1, \dots, n-1\}$. Prove that

$$\int_0^1 (f(x))^2\, dx \geq n^2.$$

Answers

Problem 1

Solution. (a) Suppose that one can find constants $a > 0$ and $b > 0$ such that $f(x) \neq ax$ for all $x \in [b, \infty)$. Since f is continuous we obtain two possible cases:

1.) $f(x) > ax$ for $x \in [b, \infty)$. Define

$$c = \min_{x \in [1,b]} \frac{f(x)}{x} = \frac{f(x_0)}{x_0}.$$

Then, for every $x \in [1, \infty)$ one should have

$$f(x) > \frac{\min(a, c)}{2} x,$$

a contradiction.

2.) $f(x) < ax$ for $x \in [b, \infty)$. Define

$$C = \max_{x \in [1,b]} \frac{f(x)}{x} = \frac{f(x_0)}{x_0}.$$

Then,

$$f(x) < 2 \max(a, C) x$$

for every $x \in [1, \infty)$ and this is again a contradiction.

(b) Choose a sequence $1 = x_1 < x_2 < \cdots < x_k < \cdots$ such that the sequence $y_k = 2^{k \cos k\pi} x_k$ is also increasing. Next define $f(x_k) = y_k$ and extend f linearly on each interval $[x_{k-1}, x_k]$: $f(x) = a_k x + b_k$ for suitable a_k, b_k. In this way we obtain an increasing continuous function f, for which $\lim\limits_{n\to\infty} \frac{f(x_{2n})}{x_{2n}} = \infty$ and $\lim\limits_{n\to\infty} \frac{f(x_{2n-1})}{x_{2n-1}} = 0$. It now follows that the continuous function $\frac{f(x)}{x}$ takes every positive value on $[1, \infty)$.

Problem 2

Solution. For each $k \geq 0$ we denote by $A_i^k = (x_i^k, y_i^k)$, $i = 1, \ldots, n$ the vertices of P_k. We may assume that the center of gravity of P_0 is $O = (0, 0)$; in other words,

$$\frac{1}{n}(x_1^0 + \cdots + x_n^0) = 0 \text{ and } \frac{1}{n}(y_1^0 + \cdots + y_n^0) = 0.$$

Since $2x_i^{k+1} = x_i^k + x_{i+1}^k$ and $2y_i^{k+1} = y_i^k + y_{i+1}^k$ for all k and i (we agree that $x_{n+j}^k = x_j^k$ and $y_{n+j}^k = y_j^k$) we see that

$$\frac{1}{n}(x_1^k + \cdots + x_n^k) = 0 \text{ and } \frac{1}{n}(y_1^k + \cdots + y_n^k) = 0$$

for all $k \geq 0$. This shows that $O = (0, 0)$ is the center of gravity of all polygons P_k.

In order to prove that O is the unique common point of all P_k's it is enough to prove the following claim:

Claim. Let R_k be the radius of the smallest ball which is centered at O and contains P_k. Then, $\lim\limits_{k\to\infty} R_k = 0$.

Proof of the Claim. Write $\|\cdot\|_2$ for the Euclidean distance to the origin O. One can easily check that there exist $\beta_1, \dots, \beta_n > 0$ and $\beta_1 + \cdots + \beta_n = 1$ such that

$$A_j^{k+n} = \sum_{i=1}^{n} \beta_i A_{j+i-1}^k$$

for all k and j. Let $\lambda = \min_{i=1,\dots,n} \beta_i$. Since $O = \sum_{i=1}^n A_{j+i-1}^k$, we have the following:

$$\begin{aligned} \|A_j^{k+n}\|_2 &= \left\| \sum_{i=1}^n (\beta_i - \lambda) A_{j+i-1}^k \right\|_2 \\ &\leq \sum_{i=1}^n (\beta_i - \lambda) \|A_{j+i-1}^k\|_2 \\ &\leq R_k \sum_{i=1}^n (\beta_i - \lambda) = R_k(1 - n\lambda). \end{aligned}$$

This means that P_{k+n} lies in the ball of radius $R_k(1-n\lambda)$ centered at O. Observe that $1 - n\lambda < 1$.

Continuing in the same way we see that P_{mn} lies in the ball of radius $R_0(1-n\lambda)^m$ centered at O. Therefore, $R_{mn} \to 0$. Since $\{R_n\}$ is decreasing, the proof is complete.

Problem 3

Solution. We will show that the only such function is $f(X) = \operatorname{rank}(X)$. Setting $Y = I_n$ we find that $f(X) \leq f(I_n)$ for all $X \in \mathcal{M}_n(\mathbb{R})$. Setting $Y = X^{-1}$ we find that $f(I_n) \leq f(X)$ for all invertible $X \in \mathcal{M}_n(\mathbb{R})$. From these facts we conclude that $f(X) = f(I_n)$ for all $X \in GL_n(\mathbb{R})$.

For $X \in GL_n(\mathbb{R})$ and $Y \in \mathcal{M}_n(\mathbb{R})$ we have

$$\begin{aligned} &f(Y) = f(X^{-1}XY) \leq f(XY) \leq f(Y), \\ &f(Y) = f(YXX^{-1}) \leq f(YX) \leq f(Y). \end{aligned}$$

Hence we have $f(XY) = f(YX) = f(Y)$ for all $X \in GL_n(\mathbb{R})$ and $Y \in \mathcal{M}_n(\mathbb{R})$. For $k = 0, 1, \dots, n$, let

$$J_k = \begin{pmatrix} I_k & O \\ O & O \end{pmatrix}.$$

It is well known that every matrix $Y \in \mathcal{M}_n(\mathbb{R})$ is equivalent to J_k for $k = \operatorname{rank}(Y)$. This means that there exist matrices $X, Z \in GL_n(\mathbb{R})$ such that $Y = XJ_kZ$. From the discussion above it follows that $f(Y) = f(J_k)$. Thus it suffices to determine the values of the function f on the matrices $J_0, J_1, \dots, J_n$. Since $J_k = J_k \cdot J_{k+1}$ we have $f(J_k) \leq f(J_{k+1})$ for $0 \leq k \leq n-1$. Surjectivity of f imples that $f(J_k) = k$ for $k = 0, 1, \dots, n$ and hence $f(Y) = \operatorname{rank}(Y)$ for all $Y \in \mathcal{M}_n(\mathbb{R})$.

Problem 4

Solution. There exists a polynomial $p(x) = a_1 + a_2x + \cdots + a_nx^{n-1}$ which satisfies

$$\int_0^1 x^k p(x)\, dx = 1 \quad \text{for all } k = 0, 1, \dots, n-1. \tag{1}$$

It follows that, for all $k = 0, 1, \ldots, n-1$,

$$\int_0^1 x^k (f(x) - p(x))\, dx = 0,$$

and hence

$$\int_0^1 p(x)(f(x) - p(x))\, dx = 0.$$

Then, we can write

$$\begin{aligned}\int_0^1 (f(x) - p(x))^2 dx &= \int_0^1 f(x)(f(x) - p(x))\, dx \\ &= \int_0^1 f^2(x)\, dx - \sum_{k=0}^{n-1} a_{k+1} \int_0^1 x^k f(x)\, dx,\end{aligned}$$

and since the first integral is non-negative we get

$$\int_0^1 f^2(x)\, dx \geq a_1 + a_2 + \cdots + a_n.$$

To complete the proof we show the following:

Claim. For the coefficients $a_1, \ldots, a_n$ of p we have

$$a_1 + a_2 + \cdots + a_n = n^2.$$

Proof of the Claim. The defining property of p can be written in the form

$$\frac{a_1}{k+1} + \frac{a_2}{k+2} + \cdots + \frac{a_n}{k+n} = 1, \qquad 0 \leq k \leq n-1.$$

Equivalently, the function

$$r(x) = \frac{a_1}{x+1} + \frac{a_2}{x+2} + \cdots + \frac{a_n}{x+n} - 1$$

has $0, 1, \ldots, n-1$ as zeros. We write r in the form

$$r(x) = \frac{q(x) - (x+1)(x+2)\cdots(x+n)}{(x+1)(x+2)\cdots(x+n)},$$

where q is a polynomial of degree $n-1$. Observe that the coefficient of x^{n-1} in q is equal to $a_1 + a_2 + \cdots + a_n$. Also, the numerator has $0, 1, \ldots, n-1$ as zeros, and since $\lim_{x\to\infty} r(x) = -1$ we must have

$$q(x) = (x+1)(x+2)\cdots(x+n) - x(x-1)\cdots(x-(n-1)).$$

This expression for q shows that the coefficient of x^{n-1} in q is $\frac{n(n+1)}{2} + \frac{(n-1)n}{2}$. It follows that

$$a_1 + a_2 + \cdots + a_n = n^2.$$

SEEMOUS 2009

South Eastern European Mathematical Olympiad for University Students

AGROS, March 6, 2009

COMPETITION PROBLEMS

Problem 1

a) Calculate the limit

$$\lim_{n\to\infty} \frac{(2n+1)!}{(n!)^2} \int_0^1 (x(1-x))^n x^k dx,$$

where $k \in \mathbb{N}$.

b) Calculate the limit

$$\lim_{n\to\infty} \frac{(2n+1)!}{(n!)^2} \int_0^1 (x(1-x))^n f(x) dx,$$

where $f : [0,1] \to \mathbb{R}$ is a continuous function.

Solution Answer: $f\left(\frac{1}{2}\right)$. Proof: Set

$$L_n(f) = \frac{(2n+1)!}{(n!)^2} \int_0^1 (x(1-x))^n f(x)\, dx .$$

A straightforward calculation (integrating by parts) shows that

$$\int_0^1 (x(1-x))^n x^k dx = \frac{(n+k)!n!}{(2n+k+1)!} .$$

Thus, $\displaystyle\int_0^1 (x(1-x))^n dx = \frac{(n!)^2}{(2n+1)!}$ and desired limit is equal to $\lim\limits_{n\to\infty} L_n(f)$. Next,

$$\lim_{n\to\infty} L_n(x^k) = \lim_{n\to\infty} \frac{(n+1)(n+2)\ldots(n+k)}{(2n+2)(2n+3)\ldots(2n+k+1)} = \frac{1}{2^k} .$$

According to linearity of the integral and of the limit, $\lim\limits_{n\to\infty} L_n(P) = P\left(\frac{1}{2}\right)$ for every polynomial $P(x)$.

Finally, fix an arbitrary $\varepsilon > 0$. A polynomial P can be chosen such that $|f(x) - P(x)| < \varepsilon$ for every $x \in [0,1]$. Then

$$|L_n(f) - L_n(P)| \le L_n(|f-P|) < L_n(\varepsilon \cdot \mathbb{I}) = \varepsilon \text{, where } \mathbb{I}(x) = 1, \text{ for every } x \in [0,1] .$$

There exists n_0 such that $\left|L_n(P) - P\left(\frac{1}{2}\right)\right| < \varepsilon$ for $n \geq n_0$. For these integers

$$\left|L_n(f) - f\left(\frac{1}{2}\right)\right| \leq |L_n(f) - L_n(P)| + \left|L_n(P) - P\left(\frac{1}{2}\right)\right| + \left|f\left(\frac{1}{2}\right) - P\left(\frac{1}{2}\right)\right| < 3\varepsilon\,,$$

which concludes the proof.

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Problem 2

Let P be a real polynomial of degree five. Assume that the graph of P has three inflection points lying on a straight line. Calculate the ratios of the areas of the bounded regions between this line and the graph of the polynomial P.

Solution Denote the inflection points by A, B, and C. Let $l : y = kx + n$ be the equation of the line that passes through them. If B has coordinates (x_0, y_0), the affine change

$$x' = x - x_0, \quad y' = kx - y + n$$

transforms l into the x-axis, and the point B—into the origin. Then without loss of generality it is sufficient to consider a fifth-degree polynomial $f(x)$ with points of inflection $(b, 0)$, $(0, 0)$ and $(a, 0)$, with $b < 0 < a$. Obviously $f''(x) = kx(x - a)(x - b)$, hence

$$f(x) = \frac{k}{20}x^5 - \frac{k(a + b)}{12}x^4 + \frac{kab}{6}x^3 + cx + d\,.$$

By substituting the coordinates of the inflection points, we find $d = 0$, $a + b = 0$ and $c = \frac{7ka^4}{60}$ and therefore

$$f(x) = \frac{k}{20}x^5 - \frac{ka^2}{6}x^3 + \frac{7ka^4}{60}x = \frac{k}{60}\,x(x^2 - a^2)(3x^2 - 7a^2)\,.$$

Since $f(x)$ turned out to be an odd function, the figures bounded by its graph and the x-axis are pairwise equiareal. Two of the figures with unequal areas are

$$\Omega_1 :\ 0 \leq x \leq a,\ 0 \leq y \leq f(x); \qquad \Omega_2 :\ a \leq x \leq a\sqrt{\frac{7}{3}},\ f(x) \leq y \leq 0.$$

We find

$$S_1 = S(\Omega_1) = \int_0^a f(x)\,dx = \frac{ka^6}{40}\,,$$

$$S_2 = S(\Omega_2) = -\int_a^{a\sqrt{\frac{7}{3}}} f(x)\,dx = \frac{4ka^6}{405}$$

and conclude that $S_1 : S_2 = 81 : 32$.

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Problem 3

Let $\mathbf{SL}_2(\mathbb{Z}) = \{A \mid A \text{ is a } 2\times 2 \text{ matrix with integer entries and } \det A = 1\}$.

a) Find an example of matrices $A, B, C \in \mathbf{SL}_2(\mathbb{Z})$ such that $A^2 + B^2 = C^2$.

b) Show that there do not exist matrices $A, B, C \in \mathbf{SL}_2(\mathbb{Z})$ such that $A^4 + B^4 = C^4$.

Solution a) Yes. Example:

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

b) No. Let us recall that every 2×2 matrix A satisfies $A^2 - (trA)\,A + (\det A)\,E = 0$ where $trA = a_{11} + a_{22}$.

Suppose that A, B, C $\in \mathbf{SL}_2(\mathbb{Z})$ and $A^4 + B^4 = C^4$. Let $a = trA,\ b = trB,\ c = trC$. Then $A^4 = (aA - E)^2 = a^2A^2 - 2aA + E = (a^3 - 2a)\,A + (1 - a^2)\,E$ and, after same expressions for B^4 and C^4 have been substituted,

$$\left(a^3 - 2a\right)A + \left(b^3 - 2b\right)B + \left(2 - a^2 - b^2\right)E = \left(c^3 - 2c\right)C + \left(1 - c^2\right)E\,.$$

Calculating traces of both sides we obtain $a^4 + b^4 - 4(a^2 + b^2) = c^4 - 4c^2 - 2$, so $a^4 + b^4 - c^4 \equiv -2 \pmod 4$. Since for every integer k: $k^4 \equiv 0 \pmod 4$ or $k^4 \equiv 1 \pmod 4$, then a and b are odd and c is even. But then $a^4 + b^4 - 4(a^2 + b^2) \equiv 2 \pmod 8$ and $c^4 - 4c^2 - 2 \equiv -2 \pmod 8$ which is a contradiction.

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Problem 4

Given the real numbers $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ we define the $n \times n$ matrices $A = (a_{ij})$ and $B = (b_{ij})$ by

$$a_{ij} = a_i - b_j \quad \text{and} \quad b_{ij} = \begin{cases} 1, & \text{if } a_{ij} \geq 0, \\ 0, & \text{if } a_{ij} < 0, \end{cases} \qquad \text{for all } i, j \in \{1, 2, \ldots, n\}.$$

Consider $C = (c_{ij})$ a matrix of the same order with elements 0 and 1 such that

$$\sum_{j=1}^{n} b_{ij} = \sum_{j=1}^{n} c_{ij}, \quad i \in \{1, 2, \ldots, n\} \quad \text{and} \quad \sum_{i=1}^{n} b_{ij} = \sum_{i=1}^{n} c_{ij}, \quad j \in \{1, 2, \ldots, n\}.$$

Show that:

a) $$\sum_{i,j=1}^{n} a_{ij}(b_{ij} - c_{ij}) = 0 \quad \text{and } B = C.$$

b) B is invertible if and only if there exists two permutations σ and τ of the set $\{1, 2, \ldots, n\}$ such that

$$b_{\tau(1)} \leq a_{\sigma(1)} < b_{\tau(2)} \leq a_{\sigma(2)} < \cdots \leq a_{\sigma(n-1)} < b_{\tau(n)} \leq a_{\sigma(n)}.$$

Solution

(a) We have that

$$\sum_{i,j=1}^{n} a_{ij}(b_{ij} - c_{ij}) = \sum_{i=1}^{n} a_i \left(\sum_{j=1}^{n} b_{ij} - \sum_{j=1}^{n} c_{ij} \right) - \sum_{j=1}^{n} b_j \left(\sum_{i=1}^{n} b_{ij} - \sum_{i=1}^{n} c_{ij} \right) = 0. \tag{1}$$

We study the sign of $a_{ij}(b_{ij} - c_{ij})$.

If $a_i \geq b_j$, then $a_{ij} \geq 0$, $b_{ij} = 1$ and $c_{ij} \in \{0, 1\}$, hence $a_{ij}(b_{ij} - c_{ij}) \geq 0$.

If $a_i < b_j$, then $a_{ij} < 0$, $b_{ij} = 0$ and $c_{ij} \in \{0, 1\}$, hence $a_{ij}(b_{ij} - c_{ij}) \geq 0$.

Using (1), the conclusion is that

$$a_{ij}(b_{ij} - c_{ij}) = 0, \quad \text{for all } i, j \in \{1, 2, \ldots, n\}. \tag{2}$$

If $a_{ij} \neq 0$, then $b_{ij} = c_{ij}$. If $a_{ij} = 0$, then $b_{ij} = 1 \geq c_{ij}$. Hence, $b_{ij} \geq c_{ij}$ for all $i, j \in \{1, 2, \ldots, n\}$ and since $\sum_{i,j=1}^{n} b_{ij} = \sum_{i,j=1}^{n} c_{ij}$ the final conclusion is that

$$b_{ij} = c_{ij}, \quad \text{for all } i, j \in \{1, 2, \ldots, n\}.$$

(b) We may assume that $a_1 \leq a_2 \leq \ldots \leq a_n$ and $b_1 \leq b_2 \leq \ldots \leq b_n$ since any permutation of $a_1, a_2, \ldots, a_n$ permutes the lines of B and any permutation of $b_1, b_2, \ldots, b_n$ permutes the columns of B, which does not change whether B is invertible or not.

- If there exists i such that $a_i = a_{i+1}$, then the lines i and $i+1$ in B are equal, so B is not invertible. In the same way, if there exists j such $b_j = b_{j+1}$, then the columns j and $j+1$ are equal, so B is not invertible.
- If there exists i such that there is no b_j with $a_i < b_j \leq a_{i+1}$, then the lines i and $i+1$ in B are equal, so B is not invertible. In the same way, if there exists j such that there is no a_i with $b_j \leq a_i < b_{j+1}$, then the columns j and $j+1$ are equal, so B is not invertible.
- If $a_1 < b_1$, then $a_1 < b_j$ for any $j \in \{1, 2, \ldots, n\}$, which means that the first line of B has only zero elements, hence B is not invertible.

Therefore, if B is invertible, then $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ separate each other

$$b_1 \leq a_1 < b_2 \leq a_2 < \ldots \leq a_{n-1} < b_n \leq a_n. \tag{3}$$

It is easy to check that if (3), then

$$B = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}$$

which is, obviously, invertible.

Concluding, B is invertible if and only if there exists a permutation $a_{i_1}, a_{i_2}, \ldots, a_{i_n}$ of $a_1, a_2, \ldots, a_n$ and a permutation $b_{j_1}, b_{j_2}, \ldots, b_{j_n}$ of $b_1, b_2, \ldots, b_n$ such that

$$b_{j_1} \leq a_{i_1} < b_{j_2} \leq a_{i_2} < \ldots \leq a_{i_{n-1}} < b_{j_n} \leq a_{i_n}.$$

South Eastern European Mathematical Olympiad for University Students

Plovdiv, Bulgaria

March 10, 2010

Problem 1. Let $f_0 : [0,1] \to \mathbb{R}$ be a continuous function. Define the sequence of functions $f_n : [0,1] \to \mathbb{R}$ by

$$f_n(x) = \int_0^x f_{n-1}(t)\,dt$$

for all integers $n \geq 1$.

a) Prove that the series $\sum_{n=1}^{\infty} f_n(x)$ is convergent for every $x \in [0,1]$.

b) Find an explicit formula for the sum of the series $\sum_{n=1}^{\infty} f_n(x)$, $x \in [0,1]$.

Solution 1. *a*) Clearly $f_n' = f_{n-1}$ for all $n \in \mathbb{N}$. The function f_0 is bounded, so there exists a real positive number M such that $|f_0(x)| \leq M$ for every $x \in [0,1]$. Then

$$|f_1(x)| \leq \int_0^x |f_0(t)|\,dt \leq Mx, \quad \forall x \in [0,1],$$

$$|f_2(x)| \leq \int_0^x |f_1(t)|\,dt \leq M\frac{x^2}{2}, \quad \forall x \in [0,1].$$

By induction, it is easy to see that

$$|f_n(x)| \leq M\frac{x^n}{n!}, \quad \forall x \in [0,1],\ \forall n \in \mathbb{N}.$$

Therefore

$$\max_{x\in[0,1]} |f_n(x)| \leq \frac{M}{n!}, \quad \forall n \in \mathbb{N}.$$

The series $\sum_{n=1}^{\infty} \frac{1}{n!}$ is convergent, so the series $\sum_{n=1}^{\infty} f_n$ is uniformly convergent on $[0,1]$.

b) Denote by $F : [0,1] \to \mathbb{R}$ the sum of the series $\sum_{n=1}^{\infty} f_n$. The series of the derivatives $\sum_{n=1}^{\infty} f_n'$ is uniformly convergent on $[0,1]$, since

$$\sum_{n=1}^{\infty} f_n' = \sum_{n=0}^{\infty} f_n$$

and the last series is uniformly convergent. Then the series $\sum_{n=1}^{\infty} f_n$ can be differentiated term by term and $F' = F + f_0$. By solving this equation, we find $F(x) = e^x \left(\int_0^x f_0(t)e^{-t}\,dt\right)$, $x \in [0,1]$.

Solution 2. We write

$$\begin{aligned}
f_n(x) &= \int_0^x dt \int_0^t dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-2}} f_0(t_{n-1})\, dt_{n-1} \\
&= \underset{0\le t_{n-1}\le \ldots \le t_1\le t\le x}{\int \ldots \int} f_0(t_{n-1})\, dt\, dt_1\, \ldots\, dt_{n-1} \\
&= \underset{0\le t\le t_1\le \ldots \le t_{n-1}\le x}{\int \ldots \int} f_0(t)\, dt\, dt_1\, \ldots dt_{n-1} \\
&= \int_0^x f_0(t)\, dt \int_t^x dt_1 \int_{t_1}^x dt_2 \ldots \int_{t_{n-3}}^x dt_{n-2} \int_{t_{n-2}}^x dt_{n-1} \\
&= \int_0^x f_0(t) \frac{(x-t)^{n-1}}{(n-1)!}\, dt.
\end{aligned}$$

Thus

$$\sum_{n=1}^{N} f_n(x) = \int_0^x f_0(t) \left(\sum_{n=1}^{N} \frac{(x-t)^{n-1}}{(n-1)!} \right) dt.$$

We have

$$e^{x-t} = \sum_{n=0}^{N-1} \frac{(x-t)^n}{n!} + e^{\theta}\, \frac{(x-t)^N}{N!}, \quad \theta \in (0, x-t),$$

$$\sum_{n=0}^{N-1} \frac{(x-t)^n}{n!} \to e^{x-t}, \quad N \to \infty.$$

Hence

$$\begin{aligned}
\left| \int_0^x f_0(t) \left(\sum_{n=0}^{N-1} \frac{(x-t)^n}{n!} \right) dt - \int_0^x f_0(t) e^{x-t} dt \right| &\le \int_0^x |f_0(t)| e^{x-t} \frac{(x-t)^N}{N!} dt \\
&\le \frac{1}{N!} \int_0^x |f_0(t)| e^{x-t}\, dt \to 0, \quad N \to \infty.
\end{aligned}$$

Problem 2. Inside a square consider circles such that the sum of their circumferences is twice the perimeter of the square.

a) Find the minimum number of circles having this property.

b) Prove that there exist infinitely many lines which intersect at least 3 of these circles.

Solution. *a*) Consider the circles $C_1, C_2, \ldots, C_k$ with diameters $d_1, d_2, ..., d_k$, respectively. Denote by s the length of the square side. By using the hypothesis, we get

$$\pi(d_1 + d_2 + \cdots + d_k) = 8s.$$

Since $d_i \le s$ for $i = 1, \ldots, k$, we have

$$8s = \pi(d_1 + d_2 + \cdots + d_k) \le \pi k s,$$

which implies $k \ge \dfrac{8}{\pi} \cong 2.54$. Hence, there are at least 3 circles inside the square.

b) Project the circles onto one side of the square so that their images are their diameters. Since the sum of the diameters is approximately $2.54s$ and there are at least three circles in the square, there exists an interval where at least three diameters are overlapping. The lines, passing through this interval and perpendicular to the side on which the diameters are projected, are the required lines.

Problem 3. Denote by $\mathcal{M}_2(\mathbb{R})$ the set of all 2×2 matrices with real entries. Prove that:

a) for every $A\in\mathcal{M}_2(\mathbb{R})$ there exist $B,\ C\in\mathcal{M}_2(\mathbb{R})$ such that $A=B^2+C^2$;

b) there do not exist $B,\ C\in\mathcal{M}_2(\mathbb{R})$ such that $\begin{pmatrix}0&1\\1&0\end{pmatrix}=B^2+C^2$ and $BC=CB$.

Solution. *a*) Recall that every 2×2 matrix A satisfies $A^2-(trA)\,A+(\det A)\,E=0$. It is clear that

$$\lim_{t\to+\infty} tr\,(A+tE)=+\infty \quad\text{and}\quad \lim_{t\to+\infty}\frac{\det(A+tE)}{tr(A+tE)}-t=\lim_{t\to+\infty}\frac{\det A-t^2}{tr\,(A+tE)}=-\infty\,.$$

Thus, for t large enough one has

$$\begin{aligned}A &= (A+tE)-tE=\frac{1}{tr(A+tE)}\,(A+tE)^2+\left(\frac{\det(A+tE)}{tr\,(A+tE)}-t\right)E\\ &= \left(\frac{1}{\sqrt{tr(A+tE)}}\,(A+tE)\right)^2+\left(\sqrt{t-\frac{\det(A+tE)}{tr\,(A+tE)}}\right)^2(-E)\\ &= \left(\frac{1}{\sqrt{tr(A+tE)}}\,(A+tE)\right)^2+\left(\sqrt{t-\frac{\det(A+tE)}{tr\,(A+tE)}}\begin{pmatrix}0&1\\-1&0\end{pmatrix}\right)^2.\end{aligned}$$

b) No. For $B,\ C\in\mathcal{M}_2(\mathbb{R})$, consider $B+iC,\ B-iC\in\mathcal{M}_2(\mathbb{C})$. If $BC=CB$ then $(B+iC)\,(B-iC)=B^2+C^2$. Thus

$$\det\left(B^2+C^2\right)=\det(B+iC)\det(B-iC)=|\,B+iC\,|^{\,2}\geq 0,$$

which contradicts the fact that $\det\begin{pmatrix}0&1\\1&0\end{pmatrix}=-1\,.$

Problem 4. Suppose that A and B are $n\times n$ matrices with integer entries, and $\det B\neq 0$. Prove that there exists $m\in\mathbb{N}$ such that the product AB^{-1} can be represented as

$$AB^{-1}=\sum_{k=1}^{m}N_k^{-1},$$

where N_k are $n\times n$ matrices with integer entries for all $k=1,\dots,m$, and $N_i\neq N_j$ for $i\neq j$.

Solution. Suppose first that $n=1$. Then we may consider the integer 1×1 matrices as integer numbers. We shall prove that for given integers p and q we can find integers $n_1,\dots,n_m$ such that $\frac{p}{q}=\frac{1}{n_1}+\frac{1}{n_2}+\cdots+\frac{1}{n_m}$ and $n_i\neq n_j$ for $i\neq j$.

In fact this is well known as the "Egyptian problem". We write $\frac{p}{q}=\frac{1}{q}+\frac{1}{q}+\cdots+\frac{1}{q}$ (p times) and ensure different denominators in the last sum by using several times the equality $\frac{1}{x}=\frac{1}{x+1}+\frac{1}{x(x+1)}$. For example, $\frac{3}{5}=\frac{1}{5}+\frac{1}{5}+\frac{1}{5}$, where we keep the first fraction, we write $\frac{1}{5}=\frac{1}{6}+\frac{1}{30}$ for the second fraction, and $\frac{1}{5}=\frac{1}{7}+\frac{1}{42}+\frac{1}{31}+\frac{1}{930}$ for the third fraction. Finally,

$$\frac{3}{5}=\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{30}+\frac{1}{31}+\frac{1}{42}+\frac{1}{930}.$$

Now consider $n > 1$.

Case 1. Suppose that A is a nonsingular matrix. Denote by λ the least common multiple of the denominators of the elements of the matrix A^{-1}. Hence the matrix $C = \lambda BA^{-1}$ is integer and nonsingular, and one has

$$AB^{-1} = \lambda C^{-1}.$$

According to the case $n = 1$, we can write

$$\lambda = \frac{1}{n_1} + \frac{1}{n_2} + \cdots + \frac{1}{n_m},$$

where $n_i \neq n_j$ for $i \neq j$. Then

$$AB^{-1} = (n_1 C)^{-1} + (n_2 C)^{-1} + \cdots + (n_m C)^{-1}.$$

It is easy to see that $n_i C \neq n_j C$ for $i \neq j$.

Case 2. Now suppose that A is singular. First we will show that

$$A = Y + Z,$$

where Y and Z are nonsingular. If $A = (a_{ij})$, for every $i = 1, 2, \ldots, n$ we choose an integer x_i such that $x_i \neq 0$ and $x_i \neq a_{ii}$. Define

$$y_{ij} = \begin{cases} a_{ij}, & \text{if } i < j \\ x_i, & \text{if } i = j \\ 0, & \text{if } i > j \end{cases} \quad \text{and} \quad z_{ij} = \begin{cases} 0, & \text{if } i < j \\ a_{ii} - x_i, & \text{if } i = j \\ a_{ij}, & \text{if } i > j. \end{cases}$$

Clearly, the matrices $Y = (y_{ij})$ and $Z = (z_{ij})$ are nonsingular. Moreover, $A = Y + Z$.

From Case 1 we have

$$YB^{-1} = \sum_{r=1}^{k} (n_r C)^{-1}, \quad ZB^{-1} = \sum_{q=1}^{l} (m_q D)^{-1},$$

where

$$YB^{-1} = \lambda C^{-1}, \quad \lambda = \sum_{r=1}^{k} \frac{1}{n_r} \quad \text{and} \quad ZB^{-1} = \mu D^{-1}, \quad \mu = \sum_{q=1}^{l} \frac{1}{m_q},$$

C and D are integer and nonsingular. Hence,

$$AB^{-1} = \sum_{r=1}^{k} (n_r C)^{-1} + \sum_{q=1}^{l} (m_q D)^{-1}.$$

It remains to show that $n_r C \neq m_q D$ for $r = 1, 2, \ldots, k$ and $q = 1, 2, \ldots, l$. Indeed, assuming that $n_r C = m_q D$ and recalling that $m_q > 0$ we find $D = \frac{n_r}{m_q} C$. Hence $ZB^{-1} = \mu D^{-1} = \frac{\mu m_q}{n_r} C^{-1}$, and then $AB^{-1} = YB^{-1} + ZB^{-1} = \lambda C^{-1} + \frac{\mu m_q}{n_r} C^{-1} = \left(\lambda + \frac{\mu m_q}{n_r}\right) C^{-1}$. We have $\lambda + \frac{\mu m_q}{n_r} > 0$, and C^{-1} is nonsingular. Then AB^{-1} is nonsingular, and therefore A is nonsingular. This is a contradiction.

Bucharest, March 4th, 2011

SOUTH EASTERN EUROPEAN MATHEMATICAL OLYMPIAD FOR UNIVERSITY STUDENTS

PROBLEMS

Problem 1 *For a given integer $n \geq 1$, let $f : [0,1] \to \mathbb{R}$ be a non-decreasing function. Prove that*

$$\int_0^1 f(x)\,\mathrm{d}x \leq (n+1)\int_0^1 x^n f(x)\,\mathrm{d}x.$$

Find all non-decreasing continuous functions for which equality holds.

Problem 2 *Let $A = (a_{ij})$ be a real $n \times n$ matrix such that $A^n \neq 0$ and $a_{ij}a_{ji} \leq 0$ for all i, j. Prove that there exist two nonreal numbers among eigenvalues of A.*

Problem 3 *Given vectors $\overline{a}, \overline{b}, \overline{c} \in \mathbb{R}^n$, show that*

$$\left(||\overline{a}||\left\langle \overline{b}, \overline{c}\right\rangle\right)^2 + \left(||\overline{b}||\left\langle \overline{a}, \overline{c}\right\rangle\right)^2 \leq ||\overline{a}||||\overline{b}||\left(||\overline{a}||||\overline{b}|| + |\left\langle \overline{a}, \overline{b}\right\rangle|\right)||\overline{c}||^2,$$

where $\langle \overline{x}, \overline{y}\rangle$ denotes the scalar (inner) product of the vectors $\overline{x}$ and $\overline{y}$ and $||\overline{x}||^2 = \langle \overline{x}, \overline{x}\rangle$.

Problem 4 *Let $f : [0,1] \to \mathbb{R}$ be a twice continuously differentiable increasing function. Define the sequences given by $L_n = \frac{1}{n}\sum\limits_{k=0}^{n-1} f\left(\frac{k}{n}\right)$ and $U_n = \frac{1}{n}\sum\limits_{k=1}^{n} f\left(\frac{k}{n}\right)$, $n \geq 1$. The interval $[L_n, U_n]$ is divided into three equal segments. Prove that, for large enough n, the number $I = \int\limits_0^1 f(x)\,\mathrm{d}x$ belongs to the middle one of these three segments.*

Each problem is 10 points worth.
Allowed time: 5 hours.

Sixth South Eastern European Mathematical Olympiad for University Students

Blagoevgrad, Bulgaria

March 8, 2012

Problem 1. Let $A = (a_{ij})$ be the $n \times n$ matrix, where a_{ij} is the remainder of the division of $i^j + j^i$ by 3 for $i, j = 1, 2, \ldots, n$. Find the greatest n for which $\det A \neq 0$.

Solution. We show that $a_{i+6,j} = a_{ij}$ for all $i, j = 1, 2, \ldots, n$. First note that if $j \equiv 0 \pmod 3$ then $j^i \equiv 0 \pmod 3$, and if $j \equiv 1$ or $2 \pmod 3$ then $j^6 \equiv 1 \pmod 3$. Hence, $j^i(j^6 - 1) \equiv 0 \pmod 3$ for $j = 1, 2, \ldots, n$, and

$$a_{i+6,j} \equiv (i+6)^j + j^{i+6} \equiv i^j + j^i \equiv a_{ij} \pmod 3,$$

or $a_{i+6,j} = a_{ij}$. Consequently, $\det A = 0$ for $n \geq 7$. By straightforward calculation, we see that $\det A = 0$ for $n = 6$ but $\det A \neq 0$ for $n = 5$, so the answer is $n = 5$.

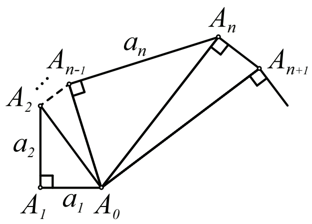
Grading of Problem 1.

5p: Concluding that $\Delta_n = 0$ for each $n \geq 7$

5p: Computing $\Delta_5 = 12,\ \Delta_6 = 0$

2p: Computing $\Delta_3 = -10,\ \Delta_4 = 4$ (in case none of the above is done)

Problem 2. Let $a_n > 0$, $n \geq 1$. Consider the right triangles $\triangle A_0A_1A_2$, $\triangle A_0A_2A_3, \ldots$, $\triangle A_0A_{n-1}A_n, \ldots$, as in the figure. (More precisely, for every $n \geq 2$ the hypotenuse A_0A_n of $\triangle A_0A_{n-1}A_n$ is a leg of $\triangle A_0A_nA_{n+1}$ with right angle $\angle A_0A_nA_{n+1}$, and the vertices A_{n-1} and A_{n+1} lie on the opposite sides of the straight line A_0A_n; also, $|A_{n-1}A_n| = a_n$ for every $n \geq 1$.)



Is it possible for the set of points $\{A_n \mid n \geq 0\}$ to be unbounded but the series $\sum_{n=2}^{\infty} m(\angle A_{n-1}A_0A_n)$ to be convergent? Here $m(\angle ABC)$ denotes the measure of $\angle ABC$.

Note. A subset B of the plane is bounded if there is a disk D such that $B \subseteq D$.

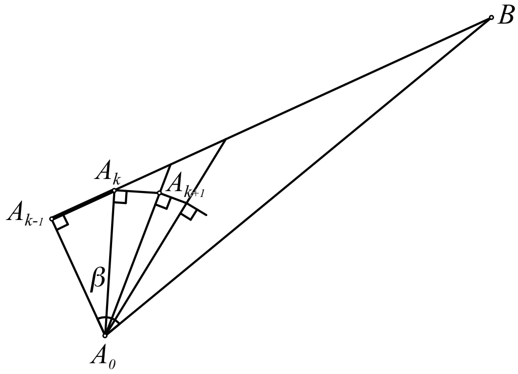
Solution. We have $|A_0A_n| = \sqrt{\sum_{i=1}^{n} a_i^2}$ and $\sum_{n=2}^{k} m(\angle A_{n-1}A_0A_n) = \sum_{n=2}^{k} \arctan \dfrac{a_n}{\sqrt{a_1^2 + \cdots + a_{n-1}^2}}$.

The set of points $\{A_n \mid n \geq 0\}$ will be unbounded if and only if the sequence of the lengths of the segments A_0A_n is unbounded. Put $a_i^2 = b_i$. Then the question can be reformulated as follows: Is it possible for a series with positive terms to be such that $\sum_{i=1}^{\infty} b_i = \infty$ and

$$\sum_{n=2}^{\infty} \arctan \sqrt{\frac{b_n}{b_1 + \cdots + b_{n-1}}} < \infty.$$

Denote $s_n = \sum_{i=1}^n b_i$. Since $\arctan x \sim x$ as $x \to 0$, the question we need to ask is whether one can have $s_n \to \infty$ as $n \to \infty$ and $\sum\limits_{n=2}^{\infty} \sqrt{\frac{s_n - s_{n-1}}{s_{n-1}}} < \infty$. Put $\sqrt{\frac{s_n - s_{n-1}}{s_{n-1}}} = u_n > 0$. Then $\frac{s_n}{s_{n-1}} = 1 + u_n^2$, $\ln s_n - \ln s_{n-1} = \ln(1+u_n^2)$, $\ln s_k = \ln s_1 + \sum_{n=2}^k \ln(1+u_n^2)$. Finally, the question is whether it is possible to have $\sum_{n=2}^\infty \ln(1+u_n^2) = \infty$ and $\sum_{n=2}^\infty u_n < \infty$. The answer is negative, since $\ln(1+x) \sim x$ as $x \to 0$ and $u_n^2 \le u_n \le 1$ for large enough n.

Different solution. Since $\sum_{n=2}^\infty m(\angle A_{n-1}A_0A_n) < \infty$, there exists some large enough k for which $\sum_{n=k}^\infty m(\angle A_{n-1}A_0A_n) \le \beta < \frac{\pi}{2}$. Then all the vertices A_n, $n \ge k-1$, lie inside the triangle $\triangle A_0A_{k-1}B$, where the side $A_{k-1}B$ of $\triangle A_0A_{k-1}B$ is a continuation of the side $A_{k-1}A_k$ of $\triangle A_0A_{k-1}A_k$ and $\angle A_{k-1}A_0B = \beta$. Consequently, the set $\{A_n \,|\, n \ge 0\}$ is bounded which is a contradiction.



Grading of Problem 2.

1p: Noting that $\{A_n \,|\, n \ge 0\}$ is unbounded $\Leftrightarrow$ $|A_0A_n|$ is unbounded **OR** expressing $|A_0A_n|$

1p: Observing that $\sum_{n=2}^\infty m(\angle A_{n-1}A_0A_n)$ is convergent $\Leftrightarrow$ A_0A_n tends to A_0B **OR** expressing the angles by arctan

8p: Proving the assertion

Problem 3.

a) Prove that if k is an even positive integer and A is a real symmetric $n \times n$ matrix such that $(\mathrm{Tr}(A^k))^{k+1} = (\mathrm{Tr}(A^{k+1}))^k$, then

$$A^n = \mathrm{Tr}(A)\, A^{n-1}.$$

b) Does the assertion from *a)* also hold for odd positive integers k?

Solution. *a)* Let $k = 2l$, $l \ge 1$. Since A is a symmetric matrix all its eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ are real numbers. We have,

$$\mathrm{Tr}(A^{2l}) = \lambda_1^{2l} + \lambda_2^{2l} + \cdots + \lambda_n^{2l} = a \tag{1}$$

and

$$\mathrm{Tr}(A^{2l+1}) = \lambda_1^{2l+1} + \lambda_2^{2l+1} + \cdots + \lambda_n^{2l+1} = b. \tag{2}$$

By (1) we get that $a \ge 0$, so there is some $a_1 \ge 0$ such that $a = a_1^{2l}$. On the other hand, the equality $a^{2l+1} = b^{2l}$ implies that $(a_1^{2l+1})^{2l} = b^{2l}$ and hence

$$b = \pm a_1^{2l+1} = (\pm a_1)^{2l+1} \quad \text{and} \quad a = a_1^{2l} = (\pm a_1)^{2l}.$$

Then equalities (1) and (2) become

$$\lambda_1^{2l} + \lambda_2^{2l} + \cdots + \lambda_n^{2l} = c^{2l} \tag{3}$$

and

$$\lambda_1^{2l+1} + \lambda_2^{2l+1} + \cdots + \lambda_n^{2l+1} = c^{2l+1}, \tag{4}$$

where $c = \pm a_1$. We consider the following cases.
Case 1. If $c = 0$ then $\lambda_1 = \cdots = \lambda_n = 0$, so $\mathrm{Tr}(A) = 0$ and we note that the characteristic polynomial of A is $f_A(x) = x^n$. We have, based on the Cayley-Hamilton Theorem, that

$$A^n = 0 = \mathrm{Tr}(A)\, A^{n-1}.$$

Case 2. If $c \neq 0$ then let $x_i = \lambda_i / c$, $i = 1, 2, \ldots, n$. In this case equalities (3) and (4) become

$$x_1^{2l} + x_2^{2l} + \cdots + x_n^{2l} = 1 \tag{5}$$

and

$$x_1^{2l+1} + x_2^{2l+1} + \cdots + x_n^{2l+1} = 1. \tag{6}$$

The equality (5) implies that $|x_i| \leq 1$ for all $i = 1, 2, \ldots, n$. We have $x^{2l} \geq x^{2l+1}$ for $|x| \leq 1$ with equality reached when $x = 0$ or $x = 1$. Then, by (5), (6), and the previous observation, we find without loss of generality that $x_1 = 1$, $x_2 = x_3 = \cdots = x_n = 0$. Hence $\lambda_1 = c$, $\lambda_2 = \cdots = \lambda_n = 0$, and this implies that $f_A(x) = x^{n-1}(x - c)$ and $\mathrm{Tr}(A) = c$. It follows, based on the Cayley-Hamilton Theorem, that

$$f_A(A) = A^{n-1}(A - cI_n) = 0 \quad \Leftrightarrow \quad A^n = \mathrm{Tr}(A)\, A^{n-1}.$$

b) The answer to the question is negative. We give the following counterexample:

$$k = 1, \qquad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}.$$

Grading of Problem 3.

3p: Reformulating the problem through eigenvalues:

$$\left(\sum \lambda_i^{2l}\right)^{2l+1} = \left(\sum \lambda_i^{2l+1}\right)^{2l} \Rightarrow \forall i:\ \lambda_i^n = (\lambda_1 + \cdots + \lambda_n)\lambda_i^{n-1}$$

4p: Only $(\lambda_i) = (0, \ldots, 0, c, 0, \ldots, 0)$ or $(0, \ldots, 0)$ are possible

3p: Finding a counterexample

Problem 4.

a) Compute

$$\lim_{n\to\infty} n \int_0^1 \left(\frac{1-x}{1+x}\right)^n dx.$$

b) Let $k \geq 1$ be an integer. Compute

$$\lim_{n\to\infty} n^{k+1} \int_0^1 \left(\frac{1-x}{1+x}\right)^n x^k\, dx.$$

Solution. *a*) The limit equals $\dfrac{1}{2}$. The result follows immediately from *b*) for $k = 0$.

b) The limit equals $\dfrac{k!}{2^{k+1}}$. We have, by the substitution $\dfrac{1-x}{1+x} = y$, that

$$\begin{aligned} n^{k+1}\int_0^1 \left(\frac{1-x}{1+x}\right)^n x^k\,dx &= 2n^{k+1}\int_0^1 y^n(1-y)^k \frac{dy}{(1+y)^{k+2}} \\ &= 2n^{k+1}\int_0^1 y^n f(y)dy, \end{aligned}$$

where

$$f(y) = \frac{(1-y)^k}{(1+y)^{k+2}}\,.$$

We observe that

$$f(1) = f'(1) = \cdots = f^{(k-1)}(1) = 0. \tag{7}$$

We integrate k times by parts $\displaystyle\int_0^1 y^n f(y)dy$, and by (7) we get

$$\int_0^1 y^n f(y)dy = \frac{(-1)^k}{(n+1)(n+2)\dots(n+k)}\int_0^1 y^{n+k} f^{(k)}(y)dy.$$

One more integration implies that

$$\begin{aligned} \int_0^1 y^n f(y)dy &= \frac{(-1)^k}{(n+1)(n+2)\dots(n+k)(n+k+1)} \\ &\quad\times\left(f^{(k)}(y)y^{n+k+1}\Big|_0^1 - \int_0^1 y^{n+k+1}f^{(k+1)}(y)dy\right) \\ &= \frac{(-1)^k f^{(k)}(1)}{(n+1)(n+2)\dots(n+k+1)} \\ &\quad+ \frac{(-1)^{k+1}}{(n+1)(n+2)\dots(n+k+1)}\int_0^1 y^{n+k+1}f^{(k+1)}(y)dy. \end{aligned}$$

It follows that

$$\lim_{n\to\infty} 2n^{k+1}\int_0^1 y^n f(y)dy = 2(-1)^k f^{(k)}(1),$$

since

$$\lim_{n\to\infty}\int_0^1 y^{n+k+1} f^{(k+1)}(y)dy = 0,$$

$f^{(k+1)}$ being continuous and hence bounded. Using Leibniz's formula we get that

$$f^{(k)}(1) = (-1)^k\frac{k!}{2^{k+2}},$$

and the problem is solved.

Grading of Problem 4.

3p: For computing *a*)

7p: For computing *b*)

SEEMOUS 2013 PROBLEMS AND SOLUTIONS

Problem 1

Find all continuous functions $f : [1, 8] \to R$, such that

$$\int_1^2 f^2(t^3)dt + 2\int_1^2 f(t^3)dt = \frac{2}{3}\int_1^8 f(t)dt - \int_1^2 (t^2-1)^2 dt.$$

Solution. Using the substitution $t = u^3$ we get

$$\frac{2}{3}\int_1^8 f(t)dt = 2\int_1^2 u^2 f(u^3)du = 2\int_1^2 t^2 f(t^3)du.$$

Hence, by the assumptions,

$$\int_1^2 \left(f^2(t^3) + (t^2-1)^2 + 2f(t^3) - 2t^2 f(t^3)\right) dt = 0.$$

Since $f^2(t^3)+(t^2-1)^2+2f(t^3)-2t^2f(t^3) = (f(t^3))^2+(1-t^2)^2+2(1-t^2)f(t^3) = \left(f(t^3)+1-t^2\right)^2 \geq 0$, we get

$$\int_1^2 \left(f(t^3)+1-t^2\right)^2 dt = 0.$$

The continuity of f implies that $f(t^3) = t^2 - 1$, $1 \leq t \leq 2$, thus, $f(x) = x^{2/3} - 1$, $1 \leq x \leq 8$.

Remark. If the continuity assumption for f is replaced by Riemann integrability then infinitely many f's would satisfy the given equality. For example if C is any closed nowhere dense and of measure zero subset of $[1, 8]$ (for example a finite set or an appropriate Cantor type set) then any function f such that $f(x) = x^{2/3} - 1$ for every $x \in [1, 8]\backslash C$ satisfies the conditions.

Problem 2

Let $M, N \in M_2(\mathbb{C})$ be two nonzero matrices such that

$$M^2 = N^2 = 0_2 \text{ and } MN + NM = I_2$$

where 0_2 is the 2×2 zero matrix and I_2 the 2×2 unit matrix. Prove that there is an invertible matrix $A \in M_2(\mathbb{C})$ such that

$$M = A\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}A^{-1} \text{ and } N = A\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}A^{-1}.$$

First solution. Consider $f, g : \mathbb{C}^2 \to \mathbb{C}^2$ given by $f(x) = Mx$ and $g(x) = Nx$.
We have $f^2 = g^2 = 0$ and $fg + gf = \mathrm{id}_{\mathbb{C}^2}$; composing the last relation (to the left, for instance) with fg we find that $(fg)^2 = fg$, so fg is a projection of $\mathbb{C}^2$.
If fg were zero, then $gf = \mathrm{id}_{\mathbb{C}^2}$, so f and g would be invertible, thus contradicting $f^2 = 0$.
Therefore, fg is nonzero. Let $u \in \mathrm{Im}(fg) \setminus \{0\}$ and $w \in \mathbb{C}^2$ such that $u = fg(w)$. We obtain $fg(u) = (fg)^2(w) = fg(w) = u$. Let $v = g(u)$. The vector v is nonzero, because otherwise we obtain $u = f(v) = 0$.
Moreover, u and v are not collinear since $v = \lambda u$ with $\lambda \in \mathbb{C}$ implies $u = f(v) = f(\lambda u) = \lambda f(u) = \lambda f^2(g(w)) = 0$, a contradiction.
Let us now consider the ordered basis $\mathcal{B}$ of $\mathbb{C}^2$ consisting of u and v.
We have $f(u) = f^2(g(u)) = 0$, $f(v) = f(g(u)) = u$, $g(u) = v$ and $g(v) = g^2(u) = 0$.
Therefore, the matrices of f and g with respect to $\mathcal{B}$ are $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, respectively.
We take A to be the change of base matrix from the standard basis of $\mathbb{C}^2$ to $\mathcal{B}$ and we are done. $\square$

Second solution. Let us denote $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ by E_{12} and $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ by E_{21}. Since $M^2 = N^2 = 0_2$ and $M, N \neq 0_2$, the minimal polynomials of both M and N are equal to x^2. Therefore, there are invertible matrices $B, C \in \mathcal{M}_2(\mathbb{C})$ such that $M = BE_{12}B^{-1}$ and $N = CE_{21}C^{-1}$.
Note that B and C are not uniquely determined. If $B_1E_{12}B_1^{-1} = B_2E_{12}B_2^{-1}$, then $(B_1^{-1}B_2)E_{12} = E_{12}(B_1^{-1}B_2)$; putting $B_1^{-1}B_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the last relation is equivalent to $\begin{pmatrix} 0 & a \\ 0 & c \end{pmatrix} = \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix}$. Consequently, $B_1E_{12}B_1^{-1} = B_2E_{12}B_2^{-1}$ if and only if there exist $a \in \mathbb{C} - \{0\}$ and $b \in \mathbb{C}$ such that

$$B_2 = B_1 \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}. \quad (*)$$

Similarly, $C_1E_{21}C_1^{-1} = C_2E_{21}C_2^{-1}$ if and only if there exist $\alpha \in \mathbb{C} - \{0\}$ and $\beta \in \mathbb{C}$ such that

$$C_2 = C_1 \begin{pmatrix} \alpha & 0 \\ \beta & \alpha \end{pmatrix}. \quad (**)$$

Now, $MN + NM = I_2$, $M = BE_{12}B^{-1}$ and $N = CE_{21}C^{-1}$ give

$$BE_{12}B^{-1}CE_{21}C^{-1} + CE_{21}C^{-1}BE_{12}B^{-1} = I_2,$$

or

$$E_{12}B^{-1}CE_{21}C^{-1}B + B^{-1}CE_{21}C^{-1}BE_{12} = I_2.$$

If $B^{-1}C = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$, the previous relation means

$$\begin{pmatrix} z & t \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ t & -y \end{pmatrix} + \begin{pmatrix} y & 0 \\ t & 0 \end{pmatrix}\begin{pmatrix} 0 & t \\ 0 & -z \end{pmatrix} = (xt - yz)I_2 \neq 0_2.$$

After computations we find this to be equivalent to $xt - yz = t^2 \neq 0$. Consequently, there are $y, z \in \mathbb{C}$ and $t \in \mathbb{C} - \{0\}$ such that

$$C = B \begin{pmatrix} t + \frac{yz}{t} & y \\ z & t \end{pmatrix}. \quad (***)$$

According to $(*)$ and $(**)$, our problem is equivalent to finding $a, \alpha \in \mathbb{C} - \{0\}$ and $b, \beta \in \mathbb{C}$ such that $C\begin{pmatrix} \alpha & 0 \\ \beta & \alpha \end{pmatrix} = B\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$. Taking relation $(***)$ into account, we need to find $a, \alpha \in \mathbb{C} - \{0\}$ and $b, \beta \in \mathbb{C}$ such that

$$B\begin{pmatrix} t + \frac{yz}{t} & y \\ z & t \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ \beta & \alpha \end{pmatrix} = B\begin{pmatrix} a & b \\ 0 & a \end{pmatrix},$$

or, B being invertible,

$$\begin{pmatrix} t + \frac{yz}{t} & y \\ z & t \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ \beta & \alpha \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}.$$

This means $\begin{cases} \alpha t + \alpha\dfrac{yz}{t} + \beta y = a \\ \alpha y = b \\ \alpha z + \beta t = 0 \\ \alpha t = a \end{cases}$,

and these conditions are equivalent to $\begin{cases} \alpha y = b \\ \alpha z = -\beta t \\ \alpha t = a \end{cases}$.

It is now enough to choose $\alpha = 1$, $a = t$, $b = y$ and $\beta = -\dfrac{z}{t}$. $\square$

Third Solution. Let f, g be as in the first solution. Since $f^2 = 0$ there exists a nonzero $v_1 \in \mathrm{Ker} f$ so $f(v_1) = 0$ and setting $v_2 = g(v_1)$ we get

$$f(v_2) = (fg + gf)(v_1) = v_1 \neq 0$$

by the assumptions (and so $v_2 \neq 0$). Also

$$g(v_2) = g^2(v_1) = 0$$

and so to complete the proof it suffices to show that v_1 and v_2 are linearly independent, because then the matrices of f, g with respect to the ordered basis (v_1, v_2) would be E_{12} and E_{21} respectively, according to the above relations. But if $v_2 = \lambda v_1$ then $0 = g(v_2) = \lambda g(v_1) = \lambda v_2$ so since $v_2 \neq 0$, λ must be 0 which gives $v_2 = 0 v_1 = 0$ contradiction. This completes the proof. $\square$

Remark. A nonelementary solution of this problem can be given by observing that the conditions on M, N imply that the correspondence $I_2 \to I_2, M \to E_{12}$ and $N \to E_{21}$ extends to an isomorphism between the subalgebras of $\mathcal{M}_2(\mathbb{C})$ generated by I_2, M, N and I_2, E_{12}, E_{21} respectively, and then one can apply Noether-Skolem Theorem to show that this isomorphism is actually conjugation by an $A \in Gl_2(\mathbb{C})$ etc.

Problem 3

Find the maximum value of

$$\int_0^1 |f'(x)|^2\,|f(x)|\,\frac{1}{\sqrt{x}}\,dx$$

over all continuously differentiable functions $f : [0,1] \to \mathbb{R}$ *with* $f(0) = 0$ *and*

$$\int_0^1 |f'(x)|^2\,dx \leq 1. \qquad (*)$$

Solution. For $x \in [0,1]$ let

$$g(x) = \int_0^x |f'(t)|^2\,dt.$$

Then for $x \in [0,1]$ the Cauchy-Schwarz inequality gives

$$|f(x)| \leq \int_0^x |f'(t)|\,dt \leq \left(\int_0^x |f'(t)|^2\,dt\right)^{1/2} \sqrt{x} = \sqrt{x} g(x)^{1/2}.$$

Thus

$$\begin{aligned}\int_0^1 |f'(x)|^2 |f(x)| \frac{1}{\sqrt{x}}\,dx &\leq \int_0^1 g(x)^{1/2} g'(x) dx = \frac{2}{3}[g(1)^{3/2} - g(0)^{3/2}] \\ &= \frac{2}{3}\left(\int_0^1 |f'(t)|^2\,dt\right)^{3/2} \leq \frac{2}{3}.\end{aligned}$$

by $(*)$. The maximum is achieved by the function $f(x) = x$. $\square$

Remark. If the condition $(*)$ is replaced by $\int_0^1 |f'(x)|^p dx \leq 1$ with $0 < p < 2$ fixed, then the given expression would have supremum equal to $+\infty$, as it can be seen by considering continuously differentiable functions that approximate the functions $f_M(x) = Mx$ for $0 \leq x \leq \dfrac{1}{M^p}$ and $\dfrac{1}{M^{p-1}}$ for $\dfrac{1}{M^p} < x \leq 1$, where M can be an arbitrary large positive real number.

Problem 4

Let $A \in M_2(Q)$ *such that there is* $n \in N$, $n \neq 0$, *with* $A^n = -I_2$. *Prove that either* $A^2 = -I_2$ *or* $A^3 = -I_2$.

First Solution. Let $f_A(x) = det(A - xI_2) = x^2 - sx + p \in \mathbb{Q}[x]$ be the characteristic polynomial of A and let λ_1, λ_2 be its roots, also known as the eigenvalues of matrix A. We have that $\lambda_1 + \lambda_2 = s \in \mathbb{Q}$ and $\lambda_1\lambda_2 = p \in \mathbb{Q}$. We know, based on Cayley-Hamilton theorem, that the matrix A satisfies the relation $A^2 - sA + pI_2 = 0_2$. For any eigenvalue $\lambda \in \mathbb{C}$ there is an eigenvector $X \neq 0$, such that $AX = \lambda X$. By induction we have that $A^n X = \lambda^n X$ and it follows that $\lambda^n = -1$. Thus, the eigenvalues of A satisfy the equalities

$$\lambda_1^n = \lambda_2^n = -1 \qquad (*).$$

Is $\lambda_1 \in \mathbb{R}$ then we also have that $\lambda_2 \in \mathbb{R}$ and from $(*)$ we get that $\lambda_1 = \lambda_2 = -1$ (and note that n must be odd) so A satisfies the equation $(A + I_2)^2 = A^2 + 2A + I_2 = 0_2$ and it follows that $-I_2 = A^n = (A + I_2 - I_2)^n = n(A + I_2) - I_2$ which gives $A = -I_2$ and hence $A^3 = -I_2$.

If $\lambda_1 \in \mathbb{C} \setminus \mathbb{R}$ then $\lambda_2 = \overline{\lambda_1} \in \mathbb{C} \setminus \mathbb{R}$ and since $\lambda_1^n = -1$ we get that $|\lambda_{1,2}| = 1$ and this implies that $\lambda_{1,2} = \cos t \pm i \sin t$. Now we have the equalities $\lambda_1 + \lambda_2 = 2\cos t = s \in \mathbb{Q}$ and $\lambda_1^n = -1$ implies that $\cos nt + i \sin nt = -1$ which in turn implies that $\cos nt = -1$. Using the equality $\cos(n+1)t + \cos(n-1)t = 2\cos t \cos nt$ we get that there is a polynomial $P_n = x^n + \cdots$ of degree n with integer coefficients such that $2\cos nt = P_n(2\cos t)$. Set $x = 2\cos t$ and observe that we have $P_n(x) = -2$ so $x = 2\cos t$ is a rational root of an equation of the form $x^n + \cdots = 0$. However, the rational roots of this equation are integers, so $x \in \mathbb{Z}$ and since $|x| \leq 2$ we get that $2\cos t = -2, -1, 0, 1, 2$.

When $2\cos t = \pm 2$ then $\lambda_{1,2}$ are real numbers (note that in this case $\lambda_1 = \lambda_2 = 1$ or $\lambda_1 = \lambda_2 = -1$) and this case was discussed above.

When $2\cos t = 0$ we get that $A^2 + I_2 = 0_2$ so $A^2 = -I_2$.

When $2\cos t = 1$ we get that $A^2 - A + I_2 = 0_2$ which implies that $(A + I_2)(A^2 - A + I_2) = 0_2$ so $A^3 = -I_2$.

When $2\cos t = -1$ we get that $A^2 + A + I_2 = 0_2$ and this implies that $(A - I_2)(A^2 + A + I_2) = 0_2$ so $A^3 = I_2$. It follows that $A^n \in \{I_2, A, A^2\}$. However, $A^n = -I_2$ and this implies that either $A = -I_2$ or $A^2 = -I_2$ both of which contradict the equality $A^3 = I_2$. This completes the proof. $\square$

Remark. The polynomials P_n used in the above proof are related to the Chebyshev polynomials, $T_n(x) = \cos(n \arccos x)$. One could also get the conclusion that $2\cos t$ is an integer by considering the sequence $x_m = 2\cos(2^m t)$ and noticing that since $x_{m+1} = x_m^2 - 2$, if x_0 were a noninteger rational $\frac{a}{b}$ $(b > 1)$ in lowest terms then the denominator of x_m *in lowest terms* would be b^{2^m} and this contradicts the fact that x_m must be periodic since t is a rational multiple of π.

Second Solution. Let $m_A(x)$ be the minimal polynomial of A. Since $A^{2n} - I_2 = (A^n + I_2)(A^n - I_2) = 0_2$, $m_A(x)$ must be a divisor of $x^{2n} - 1$ which has no multiple roots. It is well known that the monic irreducible over $\mathbb{Q}$ factors of $x^{2n} - 1$ are exactly the cyclotomic polynomials $\Phi_d(x)$ where d divides $2n$. Hence the irreducible over $\mathbb{Q}$ factors of $m_A(x)$ must be cyclotomic polynomials and since the degree of $m_A(x)$ is at most 2 we conclude that $m_A(x)$ itself must be a cyclotomic polynomial, say $\Phi_d(x)$ for some positive integer d with $\phi(d) = 1$ or 2 (where ϕ is the Euler totient function), $\phi(d)$ being the degree of $\Phi_d(x)$. But this implies that $d \in \{1, 2, 3, 4, 6\}$ and since A, A^3 cannot be equal to I_2 we get that $m_A(x) \in \{x + 1, x^2 + 1, x^2 - x + 1\}$ and this implies that either $A^2 = -I_2$ or $A^3 = -I_2$. $\square$

South Eastern European Mathematical
Olympiad for University Students
Iaşi, România - March 7, 2014

Problem 1. Let n be a nonzero natural number and $f : \mathbb{R} \to \mathbb{R} \setminus \{0\}$ be a function such that $f(2014) = 1 - f(2013)$. Let $x_1, x_2, x_3, \ldots, x_n$ be real numbers not equal to each other. If

$$\begin{vmatrix} 1+f(x_1) & f(x_2) & f(x_3) & \ldots & f(x_n) \\ f(x_1) & 1+f(x_2) & f(x_3) & \ldots & f(x_n) \\ f(x_1) & f(x_2) & 1+f(x_3) & \ldots & f(x_n) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f(x_1) & f(x_2) & f(x_3) & \ldots & 1+f(x_n) \end{vmatrix} = 0,$$

prove that f is not continuous.

Problem 2. Consider the sequence (x_n) given by

$$x_1 = 2, \qquad x_{n+1} = \frac{x_n + 1 + \sqrt{x_n^2 + 2x_n + 5}}{2}, \quad n \geq 2.$$

Prove that the sequence $y_n = \sum_{k=1}^{n} \frac{1}{x_k^2 - 1}, \quad n \geq 1$ is convergent and find its limit.

Problem 3. Let $A \in \mathcal{M}_n(\mathbb{C})$ and $a \in \mathbb{C},\ a \neq 0$ such that $A - A^* = 2aI_n$, where $A^* = (\bar{A})^t$ and $\bar{A}$ is the conjugate of the matrix A.

(a) Show that $|\det A| \geq |a|^n$

(b) Show that if $|\det A| = |a|^n$ then $A = aI_n$.

Problem 4. a) Prove that $\lim\limits_{n\to\infty} n \int\limits_0^n \frac{\operatorname{arctg} \frac{x}{n}}{x\,(x^2+1)}\,dx = \frac{\pi}{2}$.

b) Find the limit $\lim\limits_{n\to\infty} n \left(n \int\limits_0^n \frac{\operatorname{arctg} \frac{x}{n}}{x\,(x^2+1)}\,dx - \frac{\pi}{2} \right)$.

SEEMOUS 2015 Contest Problems and Solutions

Problem 1. Prove that for every $x \in (0,1)$ the following inequality holds:

$$\int_0^1 \sqrt{1+(\cos y)^2}\,dy > \sqrt{x^2+(\sin x)^2} \;.$$

Solution 1. Clearly

$$\int_0^1 \sqrt{1+(\cos y)^2}\,dy \ge \int_0^x \sqrt{1+(\cos y)^2}\,dy \,.$$

Define a function $F:[0,1]\to\mathbb{R}$ by setting:

$$F(x)=\int_0^x \sqrt{1+(\cos y)^2}\,dy - \sqrt{x^2+(\sin x)^2}\;.$$

Since $F(0)=0$, it suffices to prove $F'(x)\ge 0$. By the fundamental theorem of Calculus, we have

$$F'(x)=\sqrt{1+(\cos x)^2}-\frac{x+\sin x\cos x}{\sqrt{x^2+(\sin x)^2}}\;.$$

Thus, it is enough to prove that

$$(1+(\cos x)^2)(x^2+(\sin x)^2)\ge (x+\sin x\cos x)^2.$$

But this is a straightforward application of the Cauchy-Schwarz inequality.

Solution 2. Clearly $\int_0^1 \sqrt{1+(\cos y)^2}\,dy \ge \int_0^x \sqrt{1+(\cos y)^2}\,dy$ for each fixed $x\in(0,1)$. Observe that $\int_0^x \sqrt{1+(\cos y)^2}\,dy$ is the arc length of the function $f(y)=\sin y$ on the interval $[0,x]$ which is clearly strictly greater than the length of the straight line between the points $(0,0)$ and $(x,\sin x)$ which in turn is equal to $\sqrt{x^2+(\sin x)^2}$.

Problem 2. For any positive integer n, let the functions $f_n:\mathbb{R}\to\mathbb{R}$ be defined by $f_{n+1}(x)=f_1(f_n(x))$, where $f_1(x)=3x-4x^3$. Solve the equation $f_n(x)=0$.

Solution. First, we prove that $|x|>1 \Rightarrow |f_n(x)|>1$ holds for every positive integer *n*. It suffices to demonstrate the validity of this implication for $n=1$. But, by assuming $|x|>1$, it readily follows that $|f_1(x)|=|x||3-4x^2|\ge|3-4x^2|>1$, which completes the demonstration. We conclude that every solution of the equation $f_n(x)=0$ lies in the closed interval $[-1,1]$. For an arbitrary such x, set $x=\sin t$ where $t=\arcsin x\in[-\frac{\pi}{2},\frac{\pi}{2}]$. We clearly have $f_1(\sin t)=\sin 3t$, which gives

$$f_n(x)=\sin 3^n t=\sin(3^n \arcsin x).$$

Thus, $f_n(x)=0$ if and only if $\sin(3^n \arcsin x)=0$, i.e. only when $3^n \arcsin x=k\pi$ for some $k\in\mathbf{Z}$. Therefore, the solutions of the equation $f_n(x)=0$ are given by

$$x=\sin\frac{k\pi}{3^n},$$

where k acquires every integer value from $\frac{1-3^n}{2}$ up to $\frac{3^n-1}{2}$.

Problem 3. For an integer $n > 2$, let $A, B, C, D \in M_n(\mathbb{R})$ be matrices satisfying:

$$AC - BD = I_n,$$
$$AD + BC = O_n,$$

where I_n is the identity matrix and O_n is the zero matrix in $M_n(\mathbb{R})$.

Prove that:

a) $CA - DB = I_n$ and $DA + CB = O_n$,

b) $\det(AC) \geq 0$ and $(-1)^n \det(BD) \geq 0$.

Solution. a) We have

$$AC - BD + i(AD + BC) = I_n \Leftrightarrow (A + iB)(C + iD) = I_n,$$

which implies that the matrices $A + iB$ and $C + iD$ are inverses to one another. Thus,

$$(C + iD)(A + iB) = I_n \Leftrightarrow CA - DB + i(DA + CB) = I_n$$
$$\Leftrightarrow CA - DB = I_n, DA + CB = O_n.$$

b) We have

$$\begin{aligned}\det((A + iB)C) &= \det(AC + iBC)\\ &\overset{AD+BC=O_n}{=} \det(AC - iAD)\\ &= \det(A(C - iD).\end{aligned}$$

On the other hand,

$$\begin{aligned}\det C &\overset{(C+iD)(A+iB)=I_n}{=} \det((C + iD)(A + iB)C) = \det((C + iD)A(C - iD))\\ &= \det(A)\,|\det(C + iD)|^2.\end{aligned}$$

Thus,

$$\det(AC) = (\det A)^2 |\det(C + iD)|^2 \geq 0.$$

Similarly

$$\begin{aligned}\det((A + iB)D) &= \det(AD + iBD)\\ &\overset{AD+BC=O_n}{=} \det(-BC + iBD)\\ &= (-1)^n \det(B(C - iD)).\end{aligned}$$

This implies that

$$\begin{aligned}\det D &\overset{(C+iD)(A+iB)=I_n}{=} \det((C + iD)(A + iB)D) = (-1)^n \det((C + iD)B(C - iD))\\ &= (-1)^n \det(B)\,|\det(C + iD)|^2.\end{aligned}$$

Thus, $(-1)^n \det(BD) = (\det B)^2\,|\det(\mathrm{C} + iD)|^2 \geq 0$.

Problem 4. Let $I \subset \mathbb{R}$ be an open interval which contains 0, and $f : I \to \mathbb{R}$ be a function of class $C^{2016}(I)$ such that $f(0)=0,\ f'(0)=1,\ f''(0)=f'''(0)=\ldots=f^{(2015)}(0)=0,\ f^{(2016)}(0)<0.$

i) Prove that there is $\delta>0$ such that

$$0<f(x)<x,\quad \forall x\in(0,\delta). \tag{1.1}$$

ii) With δ determined as in *i*), define the sequence (a_n) by

$$a_1=\tfrac{\delta}{2},\ a_{n+1}=f(a_n),\ \forall n\geq 1. \tag{1.2}$$

Study the convergence of the series $\sum\limits_{n=1}^{\infty} a_n^r$, for $r\in\mathbb{R}$.

Solution. ***i)*** We claim that there exists $\alpha>0$ such that $f(x)>0$ for any $x\in(0,\alpha)$. For this, observe that, since f is of class C^1 and $f'(0)=1>0$, there exists $\alpha>0$ such that $f'(x)>0$ on $(0,\alpha)$. Since $f(0)=0$ and f is strictly increasing on $(0,\alpha)$, the claim follows.

Next, we prove that there exists $\beta>0$ such that $f(x)<x$ for any $x\in(0,\beta)$. Since $f^{(2016)}(0)<0$ and f is of class C^{2016}, there is $\beta>0$ such that $f^{(2016)}(t)<0$, for any $t\in(0,\beta)$. By the Taylor's formula, for any $x\in(0,\beta)$, there is $\theta\in[0,1]$ such that

$$f(x)=f(0)+\frac{f'(0)}{1!}x+\ldots+\frac{f^{(2015)}(0)}{2015!}x^{2015}+\frac{f^{(2016)}(\theta x)}{2016!}x^{2016}, \tag{1.3}$$

hence

$$g(x)=\frac{f^{(2016)}(\theta x)}{2016!}x^{2016}<0,\quad \forall x\in(0,\beta).$$

Taking $\delta=\min\{\alpha,\beta\}>0$, the item *i)* is completely proven.

ii) We will prove first that the sequence (a_n) given by (1.2) converges to 0. Indeed, by relation (1.1) it follows that

$$0<a_{n+1}<a_n,\ \forall n\geq 1,$$

hence the sequence (a_n) is strictly decreasing and lower bounded by 0. It follows that (a_n) converges to some $\ell\in[0,\frac{\delta}{2})$. Passing to the limit in (1.2), one gets $\ell=f(\ell)$. Taking into account (1.1), we deduce that $\ell=0$.

In what follows, we calculate

$$\lim_{n\to\infty} na_n^{2015}.$$

From $a_n\downarrow 0$, using the Stolz-Cesàro Theorem, we conclude that

$$\lim_{n\to\infty} na_n^{2015}=\lim_{n\to\infty}\frac{n}{\frac{1}{a_n^{2015}}}=\lim_{n\to\infty}\frac{(n+1)-n}{\frac{1}{a_{n+1}^{2015}}-\frac{1}{a_n^{2015}}}=\lim_{n\to\infty}\frac{1}{\frac{1}{f(a_n)^{2015}}-\frac{1}{a_n^{2015}}}$$

$$=\lim_{x\to 0}\frac{1}{\frac{1}{f(x)^{2015}}-\frac{1}{x^{2015}}}=\lim_{x\to 0}\frac{(xf(x))^{2015}}{x^{2015}-f(x)^{2015}}.$$

Observe that, by (1.3) $\dfrac{(xf(x))^{2015}}{x^{2015}-f(x)^{2015}} = \dfrac{\left(x^2 + \dfrac{f^{(2016)}(\theta x)}{2016!}x^{2017}\right)^{2015}}{-\dfrac{f^{(2016)}(\theta x)}{2016!}x^{2016}(x^{2014}+x^{2013}f(x)+\ldots+f(x)^{2014})}$.

Since f is of class C^{2016}, $\lim\limits_{x\to 0} f^{(2016)}(\theta x) = f^{(2016)}(0)$ and

$$\lim_{x\to 0}\frac{(xf(x))^{2015}}{x^{2015}-f(x)^{2015}} = -\frac{2016!}{2015 f^{(2016)}(0)} > 0.$$

It means, by the comparison criterion, that the series $\sum\limits_{n=1}^{\infty} a_n^r$ and $\sum\limits_{n=1}^{\infty}\frac{1}{n^{\frac{r}{2015}}}$ converge and/or diverge simultaneously, hence the series $\sum\limits_{n=1}^{\infty} a_n^r$ converges for $r > 2015,$ and diverges for $r \le 2015.$

	SEEMOUS 2016 **South Eastern European** **Mathematical Olympiad for University Students** **Protaras, Cyprus** **1-6 March 2016**
Mathematical Society of South Eastern Europe **Cyprus Mathematical Society**	

LANGUAGE: ENGLISH

COMPETITION PROBLEMS

Do all problems 1-4. Each problem is worth 10 points. All answers are written in the booklet provided, following the rules written in the Olympiad programme.

Problem1.

Let f be a continuous and decreasing real valued function, defined on $\left[0, \frac{\pi}{2}\right]$.

Prove the inequalities

$$\int_{\frac{\pi}{2}-1}^{\frac{\pi}{2}} f(x)dx \leq \int_0^{\frac{\pi}{2}} f(x)\cos x\, dx \leq \int_0^1 f(x)dx$$

when do equalities hold?

Problem 2.

a) Prove that for every matrix $X \in M_2(\mathbb{C})$ there exists a matrix $Y \in M_2(\mathbb{C})$ such that $Y^3 = X^2$.
b) Prove that there exists a matrix $A \in M_3(\mathbb{C})$ such that $Z^3 \neq A^2$ for all $Z \in M_3(\mathbb{C})$.

Problem3.

Let $A_1, A_2, \dots, A_k$ be idempotent matrices ($A_i^2 = A_i$) in $M_n(\mathbb{R})$. Prove that

$$\sum_{i=1}^{k} N(A_i) \geq \operatorname{rank}\left(I - \prod_{i=1}^{k} A_i\right)$$

where $N(A_i) = n - \operatorname{rank}(A_i)$ and $M_n(\mathbb{R})$ is the set of square $n \times n$ matrices with real entries.

Problem4.

Let $n \geq 1$ be an integer and let

$$I_n = \int_0^\infty \frac{\arctan x}{(1+x^2)^n}\, dx$$

Prove that

a) $\displaystyle\sum_{n=1}^{\infty} \frac{I_n}{n} = \frac{\pi^2}{6}$

b) $\displaystyle\int_0^\infty \arctan x \cdot \ln\left(1 + \frac{1}{x^2}\right) dx = \frac{\pi^2}{6}$

Seemous 2017, February 28 – March 5, 2017, Ohrid, Republic of Macedonia

Problems (time for work: 5 hours).

$\boxed{\mathbf{1.}}$ Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}_2(\mathbb{R})$ such that $a^2 + b^2 + c^2 + d^2 < \frac{1}{5}$.
Show that $I + A$ is invertible.

$\boxed{\mathbf{2.}}$ Let $A, B \in \mathcal{M}_n(\mathbb{R})$.
a) Show that there exists $a > 0$ such that for every $\varepsilon \in (-a, a)$, $\varepsilon \neq 0$, the matrix equation

$$AX + \varepsilon X = B, \quad X \in \mathcal{M}_n(\mathbb{R})$$

has a unique solution $X(\varepsilon) \in \mathcal{M}_n(\mathbb{R})$.
b) Prove that if $B^2 = I_n$, and A is diagonalizable then

$$\lim_{\varepsilon \to 0} \operatorname{Tr}(BX(\varepsilon)) = n - \operatorname{rank}(A).$$

$\boxed{\mathbf{3.}}$ Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Prove that

$$\int_0^4 f(x(x-3)^2)\,\mathrm{d}x = 2\int_1^3 f(x(x-3)^2)\,\mathrm{d}x$$

$\boxed{\mathbf{4.}}$ **a)** Let $n \geq 0$ be an integer. Calculate $\int_0^1 (1-t)^n e^t\,\mathrm{d}t$.
b) Let $k \geq 0$ be a fixed integer and let $(x_n)_{n \geq k}$ be a sequence defined by

$$x_n = \sum_{i=k}^{n} \binom{i}{k} \left(e - 1 - \frac{1}{1!} - \frac{1}{2!} - \cdots - \frac{1}{i!} \right).$$

Prove that the sequence converges and find its limit.

Solutions

$\boxed{\mathbf{1.}}$ Obvious. $\|A\| < 1 \implies (I \pm A)$ are invertible. Or, prove directly: $\det(A + I) > 0$.
Note: $(I - A)^{-1} = \sum_{k=0}^{\infty} A^k$.

$\boxed{\mathbf{2.}}$ **a)** The equation is $(A + \varepsilon I)X = B$. Take $a = \min\{|\lambda| : \lambda \in \sigma_p(A)\setminus\{0\}\}$. Then $\exists (A + \varepsilon I)^{-1}$ for $0 < |\varepsilon| < a$.
b) $X(\varepsilon) = (A + \varepsilon I)^{-1}B \implies BX(\varepsilon) = B(A + \varepsilon I)^{-1}B = B^{-1}(A + \varepsilon I)^{-1}B \implies$
$\operatorname{Tr}(BX(\varepsilon)) = \operatorname{Tr}\left((A + \varepsilon I)^{-1}\right) = \sum_{k=1}^{n} \frac{1}{\lambda_k + \varepsilon}$, where λ_k are the eigenvalues (with multiplicities).
$\lim_{\varepsilon \to 0} \varepsilon \operatorname{Tr}(BX(\varepsilon)) = \lim_{\varepsilon \to 0} \sum_{k=1}^{n} \frac{\varepsilon}{\lambda_k + \varepsilon} = \operatorname{card}\{k \in \{1, \ldots, n\} : \lambda_k = 0\}$.
If A is diagonalizable this is exactly $n - \operatorname{rank}(A)$.
[It is enough for A to satisfy the condition that the algebraic and geometric multiplicity for 0 be the same]

$\boxed{\mathbf{3.}}$ Denote $p : [0,4] \to [0,4]$, $p(x) = x(x-3)^2$ and the intervals $J_1 = [0,1]$, $J_2 = [1,3]$ and $J_3 = [3,4]$.
Let p_k be the restriction of p to J_k. p_k is bijective, and denote by $q_k : [0,4] \to J_k$ its inverse.
Notice that $p_1' > 0, p_2' < 0, p_3' > 0$ in $\mathring{J}_k$, so $q_1' > 0, q_2' < 0, q_3' > 0$ in the open interval $(0,4)$.
Changing the variables $p_k(x) = y$ we obtain:

$$\int_{J_k} f(p(x))\,\mathrm{d}x = \int_0^4 f(y)\,\left|q_k'(y)\right|\,\mathrm{d}y, \; k = 1 \ldots 3. \tag{*}$$

Now, $q_k(y)$ are the roots of the equation $p(x) = y$, i.e. $x^3 - 6x^2 + 9x - y = 0$.
It follows that $q_1 + q_2 + q_3 = 6$, hence $q_1' + q_2' + q_3' = 0$ in $(0, 4)$
Since only q_2' is negative: $|q_k'| - |q_k'| + |q_k'| = 0$ in $(0, 4)$
Using (*) it results:
$\int_{J_1} f(p(x))\,\mathrm{d}x - \int_{J_2} f(p(x))\,\mathrm{d}x + \int_{J_3} f(p(x))\,\mathrm{d}x = 0$
which is exactly the desired conclusion.

Note. The integrals in the right hand side of (*) are improper. If we want to avoid this, then we may apply the change of variables only for smaller intervals e.g. $J_k(\varepsilon) = p_k^{-1}([\varepsilon, 4 - \varepsilon])$ and take $\varepsilon \to 0_+$ at the end.

$\boxed{\mathbf{4.}}$ **a)** Denote by I_k the integral.
$I_0 = e - 1$. An integration by parts gives $I_n = -1 + nI_{n-1}$, $n \geq 1$.
$I_n/n! = -1/n! + I_{n-1}/(n-1)!$ After a telescoping summation we obtain
$I_n/n! - I_0 = -\sum_{k=1}^{n} 1/k!$

$$I_n = n!\left(e - \sum_{k=0}^{n} \frac{1}{k!}\right)$$

b) $x_n = \sum_{i=k}^{n} \binom{i}{k} I_i/i! = \sum_{i=k}^{n} \frac{I_i}{k!(i-k)!}$. The sequence is increasing, so its limit exists and
$\lim_{n\to\infty} x_n = \sum_{i=k}^{\infty} \frac{I_i}{k!(i-k)!} =$
$\frac{1}{k!}\sum_{i=0}^{\infty} \frac{I_{k+i}}{i!} =$
$\frac{1}{k!}\sum_{i=0}^{\infty} \frac{I_{k+i}}{i!} =$
$\frac{1}{k!}\sum_{i=0}^{\infty} \frac{1}{i!}\int_0^1 (1-t)^{k+i} e^t\,\mathrm{d}t =$ [by the theorem of monotone convergence]
$\frac{1}{k!}\int_0^1 \sum_{i=0}^{\infty} \frac{(1-t)^{k+i}}{i!} e^t\,\mathrm{d}t =$
$\frac{1}{k!}\int_0^1 (1-t)^k \sum_{i=0}^{\infty} \frac{(1-t)^i}{i!} e^t\,\mathrm{d}t =$
$\frac{1}{k!}\int_0^1 (1-t)^k e^{1-t} e^t\,\mathrm{d}t = \frac{e}{(k+1)!}$.

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SEEMOUS 2018

FEBRUARY 27-MARCH 4, IAŞI, ROMÂNIA

South Eastern European Mathematical Olympiad for University Students
Iaşi, Romania
March 1, 2018

COMPETITION PROBLEMS

Problem 1. Let $f : [0,1] \to (0,1)$ be a Riemann integrable function. Show that

$$\frac{2\displaystyle\int_0^1 x f^2(x)\,\mathrm{d}x}{\displaystyle\int_0^1 (f^2(x)+1)\,\mathrm{d}x} < \frac{\displaystyle\int_0^1 f^2(x)\,\mathrm{d}x}{\displaystyle\int_0^1 f(x)\,\mathrm{d}x}.$$

Problem 2. Let $m, n, p, q \geq 1$ and let the matrices $A \in \mathcal{M}_{m,n}(\mathbb{R})$, $B \in \mathcal{M}_{n,p}(\mathbb{R})$, $C \in \mathcal{M}_{p,q}(\mathbb{R})$, $D \in \mathcal{M}_{q,m}(\mathbb{R})$ be such that

$$A^t = BCD, \quad B^t = CDA, \quad C^t = DAB, \quad D^t = ABC.$$

Prove that $(ABCD)^2 = ABCD$.

Problem 3. Let A, $B \in \mathcal{M}_{2018}(\mathbb{R})$ such that $AB = BA$ and $A^{2018} = B^{2018} = I$, where I is the identity matrix. Prove that if $\operatorname{Tr}(AB) = 2018$, then $\operatorname{Tr} A = \operatorname{Tr} B$.

Problem 4. (a) Let $f : \mathbb{R} \to \mathbb{R}$ be a polynomial function. Prove that

$$\int_0^\infty \mathrm{e}^{-x} f(x)\,\mathrm{d}x = f(0) + f'(0) + f''(0) + \cdots .$$

(b) Let f be a function which has a Taylor series expansion at 0 with radius of convergence $R = \infty$. Prove that if $\displaystyle\sum_{n=0}^{\infty} f^{(n)}(0)$ converges absolutely then $\displaystyle\int_0^\infty \mathrm{e}^{-x} f(x)\,\mathrm{d}x$ converges and

$$\sum_{n=0}^{\infty} f^{(n)}(0) = \int_0^\infty \mathrm{e}^{-x} f(x)\,\mathrm{d}x.$$

SEEMOUS
BULGARIA • DEVIN • MARCH 12-17 • 2019

Problem (1). *We shall call the numerical sequence* $\{x_n\}$ *a "Devin" sequence if* $0 \le x_n \le 1$ *and for each function* $f \in \mathrm{C}[0,1]$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f(x_i) = \int_0^1 f(x)\mathrm{d}x.$$

Prove that the numerical sequence $\{x_n\}$ *is a "Devin" sequence if and only if* $\forall k \ge 0$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} x_i^k = \frac{1}{k+1}\cdot$$

Problem (2). *Let* m *and* n *be positive integers. Prove that for any matrices* $A_1, A_2, \ldots, A_m \in \mathcal{M}_n(\mathbb{R})$ *there exist* $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m \in \{-1, 1\}$ *such that*

$$\mathrm{Tr}\left((\varepsilon_1 A_1 + \varepsilon_2 A_2 + \ldots + \varepsilon_m A_m)^2\right) \ge \mathrm{Tr}\left(A_1^2\right) + \mathrm{Tr}\left(A_2^2\right) + \ldots + \mathrm{Tr}\left(A_m^2\right).$$

Problem (3). *Let* $n \ge 2$ *and* $A, B \in \mathcal{M}_n(\mathbb{C})$ *such that* $B^2 = B$. *Prove that*

$$\mathrm{rank}(AB - BA) \le \mathrm{rank}(AB + BA).$$

Problem (4). *(a) Let* $n \ge 1$ *be an integer. Calculate* $\int_0^1 x^{n-1} \ln x \, dx$.

(b) Calculate

$$\sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{(n+1)^2} - \frac{1}{(n+2)^2} + \frac{1}{(n+3)^2} - \cdots \right).$$

SEEMOUS 2020 SOLUTIONS

Thessaloniki, Greece
March 3–8, 2020

Problem 1. Consider $A \in \mathcal{M}_{2020}(\mathbb{C})$ such that

$$(1) \qquad \begin{aligned} A + A^{\times} &= I_{2020}, \\ A \cdot A^{\times} &= I_{2020}, \end{aligned}$$

where $A^{\times}$ is the adjugate matrix of A, i.e., the matrix whose elements are $a_{ij} = (-1)^{i+j} d_{ji}$, where d_{ji} is the determinant obtained from A, eliminating the line j and the column i.

Find the maximum number of matrices verifying (1) such that any two of them are not similar.

Solution. It is known that

$$A \cdot A^{\times} = \det A \cdot I_{2020},$$

hence, from the second relation we get $\det A = 1$, so A is invertible. Next, multiplying in the first relation by A, we get

$$A^2 - A + I_{2020} = \mathcal{O}_{2020}.$$

It follows that the minimal polynomial of A divides

$$X^2 - X + 1 = (X - \omega)(X - \bar{\omega}),$$

where

$$\omega = \frac{1}{2} + i\frac{\sqrt{3}}{2} = \cos\frac{\pi}{3} + i \sin\frac{\pi}{3}.$$

Because the factors of the minimal polynomial of A are of degree 1, it follows that A is diagonalizable, so A is similar to a matrix of the form

$$A_k = \begin{pmatrix} \omega I_k & \mathcal{O}_{k,2020-k} \\ \mathcal{O}_{2020-k,k} & \bar{\omega} I_{n-k} \end{pmatrix}, \quad k \in \{0, 1, ..., 2020\}.$$

But $\det A = 1$, so we must have

$$\begin{aligned} \omega^k \bar{\omega}^{2020-k} &= 1 \Leftrightarrow \omega^{2k-2020} = 1 \Leftrightarrow \cos\frac{(2k-2020)\pi}{3} + i \sin\frac{(2k-2020)\pi}{3} = 1 \\ &\Leftrightarrow \cos\frac{(2k+2)\pi}{3} + i \sin\frac{(2k+2)\pi}{3} = 1 \\ &\Leftrightarrow k = 3n + 2 \in \{0, ..., 2020\} \Leftrightarrow k \in \{2, 5, 8, ..., 2018\} \end{aligned}$$

Two matrices that verify the given relations are not similar if and only if the numbers k_1, k_2 corresponding to those matrices are different, so the required maximum number of matrices is 673. $\square$

Problem 2. Let $k > 1$ be a real number. Calculate:

(a) $L = \lim_{n\to\infty} \int_0^1 \left(\frac{k}{\sqrt[n]{x}+k-1}\right)^n \mathrm{d}x.$

(b) $\lim_{n\to\infty} n\left[L - \int_0^1 \left(\frac{k}{\sqrt[n]{x}+k-1}\right)^n \mathrm{d}x\right].$

Proof. **(a)** The limit equals $\boxed{\frac{k}{k-1}}$.
Using the substitution $x = y^n$ we have that

$$I_n = \int_0^1 \left(\frac{k}{\sqrt[n]{x}+k-1}\right)^n \mathrm{d}x = nk^n \int_0^1 \left(\frac{y}{y+k-1}\right)^{n-1} \frac{\mathrm{d}y}{y+k-1}.$$

Using the substitution $\frac{y}{y+k-1} = t \Rightarrow y = \frac{(k-1)t}{1-t}$ we get, after some calculations, that

$$I_n = nk^n \int_0^{\frac{1}{k}} \frac{t^{n-1}}{1-t}\,\mathrm{d}t.$$

We integrate by parts and we have that

$$I_n = \frac{k}{k-1} - k^n \int_0^{\frac{1}{k}} \frac{t^n}{(1-t)^2}\,\mathrm{d}t.$$

It follows that $\lim_{n\to\infty} I_n = \frac{k}{k-1}$ since

$$0 < k^n \int_0^{\frac{1}{k}} \frac{t^n}{(1-t)^2}\,\mathrm{d}t < \frac{k^{n+2}}{(k-1)^2} \int_0^{\frac{1}{k}} t^n\,\mathrm{d}t = \frac{k}{(n+1)(k-1)^2}.$$

(b) The limit equals $\boxed{\frac{k}{(k-1)^2}}$.
We have that

$$\frac{k}{k-1} - I_n = k^n \int_0^{\frac{1}{k}} \frac{t^n}{(1-t)^2}\,\mathrm{d}t.$$

We integrate by parts and we have that

$$\frac{k}{k-1} - I_n = \frac{1}{n+1}\cdot\frac{k}{(k-1)^2} - \frac{2k^n}{n+1}\int_0^{\frac{1}{k}} \frac{t^{n+1}}{(1-t)^3}\,\mathrm{d}t.$$

This implies that

$$\lim_{n\to\infty} n\left[\frac{k}{k-1} - \int_0^1 \left(\frac{k}{\sqrt[n]{x}+k-1}\right)^n \mathrm{d}x\right] = $$
$$= \lim_{n\to\infty}\left[\frac{n}{n+1}\cdot\frac{k}{(k-1)^2} - \frac{2k^n n}{n+1}\int_0^{\frac{1}{k}} \frac{t^{n+1}}{(1-t)^3}\,\mathrm{d}t\right].$$

Thus

$$\lim_{n\to\infty} n\left[\frac{k}{k-1} - \int_0^1 \left(\frac{k}{\sqrt[n]{x}+k-1}\right)^n \mathrm{d}x\right] = \frac{k}{(k-1)^2},$$

since

$$0 < k^n \int_0^{\frac{1}{k}} \frac{t^{n+1}}{(1-t)^3}\,\mathrm{d}t < \frac{k^{n+3}}{(k-1)^3}\int_0^{\frac{1}{k}} t^{n+1}\,\mathrm{d}t = \frac{k}{(k-1)^3(n+2)}.$$

□

Problem 3. Let n be a positive integer, $k \in \mathbb{C}$ and $A \in \mathcal{M}_n(\mathbb{C})$ such that $\operatorname{Tr} A \neq 0$ and

$$\operatorname{rank} A + \operatorname{rank}\left((\operatorname{Tr} A) \cdot I_n - kA\right) = n.$$

Find rank A.

Proof. For simplicity, denote $\alpha = \operatorname{Tr} A$. Consider the block matrix:

$$M = \left[\begin{array}{c|c} A & 0 \\ \hline 0 & \alpha I_n - kA \end{array}\right].$$

We perform on M a sequence of elementary transformations on rows and columns (that do not change the rank) as follows:

$$M \xrightarrow{R_1} \left[\begin{array}{c|c} A & 0 \\ \hline A & \alpha I_n - kA \end{array}\right] \xrightarrow{C_1} \left[\begin{array}{c|c} A & kA \\ \hline A & \alpha I_n \end{array}\right] \xrightarrow{R_2}$$
$$\xrightarrow{R_2} \left[\begin{array}{c|c} A - \frac{k}{\alpha}A^2 & 0 \\ \hline A & \alpha I_n \end{array}\right] \xrightarrow{C_2} \left[\begin{array}{c|c} A - \frac{k}{\alpha}A^2 & 0 \\ \hline 0 & \alpha I_n \end{array}\right] = N$$

where

R_1 : is the left multiplication by $\left[\begin{array}{c|c} I_n & 0 \\ \hline I_n & I_n \end{array}\right]$;

C_1 : is the right multiplication by $\left[\begin{array}{c|c} I_n & kI_n \\ \hline 0 & I_n \end{array}\right]$;

R_2 : is the left multiplication by $\left[\begin{array}{c|c} I_n & -\frac{k}{\alpha}A \\ \hline 0 & I_n \end{array}\right]$;

C_2 : is the right multiplication by $\left[\begin{array}{c|c} I_n & 0 \\ \hline -\frac{1}{\alpha}A & I_n \end{array}\right]$.

It follows that

$$\operatorname{rank} A + \operatorname{rank}(\alpha I_n - kA) = \operatorname{rank} M = \operatorname{rank} N = \operatorname{rank}\left(A - \frac{k}{\alpha}A^2\right) + n.$$

Note that

$$\operatorname{rank}\left(A - \frac{k}{\alpha}A^2\right) = 0 \Leftrightarrow A - \frac{k}{\alpha}A^2 = 0 \Leftrightarrow \underbrace{\frac{k}{\alpha}A}_{B} = \left(\frac{k}{\alpha}A\right)^2 \Leftrightarrow B = B^2$$
$$\Rightarrow \operatorname{rank} B = \operatorname{Tr} B = \operatorname{Tr}\left(\frac{k}{\alpha}A\right) = \frac{k}{\alpha}\operatorname{Tr} A = k$$

so finally $\operatorname{rank} A = \operatorname{rank} B = k$. $\square$

Problem 4. Consider $0 < a < T$, $D = \mathbb{R} \setminus \{kT + a \mid k \in \mathbb{Z}\}$, and let $f : D \to \mathbb{R}$ a $T-$periodic and differentiable function which satisfies $f' > 1$ on $(0, a)$ and

$$f(0) = 0, \ \lim_{\substack{x\to a\\ x<a}} f(x) = +\infty \text{ and } \lim_{\substack{x\to a\\ x<a}} \frac{f'(x)}{f^2(x)} = 1.$$

(a) Prove that for every $n \in \mathbb{N}^*$, the equation $f(x) = x$ has a unique solution in the interval $(nT, nT + a)$, denoted x_n.

(b) Let $y_n = nT + a - x_n$ and $z_n = \displaystyle\int_0^{y_n} f(x)dx$. Prove that $\lim\limits_{n\to\infty} y_n = 0$ and study the convergence of the series $\sum_{n=1}^{\infty} y_n$ and $\sum_{n=1}^{\infty} z_n$.

Proof. **(1)** Observe first that, for every $n \in \mathbb{N}^*$, $f(nT) = 0$ and $\lim\limits_{\substack{x\to nT+a\\ x<nT+a}} f(x) = +\infty$, hence the equation $f(x) = x$ has at least one solution in the interval $(nT, nT + a)$.

Now, consider the function $g(x) = f(x) - x$ on $(nT, nT + a)$ and observe that if there would exist two solutions of the equation $f(x) = x$, say $x_n^1 < x_n^2$, by Rolle's Theorem, there exists $r_n \in (x_n^1, x_n^2) \subset (nT, nT + a)$ such that $g'(r_n) = f'(r_n) - 1 = 0$, a contradiction, since $f' > 1$ on $(nT, nT + a)$ by periodicity.

(2) Observe that for any n, f is strictly increasing on $(nT, nT + a)$. We prove that (y_n) is decreasing. By contradiction, suppose that $y_n < y_{n+1}$ for some n. Then $T + x_n > x_{n+1}$, and by the monotonicity of f that

$$x_n = f(x_n) = f(x_n + T) > f(x_{n+1}) = x_{n+1},$$

an obvious contradiction.

Since $y_n \in (0, a)$ for every n, it follows that (y_n) it converges. Then there exists $\overline{y} \geq 0$ such that $y_n \to \overline{y}$. Suppose, by contradiction, that $\overline{y} > 0$. Observe that $\overline{y} < a$. Since $x_n - nT \to a - \overline{y}$ for $n \to \infty$, it follows by the continuity of f on $(-T, a)$ that $f(x_n - nT) \to f(a - \overline{y}) \in \mathbb{R}$ for $n \to \infty$. But $f(x_n - nT) = f(x_n) = x_n \to \infty$, hence we obtain a contradiction. Therefore, $y_n \to 0$.

Next, we will prove that

$$\lim_{n\to\infty} n \cdot y_n = \frac{1}{T},$$

hence $\sum_{n=1}^{\infty} y_n$ diverges by a comparison test.

For that, observe that

$$\lim_{n\to\infty} n \cdot y_n = \lim_{n\to\infty} \frac{nT}{Tx_n} \cdot x_n y_n = \frac{1}{T} \lim_{n\to\infty} x_n y_n.$$

Moreover,

$$\begin{aligned} \lim_{n\to\infty} x_n y_n &= \lim_{n\to\infty} f(x_n) \cdot y_n = \lim_{n\to\infty} f(nT + a - y_n) \cdot y_n \\ &= \lim_{n\to\infty} \frac{y_n}{\frac{1}{f(a-y_n)}} = -\lim_{n\to\infty} \frac{(a - y_n) - a}{\frac{1}{f(a-y_n)}}. \end{aligned}$$

But $a - y_n$ converges increasingly to a so the previous limit is

$$-\lim_{\substack{x\to a\\ x<a}} \frac{x - a}{\frac{1}{f(x)}} = -\lim_{\substack{x\to a\\ x<a}} \frac{1}{-\frac{f'(x)}{f^2(x)}} = 1.$$

For the second series, observe that for every n, there is $c_n \in (0, y_n)$ such that $z_n = y_n \cdot f(c_n)$. Since f is increasing on $(0, a)$,

$$z_n \leq y_n \cdot f(y_n) = y_n^2 \cdot \frac{f(y_n)}{y_n}.$$

But f is differentiable at 0, and $\dfrac{f(y_n)}{y_n} \to f'(0) \geq 0$ for $n \to \infty$, hence there exists $M > 0$ such that, for any large n,

$$\frac{f(y_n)}{y_n} \leq M.$$

Then there exist $n_0 \in \mathbb{N}$ and $K > 0$ such that

$$0 \leq z_n \leq \frac{K}{n^2}, \quad \forall n \geq n_0.$$

By a comparison test, $\sum_{n=1}^{\infty} z_n$ converges. $\square$

SEEMOUS
CYPRUS 2021
Mathematical Olympiad for University Students

Language: **English**

Wednesday, July 21, 2021

Problem 1. Let $f : [0,1] \longrightarrow \mathbb{R}$ be a continuous strictly increasing function such that

$$\lim_{x\to 0^+} \frac{f(x)}{x} = 1\,.$$

(a) Prove that the sequence $(x_n)_{n\geqslant 1}$ defined by

$$x_n = f\left(\frac{1}{1}\right) + f\left(\frac{1}{2}\right) + \cdots + f\left(\frac{1}{n}\right) - \int_1^n f\left(\frac{1}{x}\right)\,\mathrm{d}x$$

is convergent.

(b) Find the limit of the sequence $(y_n)_{n\geqslant 1}$ defined by

$$y_n = f\left(\frac{1}{n+1}\right) + f\left(\frac{1}{n+2}\right) + \cdots + f\left(\frac{1}{2021n}\right).$$

Problem 2. Let $n \geqslant 2$ be a positive integer and let $A \in \mathcal{M}_n(\mathbb{R})$ be a matrix such that $A^2 = -I_n$. If $B \in \mathcal{M}_n(\mathbb{R})$ and $AB = BA$, prove that $\det B \geqslant 0$.

Problem 3. Let $A \in \mathcal{M}_n(\mathbb{C})$ be a matrix such that $(AA^*)^2 = A^*A$, where $A^* = \left(\overline{A}\right)^t$ denotes the Hermitian transpose (i.e., the conjugate transpose) of A.

(a) Prove that $AA^* = A^*A$.

(b) Show that the non-zero eigenvalues of A have modulus one.

Problem 4. For $p \in \mathbb{R}$, let $(a_n)_{n\geqslant 1}$ be the sequence defined by

$$a_n = \frac{1}{n^p}\int_0^n \left|\sin(\pi x)\right|^x \mathrm{d}x\,.$$

Determine all possible values of p for which the series $\sum_{n=1}^{\infty} a_n$ converges.

Language: English *Time: 5 hours*

Each problem is worth 10 points

Solution - Problem 1a:

We write
$$x_n = \sum_{k=1}^{n-1} \left(f\left(\frac{1}{k}\right) - \int_k^{k+1} f\left(\frac{1}{x}\right)\, \mathrm{d}x \right) + f\left(\frac{1}{n}\right).$$
Because f is increasing, for all $k \geqslant 1$ and $x \in [k, k+1]$ we have
$$f\left(\frac{1}{k+1}\right) \leqslant f\left(\frac{1}{x}\right) \leqslant f\left(\frac{1}{k}\right)$$
and therefore
$$f\left(\frac{1}{k+1}\right) \leqslant \int_k^{k+1} f\left(\frac{1}{x}\right) \mathrm{d}x \leqslant f\left(\frac{1}{k}\right) \tag{1}$$
Summing up for $k = 1$ up to $n-1$ we obtain
$$f\left(\frac{1}{n}\right) \leqslant x_n \leqslant f(1).$$
Since f is increasing then x_n is bounded below by $f(0)$.

It is easy to see that x_n is decreasing since using (1) we have:
$$x_{n+1} - x_n = f\left(\frac{1}{n+1}\right) - \int_n^{n+1} f\left(\frac{1}{x}\right)\, \mathrm{d}x \leqslant 0.$$
We conclude that (x_n) is convergent to some $\ell \in \mathbb{R}$.

Solution 1 - Problem 1b:

Since $\lim\limits_{x\to 0^+} \frac{f(x)}{x} = 1$, given $\varepsilon > 0$, there is a $\delta > 0$ such that $1-\varepsilon < \frac{f(x)}{x} < 1+\varepsilon$ for every $0 < x < \delta$. In particular, for every $n > \frac{1}{\delta}$ and every $k \geqslant 1$ we have $0 < \frac{1}{n+k} < \frac{1}{n} < \delta$ and therefore
$$(1-\varepsilon)\frac{1}{n+k} < f\left(\frac{1}{n+k}\right) < (1+\varepsilon)\frac{1}{n+k}.$$
Summing up the above inequalities from $k = 1$ to $2020n$ we get
$$(1-\varepsilon)S_n < f\left(\frac{1}{n+1}\right) + f\left(\frac{1}{n+2}\right) + \cdots + f\left(\frac{1}{2021n}\right) < (1+\varepsilon)S_n,$$
where
$$S_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2021n}.$$
It is well-known that $\lim\limits_{n\to\infty} S_n = \ln(2021)$ so since ε is arbitrary, we get that $\lim\limits_{n\to\infty} y_n = \ln 2021$.

Solution 2 - Problem 1b:

Since

$$y_n = x_{2021n} - x_n + \int_n^{2021n} f\left(\frac{1}{x}\right) \mathrm{d}x\,,$$

from part (a), it is enough to find

$$\lim_{n\to\infty} \int_n^{2021n} f\left(\frac{1}{x}\right) \mathrm{d}x.$$

With the change of variable $x = \dfrac{1}{t}$ we obtain

$$\int_n^{2021n} f\left(\frac{1}{x}\right) \mathrm{d}x = \int_{\frac{1}{2021n}}^{\frac{1}{n}} \frac{f(t)}{t^2}\, \mathrm{d}t\,.$$

Since $\lim\limits_{x\to 0^+} \frac{f(x)}{x} = 1$, given $\varepsilon > 0$, there is a $\delta > 0$ such that $1-\varepsilon < \frac{f(x)}{x} < 1+\varepsilon$ for every $0 < x < \delta$. In particular, for every $n > \frac{1}{\delta}$, we have $0 < \frac{1}{2021n} < \frac{1}{n} < \delta$ and therefore

$$(1-\varepsilon)\int_{\frac{1}{2021n}}^{\frac{1}{n}} \frac{1}{t}\, \mathrm{d}t \leqslant \int_{\frac{1}{2021n}}^{\frac{1}{n}} \frac{f(t)}{t^2}\, \mathrm{d}t \leqslant (1+\varepsilon)\int_{\frac{1}{2021n}}^{\frac{1}{n}} \frac{1}{t}\, \mathrm{d}t\,.$$

Since ε is arbitrary, and since

$$\int_{\frac{1}{2021n}}^{\frac{1}{n}} \frac{1}{t}\, \mathrm{d}t = \ln(2021n) - \ln n = \ln 2021\,,$$

we conclude that

$$\lim_{n\to\infty} y_n = \ln 2021\,.$$

Solution - Problem 2:

Since $A^2 = -I_n$, the only possible eigenvalues of A are $\pm i$. Since also $A \in \mathcal{M}_n(\mathbb{R})$ then $n = 2k$ and A has k eigenvalues equal to i and k eigenvalues equal to $-i$. Its minimal polynomial is x^2+1 which has distinct roots, therefore A is diagonalizable and is therefore similar to

$$X = \begin{bmatrix} iI_k & 0_k \\ 0_k & -iI_k \end{bmatrix}.$$

Similarly, if $P = \begin{bmatrix} 0_k & I_k \\ -I_k & 0_k \end{bmatrix}$, then P is also a real matrix with $P^2 = -I_n$ and so P is also similar to X. Therefore A and P are similar and so there is an invertible matrix $U \in \mathcal{M}_n(\mathbb{R})$ such that $P = U^{-1}AU$. For $C = U^{-1}BU \in \mathcal{M}_n(\mathbb{R})$ we get

$$CP = U^{-1}BAU \quad \text{and} \quad PC = U^{-1}ABU\,. \tag{1}$$

Since $AB = BA$, by (1) it follows that $CP = PC$.

Writing C into block form $C = \begin{bmatrix} X & Y \\ Z & T \end{bmatrix}$, where $X, Y, Z, T \in \mathcal{M}_k(\mathbb{R})$ and using $CP = PC$, it follows that $X = T$ and $Z = -Y$. Hence $C = \begin{bmatrix} X & Y \\ -Y & X \end{bmatrix}$. We now see that

$$\begin{vmatrix} X & Y \\ -Y & X \end{vmatrix} = \begin{vmatrix} X+iY & Y-iX \\ -Y & X \end{vmatrix} = \begin{vmatrix} X+iY & (Y-iX)-i(X+iY) \\ -Y & X-iY \end{vmatrix} = \begin{vmatrix} X+iY & 0 \\ -Y & X-iY \end{vmatrix}.$$

Therefore

$$\det B = \det C = \begin{vmatrix} X & Y \\ -Y & X \end{vmatrix} = \det(X-iY)\det(X+iY) = |\det(X+iY)|^2 \geqslant 0.$$

Alternative Solution - Problem 2

Let λ be a real eigenvalue of B and let G_λ be its generalized eigenspace considered as a real vector space. I.e.

$$G_\lambda = \{\mathbf{v} \in \mathbb{R}^n : (B-\lambda I_n)^n \mathbf{v} = \mathbf{0}\}\,.$$

We have $AB^2 = (AB)B = (BA)B = B(AB) = B(BA) = B^2A$. Inductively we get $AB^k = B^kA$ for every natural number k and from this we deduce that $Ap(B) = p(B)A$ for every polynomial $p(x)$. In particular, $A(B-\lambda I_n)^n = (B-\lambda I_n)^n A$.

Now if $\mathbf{v} \in G_\lambda$, then $(B-\lambda I_n)^n(A\mathbf{v}) = A(B-\lambda I_n)^n\mathbf{v} = \mathbf{0}$, so $A\mathbf{v} \in G_\lambda$. Therefore we can define the linear map $\alpha : G_\lambda \to G_\lambda$ by $\alpha(\mathbf{v}) = A\mathbf{v}$.

Pick a basis of G_λ and let A' be the matrix of α with respect to this basis. Then $A' \in \mathcal{M}_n(\mathbb{R})$ and $(A')^2 = -I_{n'}$, where $n' = \dim(G_\lambda)$. As in the previous solution, we get that n' is even.

Since $\dim(G_\lambda)$ is even for every real eigenvalue of B and since its complex eigenvalues come in conjugate pairs, then $\det(B) \geqslant 0$.

Solution - Problem 3:

(a) The matrix AA^* is Hermitian and all its eigenvalues are non-negative real numbers.

If $\lambda \in \sigma(AA^*)$, then $\lambda^2 \in \sigma\left((AA^*)^2\right) = \sigma(A^*A) = \sigma(AA^*)$, hence $\lambda^2 \in \sigma(AA^*)$. It follows by induction that $\lambda^{2^k} \in \sigma(AA^*)$, for all $k \in \mathbb{N}$. Since $\lambda \geqslant 0$, the last relation assures us that $\lambda \in \{0,1\}$, so AA^* will have eigenvalues 0 or 1. On the other hand, since AA^* is Hermitian, it is also diagonalizable, thus

$$AA^* = U^{-1}\begin{bmatrix} I_k & O_{k,n-k} \\ O_{n-k,k} & O_{n-k} \end{bmatrix} U\,.$$

Using the above statement, we conclude that

$$A^*A = (AA^*)^2 = AA^*\,.$$

(b) Using (a), the equality of our hypothesis can be transformed into $A^*A \cdot (AA^* - I_n) = O_n$. Letting $B = A \cdot (AA^* - I_n)$ we obtain

$$B^*B = (AA^* - I_n)A^*A(AA^* - I_n) = O_n$$

which gives $B = O_n$. Thus

$$A^2A^* = A\,. \tag{1}$$

Since $A^*A = AA^*$, it follows that the matrix A is normal, hence it is a unitary diagonalizable matrix. It follows that there is an unitary matrix $U \in \mathcal{M}_n(\mathbb{C})$ such that $A = U^*DU$, where $D = \text{diag}(\lambda_1, \ldots, \lambda_n)$. Then $A^2A^* = U^*D^2UU^*\overline{D}U = U^*D^2\overline{D}U$ and using (1) we get

$$\begin{aligned} A^2A^* = A \iff D^2\overline{D} = D &\iff \lambda_i^2 \cdot \overline{\lambda_i} = \lambda_i \text{ for all } i \in \{1,2,\ldots,n\} \\ &\iff \lambda_i(|\lambda_i|^2 - 1) = 0 \text{ for all } i \in \{1,2,\ldots,n\}\,. \end{aligned}$$

Hence the conclusion.

Alternative Solution - Problem 3

(a) Let $X = AA^*$ and $Y = A^*A$. Since X is Hermitian, it is diagonalizable so $P^{-1}XP = D$ for some matrices P, D with D diagonal. Let $Z = P^{-1}YP$. The initial condition gives $Z = D^2$. Since X and Y have the same characteristic polynomial, so do $Z = D^2$ and D. As in the original proof we deduce that every entry of D must be 0 or 1. Then $Z = D$ and so $X = Y$ as required.

(b) Writing $A = U^*DU$ as in the original proof and using $(AA^*)^2 = A^*A$ (rather than $A^2A^* = A$) we get $(D\overline{D})^2 = \overline{D}D$. From this we get that $|\lambda|^4 = |\lambda|^2$ for each eigenvalue λ of A and the conclusion follows.

Solution - Problem 4:

For every positive integer n, let

$$I_n = \int_0^n \big|\sin(\pi x)\big|^x \,\mathrm{d}x = \sum_{k=0}^{n-1} \int_k^{k+1} \big|\sin(\pi x)\big|^x \,\mathrm{d}x\,.$$

Then we have

$$\sum_{k=0}^{n-1} \int_k^{k+1} \big|\sin(\pi x)\big|^{k+1} \,\mathrm{d}x < I_n < \sum_{k=0}^{n-1} \int_k^{k+1} \big|\sin(\pi x)\big|^{k} \,\mathrm{d}x\,.$$

Substituting $t = \pi x - k\pi$, we deduce that

$$\int_k^{k+1} \big|\sin(\pi x)\big|^m \,\mathrm{d}x = \frac{1}{\pi}\int_0^\pi \sin^m t \,\mathrm{d}t$$

for every nonnegative integer m. Therefore

$$\frac{1}{\pi}\sum_{k=1}^{n} J_k < I_n < \frac{1}{\pi}\sum_{k=0}^{n-1} J_k\,, \tag{1}$$

where $J_k = \displaystyle\int_0^\pi \sin^k t\,\mathrm{d}t$. For $k \geqslant 2$, integration by parts yields

$$\begin{aligned}
J_k &= \int_0^\pi (-\cos t)' \sin^{k-1} t \,\mathrm{d}t \\
&= \Big[-\cos t \sin^{k-1} t\Big]_0^\pi + (k-1)\int_0^\pi \sin^{k-2} t \cos^2 t \,\mathrm{d}t \\
&= 0 + (k-1)\int_0^\pi \sin^{k-2} t (1 - \sin^2 t) \,\mathrm{d}t \\
&= (k-1) J_{k-2} - (k-1) J_k\,,
\end{aligned}$$

whence

$$J_k = \frac{k-1}{k}\, J_{k-2}\,.$$

Since $J_0 = \pi$ and $J_1 = 2$, we obtain

$$J_{2k} = \pi\,\frac{(2k-1)!!}{(2k)!!} \quad \text{and} \quad J_{2k+1} = 2\,\frac{(2k)!!}{(2k+1)!!}\,.$$

We observe that

$$J_{2k-1}J_{2k} = \frac{2\pi}{2k} \quad \text{and} \quad J_{2k}J_{2k+1} = \frac{2\pi}{2k+1}\,.$$

Since (J_n) is a decreasing sequence, we deduce that

$$\frac{2\pi}{2k+1} = J_{2k}J_{2k+1} \leqslant J_{2k}^2 \leqslant J_{2k-1}J_{2k} = \frac{2\pi}{2k}$$

It follows that $\sqrt{2\pi}\sqrt{\frac{2k}{2k+1}} = \sqrt{2k}J_{2k} \leqslant \sqrt{2\pi}$ and therefore

$$\lim_{k\to\infty} \sqrt{2k}J_{2k} = \sqrt{2\pi}\,. \tag{2}$$

Similarly $\sqrt{2\pi}\sqrt{\frac{2k+1}{2k+2}} \leqslant \sqrt{2k+1}J_{2k+1} \leqslant \sqrt{2\pi}$ and therefore

$$\lim_{k\to\infty} \sqrt{2k+1}J_{2k+1} = \sqrt{2\pi}\,. \tag{3}$$

By (2) and (3) it follows that

$$\lim_{n\to\infty} \sqrt{n}\,J_n = \sqrt{2\pi}\,. \tag{4}$$

By virtue of (4) and the Cesàro-Stolz theorem we have

$$\begin{aligned} \lim_{n\to\infty} \frac{J_1 + \cdots + J_n}{\sqrt{n}} &= \lim_{n\to\infty} \frac{J_{n+1}}{\sqrt{n+1} - \sqrt{n}} \\ &= \lim_{n\to\infty} \left(\sqrt{n+1} + \sqrt{n}\right) J_{n+1} \\ &= 2\sqrt{2\pi}\,. \end{aligned} \tag{5}$$

Now relations (1) and (5) ensure that

$$\lim_{n\to\infty} \frac{I_n}{\sqrt{n}} = \frac{1}{\pi} \cdot 2\sqrt{2\pi} = 2\sqrt{\frac{2}{\pi}}\,.$$

Taking into consideration that

$$a_n = \frac{I_n}{n^p} = \frac{I_n}{\sqrt{n}} \cdot \frac{1}{n^{p-\frac{1}{2}}}\,,$$

we deduce that the series $\sum_{n=1}^{\infty} a_n$ has the same nature as $\sum_{n=1}^{\infty} \frac{1}{n^{p-\frac{1}{2}}}$. In conclusion, the series $\sum_{n=1}^{\infty} a_n$ converges if and only if $p > \frac{3}{2}$.

Comments.

(1) One could use Wallis' formula or Stirling's Approximation in order to deduce (4).

(2) One could avoid the use of Cesàro-Stolz as follows: By (4) we have $J_n = \Theta(\frac{1}{\sqrt{n}})$. Since also (e.g. by considering Riemann sums) $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} = \Theta(\sqrt{n})$ then $a_n = \Theta\left(\frac{1}{n^{p-1/2}}\right)$ and the conclusion follows as before.

16th South Eastern European Mathematical Olympiad for University Students, SEEMOUS 2022

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Abstract. The 16th South Eastern European Mathematical Olympiad for University Students (SEEMOUS 2022) took place between May 27 and June 1, 2022, in Palić, Serbia. We present the competition problems and their solutions, as given by the corresponding authors, members of the jury or contestants.

Keywords: Sylvester's inequality, functional equation, numerical series, spectral norm.

MSC: Primary 15A03; Secondary 15A24, 39B22, 40A05.

The Mathematical Society of South-Eastern Europe (MASSEE) has launched in 2022 the 16th South Eastern European Mathematical Competition for University Students with International Participation (SEEMOUS 2022), which is addressed to students in the first or second year of undergraduate studies, from universities in countries that are members of MASSEE or from invited countries that are not affiliated to MASSEE.

This year's competition was hosted by the Faculty of Mathematics of the University of Belgrade, Serbia, between May 27 and June 1, 2022, at the Student Resort in Palić. The number of students that participated in the contest was 61, representing 16 universities from Bulgaria, Greece, North Macedonia, Romania and Serbia. The jury awarded 7 gold medals, 12 silver medals and 21 bronze medals. The winner of the competition was Răzvan-Gabriel Petec from Babeş-Bolyai University, Cluj-Napoca, Romania.

We present the four problems from the contest and their solutions as given by the corresponding authors, members of the jury or contestants. The interested reader may also find alternative (or similar) solutions, by following the discussions on the *AoPS Online Community* website, see [1–4].

Problem 1. Let $A, B \in \mathcal{M}_n(\mathbb{C})$ such that $AB^2A = AB$. Prove that:

(a) $(AB)^2 = AB$;
(b) $(AB - BA)^3 = O_n$.

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Authors' solution. Since $AB(BA - I_n) = O_n$, the Sylvester inequality for AB and $BA - I_n$ leads to

$$\operatorname{rank}(AB) + \operatorname{rank}(BA - I_n) \leq n + \operatorname{rank}\left(AB(BA - I_n)\right) = n. \tag{1}$$

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It is true in general (see Remark 1) that

$$\operatorname{rank}(AB - I_n) = \operatorname{rank}(BA - I_n), \tag{2}$$

so (1) becomes

$$\operatorname{rank}(AB - I_n) + \operatorname{rank}(AB) \le n. \tag{3}$$

Moreover, $\ker(AB - I_n) \subseteq \operatorname{Im}(AB)$, leading to equality in the Sylvester inequality for $AB - I_n$ and AB:

$$\operatorname{rank}(AB - I_n) + \operatorname{rank}(AB) = n + \operatorname{rank}\left((AB)^2 - AB\right). \tag{4}$$

By combining (3) and (4), the conclusion (a) follows.

Next, using the identity from the hypothesis and (a), we obtain

$$\begin{aligned}
(AB - BA)^2 &= (AB)^2 + (BA)^2 - AB^2A - BA^2B \\
&= (BA)^2 - BA^2B = -BA(AB - BA), \\
(AB - BA)^3 &= -BA(AB - BA)^2 = (BA)^2(AB - BA), \\
(AB - BA)^4 &= (BA)^2(AB - BA)^2 = -(BA)^3(AB - BA) \\
&= -B(AB)^2A(AB - BA) = -B(AB)A(AB - BA) \\
&= -(BA)^2(AB - BA) = -(AB - BA)^3,
\end{aligned}$$

hence

$$(AB - BA)^4 = -(AB - BA)^3. \tag{5}$$

Let λ be any eigenvalue of $AB - BA$. Then $\lambda^4 = -\lambda^3$, by (5), so $\lambda \in \{0, -1\}$. Since $\operatorname{Tr}(AB - BA) = 0$, all the eigenvalues of $AB - BA$ must be 0, hence $(AB - BA)^n = O_n$, which by (5) leads to (b).

Alternative solution. *The following solution of* (a) *was given by Alexandru Buzea, from University Politehnica of Bucharest, and by Konstantinos Tsirkas, from National and Kapodistrian University of Athens (contestants).*

First, (3) is obtained as in the authors' solution. Next, denote by k_0 and k_1 the number of Jordan blocks of the matrix AB that correspond to the eigenvalues 0 and 1, respectively, and by J denote the Jordan canonical form of the matrix AB. Then $\operatorname{rank}(AB) = n - k_0$ and $AB = PJP^{-1}$, where $P \in \mathcal{M}_n(\mathbb{C})$ is some invertible matrix. It follows that $AB - I_n = P(J - I_n)P^{-1}$, hence $AB - I_n$ has k_1 Jordan blocks corresponding to the eigenvalue 0, so $\operatorname{rank}(AB - I_n) = n - k_1$.

Considering (3), we obtain that $n - k_0 + n - k_1 \le n$, so $k_0 + k_1 \ge n$. Consequently, the matrix AB is diagonalizable and has no eigenvalues outside $\{0, 1\}$, hence $J^2 = J$, which leads to $(AB)^2 = PJ^2P^{-1} = PJP^{-1} = AB$.

Alternative solution. *This follows the ideas from the solution of Minas Margaritis (contestant), from National and Kapodistrian University of Athens.*

Let $X = I_n - A^T B^T$ and $Y = I_n - B^T A^T$. It is well-known that $\operatorname{rank} X = \operatorname{rank} Y$ (see (2)). Since $AB(BA - I_n) = O_n$, it follows that $\left(A^T B^T - I_n\right) B^T A^T = O_n$, hence $X(I_n - Y) = O_n$, so $XY = X$, which leads to $\ker Y \subseteq \ker X$. But $\dim \ker X = n - \operatorname{rank} X = n - \operatorname{rank} Y = \dim \ker Y$, which means that $\ker X = \ker Y$. Then $X(I_n - Y) = O_n$ implies that $Y(I_n - Y) = O_n$, hence $Y^2 = Y$. Using the definition of Y, this expands immediately to $(AB)^2 = AB$.

In order to prove (b), we first obtain $(AB - BA)^3 = (BA)^2(AB - BA)$, by a similar computation as in the authors' solution. Also, by (a),

$$(BA)^3 = B(AB)^2A = BABA = (BA)^2, \tag{6}$$

which leads to $(BA)^2(AB - BA) = (BA)^2(AB - I_n)$, hence (b) is verified if and only if $(BA)^2(AB - I_n) = O_n$, which can be written as $(B^T A^T - I_n)(A^T B^T)^2 = O_n$, or $Y(I_n - X)^2 = O_n$. But since X and Y have the same kernel, it suffices to prove that $X(I_n - X)^2 = O_n$. However, $X(I_n - X)^2 = O_n$ rewrites as $\left(I_n - A^T B^T\right)\left(A^T B^T\right)^2 = O_n$, which is equivalent to $(BA)^2(I_n - BA) = O_n$, that follows from (6).

Remark 1. In order to prove the identity (2), which is a general property for all $A, B \in \mathcal{M}_n(\mathbb{C})$, we may use $\operatorname{Ker}(I_n - AB) \subseteq \operatorname{Im} A$ and $\operatorname{Ker} A \subseteq \operatorname{Im}(I_n - BA)$, which means that the equality holds in the corresponding Sylvester inequalities, i.e.,

$$\operatorname{rank}(I_n - AB) + \operatorname{rank} A = n + \operatorname{rank}(A - ABA),$$
$$\operatorname{rank} A + \operatorname{rank}(I_n - BA) = n + \operatorname{rank}(A - ABA).$$

Problem 2. Let $a, b, c \in \mathbb{R}$ be such that

$$a + b + c = a^2 + b^2 + c^2 = 1, \quad a^3 + b^3 + c^3 \neq 1.$$

We say that a function f is a *Palić function* if $f : \mathbb{R} \to \mathbb{R}$, f is continuous, and satisfies

$$f(x)+f(y)+f(z) = f(ax+by+cz)+f(bx+cy+az)+f(cx+ay+bz) \tag{P}$$

for all $x, y, z \in \mathbb{R}$.

Prove that any Palić function is infinitely many times differentiable and find all the Palić functions.

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Author's solution. Note that the given conditions imply that a, b, c are nonzero and $ab + bc + ca = 0$. Let f be a Palić function. Putting $z = 0$ in (P), we obtain

$$f(x) + f(y) + f(0) = f(ax + by) + f(bx + cy) + f(cx + ay) \tag{1}$$

for all $x, y \in \mathbb{R}$. Since f is continuous, it follows that $F(x) = \displaystyle\int_0^x f(t)\,\mathrm{d}t$ is a primitive of f. By integrating in (1) with respect to y over $[0, 1]$, it follows that

$$f(x)+\int_0^1 f(y)\,\mathrm{d}y+f(0)=\tfrac{F(ax+b)-F(ax)}{b}+\tfrac{F(bx+c)-F(bx)}{c}+\tfrac{F(cx+a)-F(cx)}{a}, \quad (2)$$

for all $x, y \in \mathbb{R}$. Since F is differentiable, it follows from (2) that f is also differentiable, hence F is twice differentiable. By repeating the argument (using (2)), we easily obtain that f is infinitely many times differentiable.

Next, we differentiate in (P) three times with respect to x to obtain

$$f'''(x) = a^3 f'''(ax+by+cz) + b^3 f'''(bx+cy+az) + c^3 f'''(cx+ay+bz),$$

then let $y = z = x$, whence $f'''(x) = (a^3+b^3+c^3)f'''(x)$, for all $x \in \mathbb{R}$. Because $a^3+b^3+c^3 \neq 1$, it follows that $f'''(x) = 0$, so any Palić function is of the type

$$f(x) = px^2 + qx + r \quad (p, q, r \in \mathbb{R}). \quad (3)$$

Replacing the expression of f from (3) in (P) leads to

$$\begin{aligned} f(ax+by+cz)+&f(bx+cy+az)+f(cx+ay+bz)\\ &= p\left(a^2+b^2+c^2\right)\left(x^2+y^2+z^2\right)\\ &\quad + 2p(ab+bc+ca)(xy+yz+xz)\\ &\quad + q(a+b+c)(x+y+z)+3r\\ &= p(x^2+y^2+z^2)+q(x+y+z)+3r = f(x)+f(y)+f(z), \end{aligned}$$

for all $x, y, z \in \mathbb{R}$, so any function f of the form (3) is a Palić function.

Alternative solution. *This is based on the solutions given by Dimitris Emmanouil (contestant) and Kyprianos–Iason Prodromidis (member of the jury), from National and Kapodistrian University of Athens.*

First, we prove the following result.

Proposition. Let $a, b, c \in \mathbb{R}$ be as in the hypothesis of the problem and let $g : \mathbb{R} \to \mathbb{R}$ be a function whose second derivative at 0 exists, such that the relations

$$g(0) = g'(0) = g''(0) = 0 \quad \text{and} \quad g(x) = g(ax)+g(bx)+g(cx), \quad \text{for all } x \in \mathbb{R},$$

hold. Then $g(x) = 0$, for all $x \in \mathbb{R}$.

Proof. Since the functions $g(-x)$, $-g(x)$, $-g(-x)$ also satisfy the same conditions as g, we can assume for the sake of contradiction that $g(x_0) = \lambda > 0$, for some $x_0 > 0$. Also, the given conditions imply that a, b, c are nonzero, so $m = \max\{|a|, |b|, |c|\} < 1$.

We construct inductively a sequence $(x_n)_{n\geq 0}$ with the following properties: $g(x_n) \geq \lambda \left(\dfrac{x_n}{x_0}\right)^2$ and $0 < |x_{n+1}| \leq m|x_n|$, for all $n \geq 0$.

Indeed, suppose $g(x_k) \geq \lambda \left(\frac{x_k}{x_0}\right)^2$ for some $k \geq 0$ (obviously, true for $k = 0$). Then

$$g(ax_k) + g(bx_k) + g(cx_k) = g(x_k) \geq \lambda \left(\frac{x_k}{x_0}\right)^2 = \lambda(a^2 + b^2 + c^2) \left(\frac{x_k}{x_0}\right)^2,$$

which leads to

$$\left(g(ax_k) - \lambda \left(\frac{ax_k}{x_0}\right)^2\right) + \left(g(bx_k) - \lambda \left(\frac{bx_k}{x_0}\right)^2\right) + \left(g(cx_k) - \lambda \left(\frac{cx_k}{x_0}\right)^2\right) \geq 0,$$

so at least one of these parentheses is non-negative. It is enough to let $x_{k+1} = ax_k$, bx_k, or cx_k, according to the term that is non-negative, to obtain the next term in the sequence. Also, it follows that $x_{k+1} \leq mx_k$, which concludes the argument. In particular, it follows that $\lim\limits_{n\to\infty} x_n = 0$.

Yet, $g(0) = g'(0) = g''(0) = 0$ implies that $\lim\limits_{x\to 0} \dfrac{g(x)}{x^2} = 0$, hence $\lim\limits_{n\to\infty} \frac{g(x_n)}{x_n^2} = 0$, which is in contradiction to $\dfrac{g(x_n)}{x_n^2} \geq \dfrac{\lambda}{x_0^2}$ for all $n \geq 0$. $\square$

Back to the initial problem, after proving that all Palić functions are infinitely many times differentiable, it is easy to check that if f is a Palić function, then so is $g(x) = f(x) - f(0) - f'(0)x - \dfrac{f''(0)}{2}x^2$ which satisfies $g(0) = g'(0) = g''(0) = 0$. By letting $y = z = 0$ in (P) for g leads to $g(x) = g(ax) + g(bx) + g(cx)$, for all $x \in \mathbb{R}$. By the Proposition above, it follows that $g = 0$, hence f is a polynomial of degree at most 2.

Problem 3. Let $\alpha \in \mathbb{C} \setminus \{0\}$ and $A \in \mathcal{M}_n(\mathbb{C})$, $A \neq O_n$, be such that

$$A^2 + (A^*)^2 = \alpha AA^*,$$

where $A^* = \left(\overline{A}\right)^T$ denotes the transpose conjugate of A.

Prove that $\alpha \in \mathbb{R}$, $|\alpha| \leq 2$, and $AA^* = A^*A$.

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Authors' solution. Let $A = (a_{ij})_{1\leq i,j\leq n}$. Applying the trace operator in the given identity, it follows that $\sum\limits_{i,j=1}^{n} a_{ij} \cdot a_{ji} + \sum\limits_{i,j=1}^{n} \overline{a_{ji}} \cdot \overline{a_{ij}} = \alpha \cdot \sum\limits_{i,j=1}^{n} a_{ij} \cdot \overline{a_{ij}}$, hence

$$2\,\mathrm{Re}\left(\sum_{i,j=1}^{n} a_{ij} \cdot a_{ji}\right) = \alpha \sum_{i,j=1}^{n} |a_{ij}|^2\,, \tag{1}$$

which leads to $\alpha \in \mathbb{R}$. Since $2\left|\operatorname{Re}(xy)\right| \leq 2\left|x\right| \cdot \left|y\right| \leq \left|x\right|^2 + \left|y\right|^2$, for all $x, y \in \mathbb{C}$, it follows by (1) that

$$|\alpha| \sum_{i,j=1}^{n} |a_{ij}|^2 = 2\left|\operatorname{Re}\left(\sum_{i,j=1}^{n} a_{ij} \cdot a_{ji}\right)\right| \leq \sum_{i,j=1}^{n} |a_{ij}|^2 + \sum_{i,j=1}^{n} |a_{ji}|^2 = 2\sum_{i,j=1}^{n} |a_{ij}|^2 .$$

Since $A \neq O_n$, $\sum_{i,j=1}^{n} |a_{ij}|^2 \neq 0$ and we get $|\alpha| \leq 2$.

Let $\varepsilon_1, \varepsilon_2 \in \mathbb{C}$ be the solutions of $z^2 - \alpha z + 1 = 0$, so $\varepsilon_1 + \varepsilon_2 = \alpha$ and $\varepsilon_1 \varepsilon_2 = 1$. Let $X = A - \varepsilon_1 A^*$ and $Y = A - \varepsilon_2 A^*$. Then $XY = A^2 + \varepsilon_1\varepsilon_2 \left(A^*\right)^2 - \varepsilon_1 A^* A - \varepsilon_2 AA^* = \alpha AA^* - \varepsilon_1 A^* A - \varepsilon_2 AA^* = \varepsilon_1 (AA^* - A^* A)$ and, similarly, $YX = \varepsilon_2 (AA^* - A^* A)$.
Then $XY = \dfrac{\varepsilon_1}{\varepsilon_2} YX = \varepsilon_1^2 YX$, so $(XY)^2 = \varepsilon_1^4 (YX)^2$. Since $\operatorname{Tr}\left((XY)^2\right) = \operatorname{Tr}\left((YX)^2\right)$, it follows that $(\varepsilon_1^4 - 1)\operatorname{Tr}\left((XY)^2\right) = 0$, so we distinguish the following cases:

- $\varepsilon_1 \in \{-i, i\}$; then $\alpha = 0$ (which contradicts the hypothesis);
- $\varepsilon_1 \in \{-1, 1\}$; then $\alpha \in \{-2, 2\}$ and the equality from the hypothesis becomes $(A \pm A^*)^2 = \pm(A^* A - AA^*)$; the equality of the traces gives $\operatorname{Tr}\left((A \pm A^*)^2\right) = 0$, which leads to $A \pm A^* = O_n$, and the conclusion follows;
- $\operatorname{Tr}\left((XY)^2\right) = 0$; then $\operatorname{Tr}\left((AA^* - A^* A)^2\right) = 0$, so $AA^* - A^* A = O_n$.

Alternative solutions. *There were several solutions proposed by the contestants for the first part of the problem. One such solution was given by Alexandru Buzea, from University Politehnica of Bucharest.*

Taking the conjugate transpose of both sides of the equality from the statement, we obtain that $(A^*)^2 + A^2 = \overline{\alpha} AA^*$, hence $(\alpha - \overline{\alpha})\, AA^* = O_n$. Since $A \neq O_n$, we obtain that $\alpha - \overline{\alpha} = 0$, so $\alpha \in \mathbb{R}$.
Next, consider the matrix norm induced by the ℓ^2–norm,

$$\|B\|_2 = \sup_{\|\mathbf{x}\|_2 = 1} \|B\mathbf{x}\|_2 , \quad \text{for all } B \in \mathcal{M}_n(\mathbb{C}) .$$

By taking the norm in both sides of the equality from the statement, it follows that $\left\|A^2 + (A^*)^2\right\|_2 = |\alpha|\, \|AA^*\|_2$, therefore $|\alpha| = \dfrac{\left\|A^2 + (A^*)^2\right\|_2}{\|AA^*\|_2}$. Considering the subadditivity of the norm and that $\|AA^*\|_2 = \|A\|_2^2 = \|A^*\|_2^2$ (see [1], p. 283), we obtain $|\alpha| \leq 2$.

Aggelos Gatos and Konstantinos Tsirkas, from National and Kapodistrian University of Athens, used a similar argument.

Consider any $\mathbf{x} \in \mathbb{C}^n$ such that $A^*\mathbf{x} \neq \mathbf{0}$. Such $\mathbf{x}$ exists, because $A \neq O_n$. Let $\langle \cdot, \cdot \rangle$ be the standard inner product on $\mathbb{C}^n$ and let $\|\cdot\|$ be the induced norm. Then

$$\begin{aligned}\langle \left(A^2 + (A^*)^2\right) \mathbf{x}, \mathbf{x}\rangle &= \langle A^2\mathbf{x}, \mathbf{x}\rangle + \langle (A^*)^2\mathbf{x}, \mathbf{x}\rangle = \langle A\mathbf{x}, A^*\mathbf{x}\rangle + \langle A^*\mathbf{x}, A\mathbf{x}\rangle \\ &= 2\,\mathrm{Re}\langle A\mathbf{x}, A^*\mathbf{x}\rangle.\end{aligned}$$

Moreover, $\langle \alpha AA^*\mathbf{x}, \mathbf{x}\rangle = \alpha\langle A^*\mathbf{x}, A^*\mathbf{x}\rangle = \alpha\|A^*\mathbf{x}\|^2$. These relations, together with the hypothesis of the problem, imply that

$$\alpha = \frac{2\,\mathrm{Re}\langle A\mathbf{x}, A^*\mathbf{x}\rangle}{\|A^*\mathbf{x}\|^2} \in \mathbb{R}.$$

Now, let $\mathbf{x}$ be a unit vector such that $\|A^*\mathbf{x}\| = \|A^*\|_2 = \|A\|_2$, where $\|\cdot\|_2$ is the matrix norm induced by the ℓ^2–norm. The Cauchy–Schwarz inequality tells us that

$$|\alpha| \leq 2\frac{|\langle A\mathbf{x}, A^*\mathbf{x}\rangle|}{\|A^*\mathbf{x}\|^2} \leq 2\frac{\|A\mathbf{x}\| \cdot \|A^*\mathbf{x}\|}{\|A^*\mathbf{x}\|^2} = 2\frac{\|A\mathbf{x}\|}{\|A^*\mathbf{x}\|} \leq 2\frac{\|A\|_2 \cdot \|\mathbf{x}\|}{\|A^*\|_2} = 2.$$

Minas Margaritis and Kyprianos–Iason Prodromidis (member of the jury), from National and Kapodistrian University of Athens, used a different approach.

Observe that $(A + A^*)^2 = (\alpha + 1)\,AA^* + A^*A$. Since $A + A^*$ is Hermitian, the eigenvalues of its square are non-negative, which means that $\mathrm{Tr}\left((A + A^*)^2\right) \geq 0$, so $(\alpha + 1)\,\mathrm{Tr}\,(AA^*) + \mathrm{Tr}\,(A^*A) \geq 0$, which leads to $(\alpha + 2)\,\mathrm{Tr}\,(AA^*) \geq 0$. Since $\mathrm{Tr}\,(AA^*) > 0$, we conclude that $\alpha \geq -2$. By using $A - A^*$ in place of $A + A^*$, it follows that $\alpha \leq 2$ in an almost identical way.

Problem 4. Let $\mathcal{F}$ be the family of all nonempty finite subsets of $\mathbb{N} \cup \{0\}$, where $\mathbb{N}$ denotes the set of positive integers. Find all positive real numbers a for which the series $\displaystyle\sum_{A\in\mathcal{F}} \frac{1}{\sum\limits_{k\in A} a^k}$ is convergent.

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Author's solution. Let $a = 2$. Since every $n \in \mathbb{N}$ can be uniquely represented in base 2, i.e., as a sum of distinct powers of 2

$$n = 2^{k_1} + 2^{k_2} + \cdots + 2^{k_p}, \quad 0 \leq k_1 < k_2 < \cdots < k_p \text{ integers},$$

the map $\varphi : \mathbb{N} \to \mathcal{F}$, with $\varphi(n) = \{k_1, k_2, \ldots, k_p\}$, is well defined and bijective, with the inverse $\varphi^{-1}(A) = \displaystyle\sum_{k\in A} 2^k$, $A \in \mathcal{F}$.

Because all the terms of the series in question are positive, they can be rearranged at our convenience. Since φ is bijective, we can write

$$\sum_{A\in\mathcal{F}} \frac{1}{\sum\limits_{k\in A} 2^k} = \sum_{n=1}^{\infty} \frac{1}{\sum\limits_{k\in\varphi(n)} 2^k} = \sum_{n=1}^{\infty} \frac{1}{n},$$

which is divergent.

If $a < 2$, then $\dfrac{1}{\sum_{k\in A} a^k} \geq \dfrac{1}{\sum_{k\in A} 2^k}$, for all $A \in \mathcal{F}$, so the series dominates the divergent harmonic series, which means it is also divergent.

Now, let $a > 2$. For every $n \in \mathbb{N} \cup \{0\}$, denote by $\mathcal{F}_n$ the family of all nonempty finite subsets of $\mathbb{N}\cup\{0\}$ whose greatest element is n. Clearly, there are 2^n sets in $\mathcal{F}_n$ and $\{\mathcal{F}_n : n \in \mathbb{N} \cup \{0\}\}$ is a partition of $\mathcal{F}$. By regrouping the terms of the series, we obtain

$$\sum_{A\in\mathcal{F}} \frac{1}{\sum\limits_{k\in A} a^k} = \sum_{n=0}^{\infty} \sum_{A\in\mathcal{F}_n} \frac{1}{\sum\limits_{k\in A} a^k} \leq \sum_{n=0}^{\infty} \sum_{A\in\mathcal{F}_n} \frac{1}{a^n} = \sum_{n=0}^{\infty} \left(\frac{2}{a}\right)^n,$$

which means that the series is dominated by a convergent geometric series, hence it is convergent.

Concluding, the series is convergent if and only if $a > 2$.

Alternative solutions. *Other solutions, proposed by members of the jury or by contestants, use the same idea of grouping the terms of the series according to the largest member of the subsets also when obtaining the divergence (for $a \leq 2$), while the convergence (for $a > 2$) follows just like in the author's solution.*

For $a \leq 2$, Mircea Rus (member of the jury), from Technical University of Cluj-Napoca, used $\sum\limits_{k\in A} a^k \leq \sum\limits_{k\in A} 2^n \leq (n+1)2^n$, for all $A \in \mathcal{F}_n$ and $n \in \mathbb{N} \cup \{0\}$, to obtain

$$\sum_{A\in\mathcal{F}} \frac{1}{\sum\limits_{k\in A} a^k} = \sum_{n=0}^{\infty} \sum_{A\in\mathcal{F}_n} \frac{1}{\sum\limits_{k\in A} a^k} \geq \sum_{n=0}^{\infty} \sum_{A\in\mathcal{F}_n} \frac{1}{(n+1)2^n} = \sum_{n=0}^{\infty} \frac{1}{n+1},$$

which is enough to prove that the series is divergent. A different domination

$$\sum_{k\in A} a^k \leq 1 + a + a^2 + \cdots + a^n, \quad \text{for all } A \in \mathcal{F}_n \text{ and } n \in \mathbb{N} \cup \{0\},$$

was given by Răzvan-Gabriel Petec (contestant), from Babeş-Bolyai University, Cluj-Napoca, who used it to obtain

$$\sum_{A\in\mathcal{F}} \frac{1}{\sum\limits_{k\in A} a^k} = \sum_{n=0}^{\infty} \sum_{A\in\mathcal{F}_n} \frac{1}{\sum\limits_{k\in A} a^k} \geq \sum_{n=0}^{\infty} \frac{2^n}{1 + a + a^2 + \cdots + a^n}.$$

It is then elementary to show that the series $\sum\limits_{n=0}^{\infty} \dfrac{2^n}{1 + a + a^2 + \cdots + a^n}$ is divergent for all $a \leq 2$.

17^{th} **South Eastern European Mathematical Olympiad for University Students**

SEEMOUS 2023

Struga, N. Macedonia **March $7^{th} - 12^{th}$, 2023**

Organizers:

Mathematical Society of Sounth Eastern Europe (MASSEE)

The Union of Mathematicians of Macedonia

Ss. Cyril and Methodius Univeristy in Skopje

Faculty of Natural Sciences and Mathematics in Skopje

SOLUTIONS OF THE PROBLEMS

Problem 1. Prove that if A and B are $n \times n$ square matrices with complex entries satisfying

$$A = AB - BA + A^2B - 2ABA + BA^2 + A^2BA - ABA^2$$

then $\det(A) = 0$.

Solution: 1. We have

$$A^k = A^kB - A^{k-1}BA + A^{k+1}B - A^kBA - A^kBA + A^{k-1}BA^2 + A^{k+1}BA - A^kBA^2.$$

Taking the trace and employing $\mathrm{tr}(MN) = \mathrm{tr}(NM)$ we deduce

$$\begin{aligned}\mathrm{tr}(A^k) = \mathrm{tr}(A^kB) - \mathrm{tr}((A^{k-1}B)A) + \mathrm{tr}(A^{k+1}B) - \mathrm{tr}((A^kB)A) - \mathrm{tr}((A^kB)A)\\ - \mathrm{tr}((A^{k-1}B)A^2) + \mathrm{tr}((A^{k+1}B)A) - \mathrm{tr}((A^kB)A^2) = 0.\end{aligned}$$

For any $k \geq 1$, $\mathrm{tr}(A^k) = 0$ and hence A is nilpotent. Therefore $\det(A) = 0$.

Solution: 2. If $\det(A) \neq 0$, multyplying the equation by A^{-1} from left (right), we get

$$I_n = B - A^{-1}BA + AB - 2BA + A^{-1}BA^2 + ABA - BA^2.$$

Taking trace and having in mind that $\mathrm{tr}(MN) = \mathrm{tr}(NM)$ we deduce:

$$\begin{aligned}n = \mathrm{tr}(I_n) = \mathrm{tr}(A(A^{-1}B)) - \mathrm{tr}((A^{-1}B)A) + \mathrm{tr}(AB) - \mathrm{tr}(BA) - \mathrm{tr}((BA^2)A^{-1}) +\\ + \mathrm{tr}(A^{-1}(BA^2)) + \mathrm{tr}(A(BA)) - \mathrm{tr}((BA)A) = 0,\end{aligned}$$

which is a contradiction. Hence $\det(A) = 0$.

Problem 2. For the sequence

$$S_n = \frac{1}{\sqrt{n^2+1^2}} + \frac{1}{\sqrt{n^2+2^2}} + \ldots + \frac{1}{\sqrt{n^2+n^2}},$$

find

$$\lim_{n\to\infty} n\left(n(\ln(1+\sqrt{2}) - S_n) - \frac{1}{2\sqrt{2}(1+\sqrt{2})}\right).$$

Solution: In what follows $O\left(x^k\right)$ stays for Cx^k where C is some constant.

$$f(x) = f(b) + f'(b)(x-b) + \frac{1}{2}f''(b)(x-b)^2 + \frac{1}{6}f'''(\theta)(x-b)^3$$

for some θ between a and b. It follows that

$$\int_a^b f(x)dx = f(b)(b-a) - \frac{1}{2}f'(b)(b-a)^2 + \frac{1}{6}f''(b)(b-a)^3 + O((b-a)^4). \qquad (1)$$

Now, let n be a positive integer. Then, for $k = 0, 1, 2, \ldots, n-1$,

$$\int_{(k-1)/n}^{k/n} f(x)dx = \frac{1}{n}f\left(\frac{k}{n}\right) - \frac{1}{2n^2}f'\left(\frac{k}{n}\right) + \frac{1}{6n^3}f''\left(\frac{k}{n}\right) + O\left(\frac{1}{n^4}\right). \qquad (2)$$

Summing over k then yields

$$\int_0^1 f(x)dx = \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) - \frac{1}{2n^2}\sum_{k=1}^{n} f'\left(\frac{k}{n}\right) + \frac{1}{6n^3}\sum_{k=1}^{n} f''\left(\frac{k}{n}\right) + O\left(\frac{1}{n^3}\right). \qquad (3)$$

Similarly, we can get

$$f(1) - f(0) = \int_0^1 f'(x)dx = \frac{1}{n}\sum_{k=1}^{n} f'\left(\frac{k}{n}\right) - \frac{1}{2n^2}\sum_{k=1}^{n} f''\left(\frac{k}{n}\right) + O\left(\frac{1}{n^2}\right), \qquad (4)$$

and

$$f'(1) - f'(0) = \int_0^1 f''(x)dx = \frac{1}{n}\sum_{k=1}^{n} f''\left(\frac{k}{n}\right) + O\left(\frac{1}{n}\right). \qquad (5)$$

Combining (3), (4) and (5) we obtain

$$\int_0^1 f(x)dx = \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) - \frac{1}{2n}(f(1)-f(0)) - \frac{1}{12n^2}(f'(1)-f'(0)) + O\left(\frac{1}{n^3}\right).$$

Now, let

$$f(x) = \frac{1}{\sqrt{1+x^2}}.$$

Then

$$\int_0^1 f(x)dx = \ln\left|x+\sqrt{1+x^2}\right|\Big|_0^1 = \ln(1+\sqrt{2}) - \ln(1) = \ln(1+\sqrt{2});$$

$$\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) = \frac{1}{n}\sum_{k=1}^{n}\frac{1}{\sqrt{1+(k/n)^2}} = \sum_{k=1}^{n}\frac{1}{\sqrt{n^2+k^2}} = S_n;$$

$$f(1)-f(0) = \frac{1}{\sqrt{2}} - 1 = \frac{1-\sqrt{2}}{\sqrt{2}} = -\frac{1}{\sqrt{2}(1+\sqrt{2})};$$

$$f'(1)-f'(0) = -\frac{1}{2\sqrt{2}} - 0 = -\frac{1}{2\sqrt{2}}.$$

Hence

$$\ln(1+\sqrt{2}) = S_n + \frac{1}{2\sqrt{2}(1+\sqrt{2})n} + \frac{1}{24\sqrt{2}n^2} + O\left(\frac{1}{n^3}\right).$$

Finally,

$$\lim_{n\to\infty} n\left(n(\ln(1+\sqrt{2})-S_n) - \frac{1}{2\sqrt{2}(1+\sqrt{2})}\right) = \frac{1}{24\sqrt{2}}.$$

Problem 3. Prove that: if A is $n \times n$ square matrix with complex entries such that $A + A^* = A^2\, A^*$, then $A = A^*$. (For any matrix M, denote by $M^* = \overline{M}^t$ the conjugate transpose of M.)

Solution: We show first that A is normal, i.e., $A\, A^* = A^*\, A$.

We have that $A + A^* = A^2\, A^*$ leads to $A = (A^2 - I_n)A^* \quad (1)$, hence $A \pm I_n = (A - I_n)(A + I_n)A^* \pm I_n$, so

$$\begin{aligned}(A - I_n)\,[(A + I_n)A^* - I_n] &= I_n \\ (A + I_n)\,[I_n - (A - I_n)A^*] &= I_n,\end{aligned}$$

which leads to $A - I_n$ and $A + I_n$ being invertible. From here, $A^2 - I_n$ is also invertible, and by (1) it follows that $A^* = (A^2 - I_n)^{-1}A$. Using the Cayley–Hamilton theorem, it follows that $(A^2 - I_n)^{-1}$ is a polynomial of $A^2 - I_n$, hence a polynomial of A, so $A^*\, A = A\, A^*$.

Since A is normal, it is unitary diagonalizable, i.e., there exist a unitary matrix $U \in \mathcal{M}_n(\mathbb{C})$ and $D = \operatorname{diag}[\lambda_1, \lambda_2, \ldots, \lambda_n]$ a diagonal matrix such that $A = UDU^*$. Then $A^* = U\overline{D}U^*$, which, by the hypothesis leads to $D + \overline{D} = D^2\overline{D}$, meaning that $\lambda_i + \overline{\lambda_i} = \lambda_i^2\overline{\lambda_i}$, for all $i \in \{1, 2, \ldots, n\}$. Then $2\operatorname{Re}\lambda_i = \lambda_i \cdot |\lambda_i|^2$, so λ_i are all real, and $D = \overline{D}$. This is now enough for $A = A^*$.

Problem 4. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous, strictly decreasing function such that $f([0,1]) \subseteq [0,1]$.

(i) For all $n \in \mathbb{N}\backslash\{0\}$, prove that there exists a unique $a_n \in (0,1)$, solution of the equation

$$f(x) = x^n.$$

Moreover, if (a_n) is the sequence defined as above, prove that $\lim\limits_{n\to\infty} a_n = 1$.

(ii) Suppose f has a continuous derivative, with $f(1) = 0$ and $f'(1) < 0$. For any $x \in \mathbb{R}$, we define

$$F(x) = \int_x^1 f(t)\ dt.$$

Study the convergence of the series $\sum_{n=1}^{\infty} (F(a_n))^{\alpha}$, with $\alpha \in \mathbb{R}$.

Solution: (i) Consider the continuous function $g : [0,1] \to \mathbb{R}$ given by $g(x) = f(x) - x^n$, and observe that $g(0) = f(0) > 0$, and $g(1) = f(1) - 1 < 0$. It follows the existence of $a_n \in (0,1)$ such that $g(a_n) = 0$. For uniqueness, observe that if would exists two solutions of the equation (4), say $a_n < b_n$, we would obtain

$$f(a_n) > f(b_n) \Leftrightarrow a_n^n > b_n^n \Leftrightarrow a_n > b_n,$$

a contradiction.

We prove that the sequence (a_n) is strictly increasing. If it would exist $n \in \mathbb{N}^*$ such that $a_n \geq a_{n+1}$, we would obtain that

$$f(a_n) \leq f(a_{n+1}) \Leftrightarrow a_n^n \leq a_{n+1}^{n+1} < a_{n+1}^n,$$

since f is strictly decreasing and $a_{n+1} \in (0,1)$. It follows that $a_n < a_{n+1}$, a contradiction. Hence, (a_n) is strictly increasing and bounded above by 1, so it converges to $\ell \in (0,1]$. Suppose, by contradiction, that $\ell < 1$. Since $f(a_n) = a_n^n$ for any n, using the continuity of f it follows that $f(\ell) = 0$ for $\ell < 1$, contradicting the fact that f is strictly decreasing with $f(1) \geq 0$. Hence, $\lim\limits_{n\to\infty} a_n = 1$.

(ii) Observe that F is well-defined, of class C^2, with $F(1) = 0$, $F'(x) = -f(x) \Rightarrow F'(1) = 0$, $F''(x) = -f'(x) \Rightarrow F''(1) > 0$. Moreover, remark that $F(x) > 0$ on $[0,1)$. Using the Taylor formula on the interval $[a_n, 1]$, it follows that for any n, there exist $c_n, d_n \in (a_n, 1)$ such that

$$\begin{aligned} F(a_n) &= F(1) + F'(1)(a_n - 1) + \frac{F''(c_n)}{2}(a_n-1)^2 = \frac{F''(c_n)}{2}(a_n-1)^2, \\ f(a_n) &= f(1) + f'(d_n)(a_n-1) = f'(d_n)(a_n-1). \end{aligned} \tag{1}$$

Hence, since $c_n \to 1$ and F is C^2, we obtain

$$\lim_{n\to\infty} \frac{(1-a_n)^2}{F(a_n)} = \frac{2}{F''(1)} \in (0,+\infty),$$

so due to the comparison test,

$$\sum_{n=1}^{\infty} \left(F\left(a_n\right)\right)^{\alpha} \sim \sum_{n=1}^{\infty} \left(1-a_n\right)^{2\alpha}.$$

But

$$\begin{aligned}
\lim_{n\to\infty} n\left(1-a_n\right) &= -\lim_{n\to\infty} n \cdot \frac{\left(a_n-1\right)}{\ln\left(1+\left(a_n-1\right)\right)} \cdot \ln a_n \\
&= -\lim_{n\to\infty} \ln a_n^n = -\lim_{n\to\infty} \ln f\left(a_n\right) = -\ln\left(\lim_{n\to\infty} f\left(a_n\right)\right) = +\infty.
\end{aligned}$$

It follows that $\sum_{n=1}^{\infty}\left(1-a_n\right)$ diverges and, furthermore, $\sum_{n=1}^{\infty}\left(1-a_n\right)^{2\alpha}$ diverges for any $2\alpha \leq 1$.

Next, consider arbitrary $\gamma \in (0,1)$. Using (1) and the fact that $d_n \to 1$, we obtain

$$\begin{aligned}
&\lim_{n\to\infty} n^{\gamma}\left(1-a_n\right) = \lim_{n\to\infty} \left[n\left(1-a_n\right)\right]^{\gamma} \cdot \left(1-a_n\right)^{1-\gamma} \\
= &\lim_{n\to\infty} \left[n\left(1-a_n\right)\right]^{\gamma} \cdot \left[\frac{f\left(a_n\right)}{-f'\left(d_n\right)}\right]^{1-\gamma} = \frac{1}{\left(-f'(1)\right)^{1-\gamma}} \cdot \lim_{n\to\infty} \left[-\ln f\left(a_n\right)\right]^{\gamma} \cdot \left[e^{\ln f\left(a_n\right)}\right]^{1-\gamma}.
\end{aligned}$$

Observe that

$$-\ln f\left(a_n\right) \to +\infty \quad \text{and} \quad \lim_{x\to+\infty} \frac{x^{\gamma}}{e^{(1-\gamma)x}} = 0,$$

hence $\lim\limits_{n\to\infty} n^{\gamma}\left(1-a_n\right) = 0$. So, if $\alpha > \dfrac{1}{2}$, we obtain that there exists $\varepsilon > 0$ such that $2\alpha > 1+\varepsilon$, hence for $\gamma := \dfrac{1+\varepsilon}{2\alpha} < 1$, we get

$$\lim_{n\to\infty} n^{2\alpha\gamma}\left(1-a_n\right)^{2\alpha} = \lim_{n\to\infty} n^{(1+\varepsilon)}\left(1-a_n\right)^{2\alpha} = 0.$$

Using the comparison test, it follows that the series $\sum_{n=1}^{\infty}\left(1-a_n\right)^{2\alpha}$ converges. In conclusion, the series $\sum_{n=1}^{\infty}\left(F\left(a_n\right)\right)^{\alpha}$ converges iff $\alpha > \frac{1}{2}$.

Mathematical Society of South Eastern Europe

Technical University "Gheorghe Asachi" of Iasi

Romanian Mathematical Society

South Eastern European Mathematical Olympiad for University Students

Iaşi, Romania - April 11, 2024

Problem 1. Let $(x_n)_{n\geq 1}$ be the sequence defined by $x_1 \in (0,1)$ and $x_{n+1} = x_n - \dfrac{x_n^2}{\sqrt{n}}$ for all $n \geq 1$.

Find the values of $\alpha \in \mathbb{R}$ for which the series $\displaystyle\sum_{n=1}^{\infty} x_n^{\alpha}$ is convergent.

Problem 2. Let $A, B \in \mathcal{M}_n(\mathbb{R})$ two real, symmetric matrices with nonnegative eigenvalues. Prove that $A^3 + B^3 = (A+B)^3$ if and only if $AB = \mathcal{O}_n$.

Problem 3. For every $n \geq 1$ define x_n by

$$x_n = \int_0^1 \ln(1 + x + x^2 + \ldots + x^n) \cdot \ln \frac{1}{1-x} \, \mathrm{d}x, \quad n \geq 1.$$

a) Show that x_n is finite for every $n \geq 1$ and $\lim\limits_{n\to\infty} x_n = 2$.

b) Calculate $\lim\limits_{n\to\infty} \dfrac{n}{\ln n}(2 - x_n)$.

Problem 4. Let $n \in \mathbb{N}$, $n \geq 2$. Find all the values $k \in \mathbb{N}$, $k \geq 1$, for which the following statement holds:

"If $A \in \mathcal{M}_n(\mathbb{C})$ is such that $A^k A^* = A$, then $A = A^*$."

(here, $A^* = \overline{A}^t$ denotes the transpose conjugate of A).

SEEMOUS 2024

APRIL 9-14, IAŞI, ROMÂNIA

South Eastern European Mathematical Olympiad for University Students

Iaşi, Romania - April 11, 2024

Solutions and marking scheme

Problem 1. Let $(x_n)_{n\geq 1}$ be the sequence defined by $x_1 \in (0,1)$ and $x_{n+1} = x_n - \frac{x_n^2}{\sqrt{n}}$ for all $n \geq 1$. Find the values of $\alpha \in \mathbb{R}$ for which the series $\sum_{n=1}^{\infty} x_n^{\alpha}$ is convergent.

Solution:

By induction we deduce that $x_n \in (0,1)$ for all $n \geq 1$. Let $n \geq 1$. From $x_n - x_{n+1} = \frac{x_n^2}{\sqrt{n}}$ for all $n \geq 1$ we deduce that $1 - \frac{x_{n+1}}{x_n} = \frac{x_n}{\sqrt{n}}$ and since $0 < \frac{x_n}{\sqrt{n}} < \frac{1}{\sqrt{n}}$, $\forall n \geq 1$ we deduce that $\lim_{n\to\infty} \frac{x_n}{\sqrt{n}} = 0$ and hence $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = 1$. Now let $n \geq 1$. By the recurence relation we have $\frac{1}{x_{n+1}} - \frac{1}{x_n} = \frac{x_n - x_{n+1}}{x_n x_{n+1}} = \frac{x_n}{x_{n+1}} \cdot \frac{1}{\sqrt{n}}$ which implies that

$$\lim_{n\to\infty} \frac{\frac{1}{x_{n+1}} - \frac{1}{x_n}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{x_n}{x_{n+1}} = 1.$$

Since $\lim_{n\to\infty} \left(1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n-1}}\right) = \infty$ by the Stolz-Cesaro lemma it follows that

$$\lim_{n\to\infty} \frac{\frac{1}{x_n}}{1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n-1}}} = 1.$$

Now if we use that, again by the Stolz-Cesaro lemma

$$\lim_{n\to\infty} \frac{\sqrt{n}}{1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n-1}}} = \lim_{n\to\infty} \frac{\sqrt{n+1} - \sqrt{n}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{2}$$

we get $\lim_{n\to\infty} \frac{\frac{1}{x_n}}{\sqrt{n}} = 2$ and hence $\lim_{n\to\infty} \frac{x_n^{\alpha}}{\frac{1}{n^{\frac{\alpha}{2}}}} = 2^{-\alpha}$. By the comparison criterion for the positive series it follows that the series $\sum_{n=1}^{\infty} x_n^{\alpha}$ is convergent if and only if the $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{\alpha}{2}}}$ is convergent that is if and only if $\frac{\alpha}{2} > 1$, $\alpha > 2$.

Marking scheme:

I) Proving that $(x_n)_{n\geq 1}$ is convergent. **1p**

II) Proving that $x_n \to 0$. **2p**

III) Proving that there exists a constant $c_1 > 0$ such that $x_n \leq \dfrac{c_1}{\sqrt{n}}$. **3p**

IV) Proving that there exists a constant $c_2 > 0$ such that $x_n \geq \dfrac{c_2}{\sqrt{n}}$. **3p**

V) Conclusion. **1p**

First remark: Using the Stolz-Cesaro lemma to prove that $x_n \sim \dfrac{1}{\sqrt{n}}$ generates **6p**, since it replaces parts III and IV from the previous mentioned mark scheme.
Second remark: Using the Stolz-Cesaro lemma without arguing that the denominator is increasing and unbounded generates only **5p**.
Third remark: Claiming that $x_n \sim \dfrac{1}{\sqrt{n}}$ without a proof will only generate **1p**, which is **not** additive with V.

Problem 2. Let $A, B \in \mathcal{M}_n(\mathbb{R})$ two real, symmetric matrices with nonnegative eigenvalues. Prove that $A^3 + B^3 = (A+B)^3$ if and only if $AB = \mathcal{O}_n$.

Solution (Author): If $AB = \mathcal{O}_n$ then

$$AB = \mathcal{O}_n = (AB)^T = B^T A^T = BA$$

therefore A and B commute and

$$(A+B)^3 = A^3 + B^3 + 3AB(A+B) = A^3 + B^3.$$

Assume now that $A^3 + B^3 = (A+B)^3$. Since the trace operator is linear and invariant under cyclic permutations it follows that

$$\operatorname{Tr}(ABA) + \operatorname{Tr}(BAB) = 0. \tag{1}$$

We recall that a real, symmetric matrix M has nonnegative eigenvalues $\lambda_1, ..., \lambda_n$ i.e. M is positive semidefinite if and only if M can be decomposed as a product $M = Q^T Q$ for some real matrix Q. Moreover, if for such a matrix $\operatorname{Tr} M = 0$ then $M = \mathcal{O}_n$. Let $U, V \in \mathcal{M}_n(\mathbb{R})$ such that $A = U^T U$ and $B = V^T V$. Then, using the symmetry of A and B we get

$$ABA = AV^T VA = (VA)^T (VA) \qquad BAB = BU^T UB = (UB)^T (UB)$$

so $\operatorname{Tr}(ABA) \geq 0$ and $\operatorname{Tr}(BAB) \geq 0$. From (1) it follows that we must have $\operatorname{Tr}(ABA) = \operatorname{Tr}(BAB) = 0$ and therefore $ABA = BAB = \mathcal{O}_n$.
In particular, for every $x \in \mathbb{R}^n$ we have

$$\|VAx\|^2 = x^T (VA)^T (VA) x = x^T ABA x = 0$$

so $VA = \mathcal{O}_n$. Again, for every $x \in \mathbb{R}^n$

$$\|ABx\|^2 = x^T (AB)^T (AB) x = x^T V^T (VA) ABx = 0$$

and, finally, we find $AB = \mathcal{O}_n$.

Alternative solution (2). For every $x \in \mathbb{R}^n$ we have, on account of B being positive semidefinite $< Bx, x > \geq 0$ and equality holds only for $x \in \ker B$. But then $(ABA)^T = ABA$ and

$$< ABAx, x > = < BAx, Ax > \geq 0$$

so ABA is positive semidefinite and $\mathrm{Tr}(ABA) \geq 0$. In the same manner we get BAB as positive semidefinite and $\mathrm{Tr}(ABA) \geq 0$ which leads to $\mathrm{Tr}(ABA) = \mathrm{Tr}(BAB) = 0$ and, next, to $ABA = BAB = \mathcal{O}_n$. Finally, for every $x \in \mathbb{R}^n$ we have

$$0 = < BABx, x > = < ABx, Bx >$$

which implies $Bx \in \ker A$, $\forall x \in \mathbb{R}^n$, which concludes the proof.

Marking scheme:

I) $AB = O_n \Rightarrow (A+B)^3 = A^3 + B^3$ **2p**

II) $\mathrm{Tr}(ABA) + \mathrm{Tr}(BAB) = 0$ **2p**

III) $ABA = BAB = O_n$ **4p**

IV) Conclusion $AB = O_n$. **2p**

Problem 3. For every $n \geq 1$ define x_n by

$$x_n = \int_0^1 \ln(1 + x + x^2 + \ldots + x^n) \cdot \ln \frac{1}{1-x} \, \mathrm{d}x, \quad n \geq 1.$$

a) Show that x_n is finite for every $n \geq 1$ and $\lim_{n\to\infty} x_n = 2$.

b) Calculate $\lim_{n\to\infty} \frac{n}{\ln n}(2 - x_n)$.

Solution. a) For all $n \geq 1$ and $x \in [0, 1)$,

$$\frac{1}{1-x} \geq 1 \quad \text{and} \quad 0 \leq \ln(1 + x + x^2 + \ldots + x^n) \cdot \ln \frac{1}{1-x} \leq \ln n \cdot \ln \frac{1}{1-x}.$$

Since $\int_0^1 \ln \frac{1}{1-x} \, \mathrm{d}x$ is convergent (to 1, by a direct computation), it follows that the sequence is well-defined.

Next, the sequence of functions $f_n(x) = \ln(1 + x + x^2 + \ldots + x^n) \cdot \ln \frac{1}{1-x}$ satisfies:

$$0 \leq f_n(x) \leq f_{n+1}(x), \quad \text{for all } x \in [0,1) \text{ and } n \geq 1,$$

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \left(\ln \frac{1 - x^{n+1}}{1-x} \cdot \ln \frac{1}{1-x} \right) = \ln^2 \frac{1}{1-x}, \quad \text{for all } x \in [0,1).$$

It follows by *the Lebesgue-Beppo-Levi theorem* (of *monotone convergence*) that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \int_0^1 f_n(x) \, \mathrm{d}x = \int_0^1 \ln^2 \frac{1}{1-x} \, \mathrm{d}x = 2$$

(the last equality follows by an elementary computation).

b) From (a),

$$2 - x_n = \int_0^1 \left(\ln^2 \frac{1}{1-x} - \ln \frac{1 - x^{n+1}}{1-x} \cdot \ln \frac{1}{1-x} \right) \mathrm{d}x = \int_0^1 \ln(1 - x^{n+1}) \cdot \ln(1-x) \, \mathrm{d}x$$

and with the change of variable $y = x^{n+1}$, it follows that

$$2 - x_n = \frac{1}{n+1} \int_0^1 \ln(1-y) \cdot \ln\left(1 - y^{\frac{1}{n+1}}\right) \cdot y^{\frac{1}{n+1}-1} \, dy.$$

By shifting the index, for convenience, it follows that

$$\begin{aligned}
\lim_{n\to\infty} \frac{n}{\ln n}(2 - x_n) &= \lim_{n\to\infty} \frac{n-1}{\ln(n-1)}(2 - x_{n-1}) \\
&= \lim_{n\to\infty} \frac{n-1}{n} \cdot \lim_{n\to\infty} \frac{\ln n}{\ln(n-1)} \cdot \lim_{n\to\infty} \frac{1}{\ln n} \int_0^1 \ln(1-y) \cdot \ln\left(1 - y^{\frac{1}{n}}\right) \cdot y^{\frac{1}{n}-1} \, dy \\
&= \lim_{n\to\infty} \int_0^1 \frac{\ln(1-y)}{y} \cdot \frac{y^{\frac{1}{n}} \ln(1 - y^{\frac{1}{n}})}{\ln n} \, dy.
\end{aligned}$$

We want to verify the conditions in the *Lebesgue dominated convergence theorem*, so consider

$$g_n(y) = \frac{\ln(1-y)}{y} \cdot \frac{y^{\frac{1}{n}} \ln(1 - y^{\frac{1}{n}})}{\ln n}, \text{ for } y \in (0,1), \text{ and } n \geq 2.$$

The pointwise convergence follows in a standard manner: we start from

$$\lim_{n\to\infty} \frac{y^{\frac{1}{n}} - 1}{\frac{1}{n}} = \ln y, \quad \text{hence} \quad \lim_{n\to\infty} n\left(1 - y^{\frac{1}{n}}\right) = \ln\frac{1}{y} > 0,$$

which leads to

$$\lim_{n\to\infty} \left(\ln\left(1 - y^{\frac{1}{n}}\right) + \ln n\right) = \ln\left(\ln\frac{1}{y}\right).$$

Then

$$\begin{aligned}
\lim_{n\to\infty} g_n(y) &= \frac{\ln(1-y)}{y} \cdot \lim_{n\to\infty} y^{\frac{1}{n}} \cdot \lim_{n\to\infty} \frac{\ln\left(1 - y^{\frac{1}{n}}\right)}{\ln n} \\
&= \frac{\ln(1-y)}{y} \cdot \lim_{n\to\infty} \left(\frac{\ln\left(1 - y^{\frac{1}{n}}\right) + \ln n}{\ln n} - 1\right) \\
&= \frac{\ln(1-y)}{y}\left(\ln\left(\ln\frac{1}{y}\right) \cdot \frac{1}{\infty} - 1\right) = -\frac{\ln(1-y)}{y}, \quad \text{for all } y \in (0,1).
\end{aligned}$$

To check *the domination condition*, let $g(t) = -\ln(1-t) = \ln\dfrac{1}{1-t}$, for $t \in [0,1)$. Note that g is positive. Since $0 \leq y^{\frac{1}{n}} \leq 1$, it follows that

$$0 \leq g_n(y) \leq \frac{\ln(1-y)}{y} \cdot \frac{\ln\left(1 - y^{\frac{1}{n}}\right)}{\ln n} = \frac{g(y)}{y} \cdot \frac{g\left(y^{\frac{1}{n}}\right)}{\ln n}, \quad \text{for all } n \geq 2 \text{ and } y \in (0,1). \tag{1}$$

From

$$g(t) - g(t^n) = \ln\frac{1-t^n}{1-t} = \ln(1 + t + \ldots + t^{n-1}) \leq \ln n, \quad \text{for all } t \in (0,1) \text{ and } n \geq 1,$$

it follows that $g\left(y^{\frac{1}{n}}\right) - g(y) \leq \ln n$, hence

$$\frac{g\left(y^{\frac{1}{n}}\right)}{\ln n} \leq 1 + \frac{g(y)}{\ln n} \leq 1 + g(y), \quad \text{for all } n \geq 3. \tag{2}$$

Combining (1) and (2) and replacing g, we finally obtain

$$0 \leq g_n(y) \leq \frac{\ln^2(1-y) - \ln(1-y)}{y}, \quad \text{for all } n \geq 3 \text{ and } y \in (0,1).$$

It is an elementary exercise to check that $\displaystyle\int_0^1 \frac{\ln^2(1-y)-\ln(1-y)}{y}\,\mathrm{d}y$ is convergent, which concludes the proof of the domination condition and establishes that

$$L=\lim_{n\to\infty}\frac{n}{\ln n}(2-x_n)=-\int_0^1\frac{\ln(1-y)}{y}\,\mathrm{d}y=\frac{\pi^2}{6},$$

where the last equality is a well know result, that can be obtained by integrating the Maclaurin series of $-\dfrac{\ln(1-y)}{y}$ and then using Euler's identity $\sum\limits_{n\geq 1}\dfrac{1}{n^2}=\dfrac{\pi^2}{6}$.

Marking scheme:

a)

• The convergence of the integral defining x_n .... **1** p

• Apply a convergence theorem (e.g., *Beppo–Levi monotone convergence*) for the sequence of functions

$$f_n(x)=\ln(1+x+x^2+\ldots+x^n)\cdot\ln\frac{1}{1-x},\qquad x\in[0,1)\text{ and }n\geq 1$$

to obtain that $\displaystyle\lim_{n\to\infty}x_n=\int_0^1\lim_{n\to\infty}f_n\,\mathrm{d}x=\int_0^1\ln^2\frac{1}{1-x}\,\mathrm{d}x$ .... **1** p

• Compute $\displaystyle\int_0^1\ln^2\frac{1}{1-x}\,\mathrm{d}x=2$ .... **1** p

b)

• Obtain $\displaystyle 2-x_n=\int_0^1\ln(1-x^{n+1})\cdot\ln(1-x)\,\mathrm{d}x$ .... **1** p

• Use the change of variable $y=x^{n+1}$, and rewrite

$\displaystyle\frac{n}{\ln n}(2-x_n)=\frac{n}{n+1}\cdot\frac{\ln(n+1)}{\ln n}\int_0^1\frac{\ln(1-y)}{y}\cdot\frac{y^{\frac{1}{n+1}}\ln(1-y^{\frac{1}{n+1}})}{\ln(n+1)}\,\mathrm{d}y$ .... **1** p

• For the sequence of functions

$$g_n(y)=\frac{\ln(1-y)}{y}\cdot\frac{y^{\frac{1}{n}}\ln(1-y^{\frac{1}{n}})}{\ln n},\qquad\text{for } y\in(0,1)\text{ and } n\geq 2$$

compute $\displaystyle\lim_{n\to\infty}g_n(y)=-\frac{\ln(1-y)}{y}$, for all $y\in(0,1)$ .... **2** p

• Apply a convergence theorem to obtain that

$\displaystyle\lim_{n\to\infty}\frac{n}{\ln n}(2-x_n)=\int_0^1\lim_{n\to\infty}g_n(y)\,\mathrm{d}y=-\int_0^1\frac{\ln(1-y)}{y}\,\mathrm{d}y$ .... **2** p*

(***0.5** p for choosing a convergence theorem and stating the conditions that need to be verified, without completing the corresponding computations)

E.g., use the *Lebesgue dominated convergence* with the domination

$$0\leq g_n(y)\leq\frac{\ln^2(1-y)-\ln(1-y)}{y},\qquad\text{for all } n\geq 3\text{ and } y\in(0,1)$$

and check that $\displaystyle\int_0^1\frac{\ln^2(1-y)-\ln(1-y)}{y}\,\mathrm{d}y$ is convergent.

• Concluding, $\displaystyle\lim_{n\to\infty}\frac{n}{\ln n}(2-x_n)=-\int_0^1\frac{\ln(1-y)}{y}\,\mathrm{d}y=\frac{\pi^2}{6}$ .... **1** p*

(***0.5** p for the value of the integral, without proof)

Problem 4. Let $n \in \mathbb{N}$, $n \geq 2$. Find all the values $k \in \mathbb{N}$, $k \geq 1$, for which the following statement holds:

$$\text{“If } A \in \mathcal{M}_n(\mathbb{C}) \text{ is such that } A^k A^* = A, \text{ then } A = A^*.\text{”} \tag{*}$$

(here, $A^* = \overline{A}^t$ denotes the transpose conjugate of A).

Solution (Author). First, we limit the range of the possible values for k, by choosing $A = \varepsilon I_n$, with suitable $\varepsilon \in \mathbb{C}$, $|\varepsilon| = 1$, such that the implication in (*) is false, so we ask that $A^k A^* = A$, but $A \neq A^*$. Then $\varepsilon I_n = A = A^k A^* = \varepsilon^k \overline{\varepsilon} I_n = \varepsilon^{k-1} I_n$ and $\varepsilon I_n = A \neq A^* = \overline{\varepsilon} I_n$, which are equivalent to $\varepsilon^{k-2} = 1$ and $\varepsilon \notin \mathbb{R}$. In consequence,

- if $k = 2$, then let $\varepsilon = i$.
- if $k \geq 5$, then let $\varepsilon = \cos \dfrac{2\pi}{k-2} + i \sin \dfrac{2\pi}{k-2} \notin \mathbb{R}$ (since $\dfrac{2\pi}{k-2} \in (0, \pi)$).

This means that $k \in \{1, 3, 4\}$. We prove next that the statement (*) is true for these values of k.
For $k = 1$, if $A \cdot A^* = A$, then $A^* = (A \cdot A^*)^* = (A^*)^* \cdot A^* = A \cdot A^* = A$, so (*) is true.
For $k \in \{3, 4\}$, we provide two methods.

First method.
$A^k A^* = A$ implies that $\operatorname{rank} A = \operatorname{rank}\left(A^k A^*\right) \leq \operatorname{rank} A^k \leq \operatorname{rank} A$, so $\operatorname{rank} A^k = \operatorname{rank} A = \operatorname{rank} A^*$. By the rank–nullity theorem, it follows that $\dim\ker A^k = \dim\ker A = \dim\ker A^*$. Since $\operatorname{Ker} A^* \subseteq \operatorname{Ker} A$ (by $A^k A^* = A$) and $\operatorname{Ker} A \subseteq \operatorname{Ker} A^k$, we obtain

$$\operatorname{Ker} A^* = \operatorname{Ker} A^k = \operatorname{Ker} A. \tag{1}$$

Next, $A^k A^* A^{k-1} = A \cdot A^{k-1} = A^k$, so $A^k \left(A^* A^{k-1} - I_n\right) = O_n$, then $A^* \left(A^* A^{k-1} - I_n\right) = O_n$, by (1), hence

$$\left(A^*\right)^2 A^{k-1} = A^*. \tag{2}$$

For $k = 3$, (2) becomes $\left(A^*\right)^2 A^2 = A^*$, so $A = \left(\left(A^*\right)^2 A^2\right)^* = \left(A^*\right)^2 A^2 = A^*$, which means that the statement (*) is true.

For $k = 4$, (2) becomes $\left(A^*\right)^2 A^3 = A^*$, so $\left(A^*\right)^2 A^4 A^* = \left(A^*\right)^2 A^3 \cdot A A^* = A^* A A^*$. At the same time, $\left(A^*\right)^2 A^4 A^* = \left(A^*\right)^2 A$, so $\left(A^*\right)^2 A = A^* A A^*$, which leads to $\left(A^*\right)^2 A^2 = \left(A^* A\right)^2$. With $B = A^* A - A A^*$, we have $B^* = B$ and

$$\operatorname{Tr} BB^* = \operatorname{Tr} B^2 = \operatorname{Tr}\left(A^* A - A A^*\right)^2 = 2\left(\operatorname{Tr}\left(A^* A\right)^2 - \operatorname{Tr}\left(\left(A^*\right)^2 A^2\right)\right) = 0,$$

hence $B = O_n$. This proves that $A^* A = A A^*$ (i.e., A is normal), so A is unitarily diagonalizable, $A = U^* D U$, $\quad D = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ with $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}$, $\quad U \in \mathcal{M}_n(\mathbb{C})$ with $U^{-1} = U^*$.Then $A^* = U^* \overline{D} U$, and $A^4 A^* = A$ becomes $D^4 \overline{D} = D$, which means that $\lambda_i^4 \overline{\lambda_i} = \lambda_i$, for all $i = 1, 2, \ldots, n$. It follows that $\lambda_i \in \{-1, 0, 1\}$, for all $i = 1, 2, \ldots, n$, so $\overline{D} = D$, therefore $A^* = A$, which means that the statement (*) is true.

Second method. We continue from relation (1) (from the first method).
It is true in general, for any matrix $A \in \mathcal{M}_n(\mathbb{C})$, that $\operatorname{Ker} A^* \perp \operatorname{Im} A$ [indeed, if $Y \in \operatorname{Ker} A^*$ and $Z = AX \in \operatorname{Im} A$, then $\langle Z, Y \rangle = \langle AX, Y \rangle = \langle X, A^* Y \rangle = \langle X, O \rangle = 0$].

Next, by (1), it follows that $\operatorname{Ker} A \perp \operatorname{Im} A$, so $\mathbb{C}^n = \operatorname{Ker} A \oplus \operatorname{Im} A$.

Consider an orthonormal basis in $\operatorname{Ker} A$ and an orthonormal basis in $\operatorname{Im} A$, which together give an orthonormal basis in $\mathbb{C}^n$ such that $A = U^* A_1 U$, where $A_1 = \begin{bmatrix} B & O \\ O & O \end{bmatrix}$ with $B \in \mathcal{M}_m(\mathbb{C})$ invertible, and $U \in \mathcal{M}_n(\mathbb{C})$ with $U^{-1} = U^*$. Then the relation $A^k A^* = A$ becomes $B^k B^* = B$, hence $B^* = (B^{-1})^{k-1}$. From the Cayley-Hamilton theorem, it follows that $B^{-1} = f(B)$ for some polynomial f of degree at most $n-1$, so $B^* = (f(B))^{k-1}$, which leads to $B^* B = B B^*$ (B is normal). Just like in the previous approach, B is unitarily diagonalizable, $B = V^* D V$, $\quad D = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_m)$ with $\lambda_1, \lambda_2, \ldots, \lambda_m \neq 0$, $\quad V \in \mathcal{M}_m(\mathbb{C})$ with $V^{-1} = V^*$.Then $B^* = V^* \overline{D} V$, and the relation $B^k B^* = B$ becomes $D^k \overline{D} = D$, which leads to $\lambda_i^{k-1} \overline{\lambda_i} = 1$, for all i. It follows that $|\lambda_i| = 1$ and $\lambda_i^{k-2} = 1$, for all i. When $k = 3$ or $k = 4$, then $\lambda_i \in \{-1, 1\}$ for all i, so $\overline{D} = D$, therefore $B^* = B$, then $A^* = A$, which means that the statement (*) is true.

Conclusion: $k \in \{1, 3, 4\}$.

Marking scheme:

1. Solve case $k = 1$ 1p
2. Eliminate $k = 2$ and $k \geq 5$ 3p
3. Find the relation $\operatorname{Ker} A^* = \operatorname{Ker} A^k = \operatorname{Ker} A$ 2p

Now, we solve cases $k \in \{3, 4\}$ with two methods.

First method

1. Find the relation $(A^*)^2 A^{k-1} = A^*$. 1p
2. Solve case $k = 3$ 1p
3. Solve case $k = 4$ 2p

Second method

1. $\mathbb{C}^n = \operatorname{Ker} A \oplus \operatorname{Im} A$ 1p
2. Consider an orthonormal basis in $\operatorname{Ker} A$ and an orthonormal basis in $\operatorname{Im} A$ using these basis to write $A = U^* A_1 U$, where $A_1 = \begin{bmatrix} B & O \\ O & O \end{bmatrix}$ with $B \in \mathcal{M}_m(\mathbb{C})$ invertible
   .... 1p
3. Find B is normal and therefore $B^* = B$ from which the conclusion folows for $k \in \{3, 4\}$
   .... 2p

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