Notes on the von Neumann algebra of a group

Ioannis Emmanouil

Contents

1 The strong and weak operator topologies on $\mathcal{B}(\mathcal{H})$

Let $\mathcal{H}$ be a Hilbert space. Besides the operator norm topology, the algebra $\mathcal{B}(\mathcal{H})$ can be also endowed with the strong operator topology (SOT). The latter is the locally convex topology which is induced by the family of semi-norms $(Q_\xi)_{\xi\in\mathcal{H}}$, where

$$Q_\xi(a) = \| a(\xi) \|$$

for all $\xi \in \mathcal{H}$ and $a \in \mathcal{B}(\mathcal{H})$. Hence, a net of operators $(a_\lambda)_\lambda$ in $\mathcal{B}(\mathcal{H})$ is SOT-convergent to 0 if and only if $\lim_\lambda a_\lambda(\xi) = 0$ for all $\xi \in \mathcal{H}$. The weak operator topology (WOT) on $\mathcal{B}(\mathcal{H})$ is the locally convex topology which is induced by the family of semi-norms $(P_{\xi,\eta})_{\xi,\eta\in\mathcal{H}}$, where

$$P_{\xi,\eta}(a) = |< a(\xi), \eta >|$$

for all $\xi, \eta \in \mathcal{H}$ and $a \in \mathcal{B}(\mathcal{H})$. In other words, a net of operators $(a_\lambda)_\lambda$ in $\mathcal{B}(\mathcal{H})$ is WOT-convergent to 0 if and only if $\lim_\lambda < a_\lambda(\xi), \eta > = 0$ for all $\xi, \eta \in \mathcal{H}$.

Remarks 1.1 (i) Let $(a_\lambda)_\lambda$ be a net of operators on $\mathcal{H}$. Then, we have

$$\| \cdot \| - \lim_\lambda a_\lambda = 0 \Longrightarrow \text{SOT} - \lim_\lambda a_\lambda = 0 \Longrightarrow \text{WOT} - \lim_\lambda a_\lambda = 0.$$

If the Hilbert space $\mathcal{H}$ is not finite dimensional, none of the implications above can be reversed (cf. Exercise 5.1).

(ii) For any $a \in \mathcal{B}(\mathcal{H})$ we consider the left (resp. right) multiplication operator

$$L_a : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H}) \ \ (\text{resp. } R_a : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})),$$

which is defined by letting $L_a(b) = ab$ (resp. $R_a(b) = ba$) for all $b \in \mathcal{B}(\mathcal{H})$. It is easily seen that the operators L_a and R_a are WOT-continuous. On the other hand, if the Hilbert space $\mathcal{H}$ is not finite dimensional, then the multiplication in $\mathcal{B}(\mathcal{H})$ is not (jointly) WOT-continuous (cf. Exercise 5.1).

(iii) The adjoint operator

$$(\_)^* : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H}),$$

which is defined by $a \mapsto a^*$, $a \in \mathcal{B}(\mathcal{H})$, is WOT-continuous.

Proposition 1.2 *Let $(a_\lambda)_\lambda$ be a bounded net of operators on $\mathcal{H}$. Then, the following conditions are equivalent:*

(i) $WOT - \lim_\lambda a_\lambda = 0$.

(ii) There is an orthonormal basis $(e_i)_i$ of the Hilbert space $\mathcal{H}$, such that for all i, j we have $\lim_\lambda < a_\lambda(e_i), e_j > = 0$.

(iii) There is a subset $B \subseteq \mathcal{H}$, whose closed linear span is $\mathcal{H}$, such that for all $\xi, \eta \in B$ we have $\lim_\lambda < a_\lambda(\xi), \eta > = 0$.

(iv) There is a dense subset $X \subseteq \mathcal{H}$, such that $\lim_\lambda < a_\lambda(\xi), \eta > = 0$ for all $\xi, \eta \in X$.

Proof. It is clear that (i)→(ii)→(iii), whereas the implication (iii)→(iv) follows by letting X be the (algebraic) linear span of B. It only remains to show that (iv)→(i). To that end, assume that $M > 0$ is such that $\| a_\lambda \| \leq M$ for all λ and consider two vectors $\xi, \eta \in \mathcal{H}$. For any positive ϵ we may choose two vectors $\xi', \eta' \in X$, such that $\| \xi - \xi' \| < \epsilon$ and $\| \eta - \eta' \| < \epsilon$. Since

$$< a_\lambda(\xi), \eta > - < a_\lambda(\xi'), \eta' > = < a_\lambda(\xi - \xi'), \eta > + < a_\lambda(\xi'), \eta - \eta' >,$$

it follows that

$$\begin{aligned}
|{<}a_\lambda(\xi),\eta{>} - {<}a_\lambda(\xi'),\eta'{>}| &\leq |{<}a_\lambda(\xi-\xi'),\eta{>}| + |{<}a_\lambda(\xi'),\eta-\eta'{>}| \\
&\leq \|a_\lambda\| \cdot \|\xi-\xi'\| \cdot \|\eta\| + \\
&\quad \|a_\lambda\| \cdot \|\xi'\| \cdot \|\eta-\eta'\| \\
&\leq M\epsilon(\|\xi\| + \|\eta\| + \epsilon).
\end{aligned}$$

Since $\lim_\lambda {<}a_\lambda(\xi'),\eta'{>} = 0$, we may choose λ_0 such that $|{<}a_\lambda(\xi'),\eta'{>}| < \epsilon$ for all $\lambda \geq \lambda_0$. Then, $|{<}a_\lambda(\xi),\eta{>}| < \epsilon(1 + M(\|\xi\| + \|\eta\| + \epsilon))$ for all $\lambda \geq \lambda_0$ and hence $\lim_\lambda {<}a_\lambda(\xi),\eta{>} = 0$, as needed. □

Theorem 1.3 *Let $\mathcal{H}$ be a separable Hilbert space, r a positive real number and $\mathcal{B}(\mathcal{H})_r = \{a \in \mathcal{B}(\mathcal{H}) : \|a\| \leq r\}$ the closed r-ball of $\mathcal{B}(\mathcal{H})$. Then, the topological space $(\mathcal{B}(\mathcal{H})_r, WOT)$ is compact and metrizable.*

Proof. In order to prove compactness, we consider for any $\xi, \eta \in \mathcal{H}$ the closed disc

$$D_{\xi,\eta} = \{z \in \mathbf{C} : |z| \leq r\|\xi\| \cdot \|\eta\|\} \subseteq \mathbf{C}$$

and the product space $\prod_{\xi,\eta\in\mathcal{H}} D_{\xi,\eta}$. In view of Tychonoff's theorem, the latter space is compact. We now define the map

$$f : \mathcal{B}(\mathcal{H})_r \longrightarrow \prod\nolimits_{\xi,\eta\in\mathcal{H}} D_{\xi,\eta},$$

by letting $f(a) = ({<}a(\xi),\eta{>})_{\xi,\eta}$ for all $a \in \mathcal{B}(\mathcal{H})_r$. It is clear that f is a homeomorphism of $(\mathcal{B}(\mathcal{H})_r, \text{WOT})$ onto its image. Therefore, the compactness of $(\mathcal{B}(\mathcal{H})_r, \text{WOT})$ will follow, as soon as we prove that the image $\text{im}\, f$ of f is closed in $\prod_{\xi,\eta\in\mathcal{H}} D_{\xi,\eta}$. To that end, let $(z_{\xi,\eta})_{\xi,\eta}$ be an element in the closure of $\text{im}\, f$. Then, the family $(z_{\xi,\eta})_{\xi,\eta}$ is easily seen to be linear in ξ and quasi-linear in η, whereas $|z_{\xi,\eta}| \leq r\|\xi\| \cdot \|\eta\|$ for all ξ, η. Hence, there is a vector $a_\xi \in \mathcal{H}$ with $\|a_\xi\| \leq r\|\xi\|$, such that $z_{\xi,\eta} = {<}a_\xi, \eta{>}$ for all $\xi, \eta \in \mathcal{H}$. Using the linearity of the family $(z_{\xi,\eta})_{\xi,\eta}$ in the first variable, it follows that there is an operator $a \in \mathcal{B}(\mathcal{H})_r$, such that $a_\xi = a(\xi)$ for all $\xi \in \mathcal{H}$. Then, $(z_{\xi,\eta})_{\xi,\eta} = f(a) \in \text{im}\, f$, as needed.

In order to prove metrizability, we fix an orthonormal basis $(e_n)_{n=0}^\infty$ of the separable Hilbert space $\mathcal{H}$ and define for any $a, b \in \mathcal{B}(\mathcal{H})_r$

$$d_r(a,b) = \sum\nolimits_{n,m} \frac{1}{2^{n+m}} \, |{<}(b-a)(e_n), e_m{>}| \; .$$

It is easily seen that d_r is a metric on $\mathcal{B}(\mathcal{H})_r$, which induces, in view of Proposition 1.2, the weak operator topology on $\mathcal{B}(\mathcal{H})_r$. □

Our next goal is to prove a result of von Neumann, describing the closure of unital self-adjoint subalgebras of $\mathcal{B}(\mathcal{H})$ in the weak and strong operator topologies in purely algebraic terms. To that end, we consider for any subset $X \subseteq \mathcal{B}(\mathcal{H})$ the commutant

$$X' = \{a \in \mathcal{B}(\mathcal{H}) : ax = xa \text{ for all } x \in X\}.$$

The bicommutant X'' of X is the commutant of X'. It is clear that $X \subseteq X''$.

Lemma 1.4 *For any $X \subseteq \mathcal{B}(\mathcal{H})$ the commutant X' is WOT-closed.*

Proof. For any operator $x \in \mathcal{B}(\mathcal{H})$ we consider the linear endomorphisms L_x and R_x of $\mathcal{B}(\mathcal{H})$, which are given by left and right multiplication with x respectively. Then, $X' = \bigcap_{x\in X} \ker(L_x - R_x)$ and hence the result follows from Remark 1.1(ii). □

If n is a positive integer and $X \subseteq \mathcal{B}(\mathcal{H})$ a set of operators, we shall consider the set $X \cdot I_n = \{xI_n : x \in X\} \subseteq \mathbf{M}_n(\mathcal{B}(\mathcal{H})) \simeq \mathcal{B}(\mathcal{H}^n)$. Then, the following two properties are easily verified (cf. Exercise 5.2):

(i) The commutant $(X \cdot I_n)'$ of $X \cdot I_n$ in $\mathbf{M}_n(\mathcal{B}(\mathcal{H})) \simeq \mathcal{B}(\mathcal{H}^n)$ is the set $\mathbf{M}_n(X')$ of matrices with entries in the commutant X' of X in $\mathcal{B}(\mathcal{H})$.

(ii) The bicommutant $(X \cdot I_n)''$ of $X \cdot I_n$ in $\mathbf{M}_n(\mathcal{B}(\mathcal{H})) \simeq \mathcal{B}(\mathcal{H}^n)$ is the set $X'' \cdot I_n$, where X'' is the bicommutant of X in $\mathcal{B}(\mathcal{H})$.

Lemma 1.5 *Let $\mathcal{A}$ be a self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$ and $V \subseteq \mathcal{H}$ a closed $\mathcal{A}$-invariant subspace. Then:*

(i) The orthogonal complement $V^\perp$ is $\mathcal{A}$-invariant.

(ii) If p is the orthogonal projection onto V, then $p \in \mathcal{A}'$.

(iii) The subspace V is $\mathcal{A}''$-invariant.

Proof. (i) Let $\xi \in V^\perp$ and $a \in \mathcal{A}$. Then, for any vector $\eta \in V$ we have $a^*(\eta) \in \mathcal{A}V \subseteq V$ and hence $<a(\xi), \eta> = <\xi, a^*(\eta)> = 0$. Therefore, it follows that $a(\xi) \in V^\perp$.

(ii) We fix an operator $a \in \mathcal{A}$ and note that the subspaces V and $V^\perp$ are a-invariant, in view of our assumption and (i) above. It follows easily from this that the operators ap and pa coincide on both V and $V^\perp$. Hence, $ap = pa$.

(iii) Let $\xi \in V$, $a'' \in \mathcal{A}''$ and consider the orthogonal projection p onto V. In view of (ii) above, we have $a''p = pa''$ and hence $a''(\xi) = a''p(\xi) = pa''(\xi) \in V$, as needed. □

We are now ready to state and prove von Neumann's theorem.

Theorem 1.6 *(von Neumann bicommutant theorem) Let $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ be a self-adjoint subalgebra containing the identity operator. Then, $\overline{\mathcal{A}}^{SOT} = \overline{\mathcal{A}}^{WOT} = \mathcal{A}''$, where we denote by $\overline{\mathcal{A}}^{SOT}$ (resp. $\overline{\mathcal{A}}^{WOT}$) the SOT-closure (resp. WOT-closure) of $\mathcal{A}$ in $\mathcal{B}(\mathcal{H})$.*

Proof. It is clear that $\overline{\mathcal{A}}^{SOT} \subseteq \overline{\mathcal{A}}^{WOT}$. Since $\mathcal{A} \subseteq \mathcal{A}''$, it follows from Lemma 1.4 that $\overline{\mathcal{A}}^{WOT} \subseteq \mathcal{A}''$. Hence, it only remains to show that $\mathcal{A}'' \subseteq \overline{\mathcal{A}}^{SOT}$. In order to verify this, we consider an operator $a'' \in \mathcal{A}''$, a positive real number ϵ, a positive integer n and vectors $\xi_1, \ldots, \xi_n \in \mathcal{H}$. We have to show that the SOT-neighborhood

$$\mathcal{N}_{\epsilon,\xi_1,\ldots,\xi_n}(a'') = \{a \in \mathcal{B}(\mathcal{H}) : \| (a - a'')\xi_i \| < \epsilon \text{ for all } i = 1, \ldots, n\}$$

of a'' intersects $\mathcal{A}$ non-trivially. To that end, we consider the self-adjoint subalgebra $\mathcal{A} \cdot I_n \subseteq \mathbf{M}_n(\mathcal{B}(\mathcal{H}))$ acting on the Hilbert space $\mathcal{H}^n$ by left multiplication and the closed subspace

$$V = \overline{\{(a(\xi_1), \ldots, a(\xi_n)) : a \in \mathcal{A}\}} \subseteq \mathcal{H}^n.$$

It is clear that V is $\mathcal{A} \cdot I_n$-invariant. Invoking Lemma 1.5(iii) and the discussion preceding it, we conclude that the subspace V is left invariant under the action of the operator $a''I_n \in \mathbf{M}_n(\mathcal{B}(\mathcal{H}))$. Since $1 \in \mathcal{A}$, we have $(\xi_1, \ldots, \xi_n) \in V$ and hence $(a''(\xi_1), \ldots, a''(\xi_n)) \in V$. Therefore, there is an operator $a \in \mathcal{A}$, such that

$$\| (a''(\xi_1), \ldots, a''(\xi_n)) - (a(\xi_1), \ldots, a(\xi_n)) \| < \epsilon.$$

Then, $\| a''(\xi_i) - a(\xi_i) \| < \epsilon$ for all $i = 1, \ldots, n$ and hence $a \in \mathcal{N}_{\epsilon,\xi_1,\ldots,\xi_n}(a'')$, as needed. □

A von Neumann algebra of operators acting on $\mathcal{H}$ is a self-adjoint subalgebra $\mathcal{N} \subseteq \mathcal{B}(\mathcal{H})$, which is WOT-closed and contains the identity 1. Equivalently, in view of von Neumann's bicommutant theorem, a von Neumann algebra $\mathcal{N}$ is a self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$, such that $\mathcal{N} = \mathcal{N}''$. It is clear that any von Neumann algebra $\mathcal{N}$ as above is closed under the operator norm topology of $\mathcal{B}(\mathcal{H})$; in particular, $\mathcal{N}$ is a unital C^*-algebra.

Lemma 1.7 *Let $\mathcal{A}$ be a von Neumann algebra of operators acting on the Hilbert space $\mathcal{H}$. For any idempotent $e \in \mathcal{A}$ there is a projection $f \in \mathcal{A}$, such that $ef = f$ and $fe = e$.*

Proof. Since $e \in \text{Idem}(\mathcal{A})$, the subspace $V = \text{im}\, e$ is easily seen to be closed and $\mathcal{A}'$-invariant. Therefore, Lemma 1.5(ii) implies that the orthogonal projection f onto V is contained in $\mathcal{A}''$. Invoking Theorem 1.6, we conclude that $f \in \mathcal{A}$. The equalities $ef = f$ and $fe = e$ follow since e and f are idempotent operators on $\mathcal{H}$ with the same image. □

Let $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ be a self-adjoint algebra of operators containing 1 and $\mathcal{N}$ its WOT-closure. Then, any operator $a \in \mathcal{N}$ can be approximated (in the weak operator topology) by a net $(a_\lambda)_\lambda$ of operators from $\mathcal{A}$. The following result, which is cited without proof, implies that the net $(a_\lambda)_\lambda$ can be chosen to be bounded.

Theorem 1.8 *(Kaplansky density theorem) Let $\mathcal{A}$ be a self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$ containing 1 and $\mathcal{N}$ its WOT-closure. Then, for any positive real number r the r-ball $\mathcal{A}_r = \mathcal{A} \cap \mathcal{B}(\mathcal{H})_r$ of $\mathcal{A}$ is WOT-dense in the r-ball $\mathcal{N}_r = \mathcal{N} \cap \mathcal{B}(\mathcal{H})_r$ of $\mathcal{N}$.* □

2 The von Neumann algebra of a group

Given a (discrete) group G, we consider the Hilbert space $\ell^2 G$ of square summable complex-valued functions on G with canonical orthonormal basis $(\delta_g)_{g\in G}$. In other words, $\ell^2 G$ consists of vectors of the form $\sum_{g\in G} r_g \delta_g$, where the r_g's are complex numbers such that $\sum_{g\in G} |r_g|^2 < \infty$. The inner product of two vectors $\xi = \sum_{g\in G} r_g \delta_g$ and $\xi' = \sum_{g\in G} r'_g \delta_g$ is given by

$$<\xi, \xi'> = \sum\nolimits_{g\in G} r_g \overline{r'_g}.$$

For any element $g \in G$ we consider the linear endomorphism L_g of $\ell^2 G$, which is defined by letting

$$L_g\left(\sum\nolimits_{x\in G} r_x \delta_x\right) = \sum\nolimits_{x\in G} r_x \delta_{gx}$$

for any vector $\sum_{x\in G} r_x \delta_x \in \ell^2 G$. It is easily seen that $L_1 = 1$ and $L_{gh} = L_g L_h$ for all $g, h \in G$. Moreover, L_g is an isometry and hence $L_g^* = L_g^{-1} = L_{g^{-1}}$ for all $g \in G$. We shall consider the $\mathbf{C}$-linear map

$$L : \mathbf{C}G \longrightarrow \mathcal{B}(\ell^2 G),$$

which extends the map $g \mapsto L_g$, $g \in G$. For any element $a \in \mathbf{C}G$ we shall denote its image in $\mathcal{B}(\ell^2 G)$ by L_a. We note that the group algebra $\mathbf{C}G$ can be endowed with the structure of a $*$-algebra, by letting $\left(\sum_{g\in G} a_g g\right)^* = \sum_{g\in G} \overline{a_g} g^{-1}$ for all $\sum_{g\in G} \lambda_g g \in \mathbf{C}G$.

Lemma 2.1 *Let G be a group and $L : \mathbf{C}G \longrightarrow \mathcal{B}(\ell^2 G)$ the linear map defined above. Then, L is an injective $*$-algebra homomorphism and hence the subalgebra $L(\mathbf{C}G) \subseteq \mathcal{B}(\ell^2 G)$ is self-adjoint.*

Proof. It is clear that L is an algebra homomorphism. For any $a = \sum_{g \in G} a_g g \in \mathbf{C}G$, where $a_g \in \mathbf{C}$ for all $g \in G$, we have $L_a = \sum_{g \in G} a_g L_g$ and hence $L_a(\delta_1) = \sum_{g \in G} a_g \delta_g \in \ell^2 G$. It follows readily from this that L is injective. We now let $a = \sum_{g \in G} a_g g \in \mathbf{C}G$, where $a_g \in \mathbf{C}$ for all $g \in G$, and consider the associated operator $L_a = \sum_{g \in G} a_g L_g \in L(\mathbf{C}G)$. Since $L_g^* = L_{g^{-1}}$ for all $g \in G$, it follows that $L_a^* = \sum_{g \in G} \overline{a_g} L_{g^{-1}} = L_{a^*} \in L(\mathbf{C}G)$. □

We define the reduced C^*-algebra $C_r^* G$ of G to be the operator norm closure of $L(\mathbf{C}G)$ in $\mathcal{B}(\ell^2 G)$; then, $C_r^* G$ is a unital C^*-algebra We also define the group von Neumann algebra $\mathcal{N}G$ as the WOT-closure of $L(\mathbf{C}G)$ in $\mathcal{B}(\ell^2 G)$. Since $\mathcal{N}G$ is closed under the operator norm topology, it contains $C_r^* G$; hence, there are inclusions $L(\mathbf{C}G) \subseteq C_r^* G \subseteq \mathcal{N}G \subseteq \mathcal{B}(\ell^2 G)$.

Remark 2.2 Assume that the group G is finite of order n. Then, the Hilbert space $\ell^2 G$ is identified with $\mathbf{C}^n$ and hence $\mathcal{B}(\ell^2 G) \simeq \mathbf{M}_n(\mathbf{C})$. Moreover, all three topologies defined above on $\mathcal{B}(\ell^2 G)$ (i.e. operator norm topology, SOT and WOT) coincide with the standard Cartesian product topology on $\mathbf{M}_n(\mathbf{C}) \simeq \mathbf{C}^{n^2}$. Since any linear subspace is closed therein, it follows that $L(\mathbf{C}G) = C_r^* G = \mathcal{N}G$.

In the following lemma we describe certain properties that are satisfied by the operators in the von Neumann algebra $\mathcal{N}G$.

Lemma 2.3 *Let G be a group and consider an operator $a \in \mathcal{N}G$.*

(i) If $(\delta_g)_{g \in G}$ denotes the canonical orthonormal basis of $\ell^2 G$, then we have $< a(\delta_g), \delta_{hg} > \, = \, < a(\delta_1), \delta_h >$ for all $g, h \in G$.[1]

(ii) For any vector $\xi \in \ell^2 G$ and any group element $g \in G$ the family of complex numbers $(< a(\delta_1), \delta_x > \cdot < \xi, \delta_{x^{-1}g} >)_x$ is summable and

$$\sum_{x \in G} < a(\delta_1), \delta_x > \cdot < \xi, \delta_{x^{-1}g} > \, = \, < a(\xi), \delta_g > .$$

(iii) If $a(\delta_1) = 0 \in \ell^2 G$ then a is the zero operator.

Proof. (i) First of all, let us consider the case where $a = L_x$ for some $x \in G$. In that case, we have to prove that $< \delta_{xg}, \delta_{hg} > \, = \, < \delta_x, \delta_h >$. But this equality is obvious, since $xg = hg$ if and only if $x = h$. Both sides of the formula to be proved are linear and WOT-continuous in a and hence the result follows from the special case considered above, since $\mathcal{N}G$ is the WOT-closure of the linear span of the set $\{L_x : x \in G\}$.

(ii) Since $\xi = \sum_x < \xi, \delta_x > \delta_x$, it follows that $a(\xi) = \sum_x < \xi, \delta_x > a(\delta_x)$. In view of the linearity and continuity of the inner product, we conclude that

$$\begin{aligned} < a(\xi), \delta_g > &= \sum_x < \xi, \delta_x > \cdot < a(\delta_x), \delta_g > \\ &= \sum_x < \xi, \delta_x > \cdot < a(\delta_1), \delta_{gx^{-1}} > \\ &= \sum_y < \xi, \delta_{y^{-1}g} > \cdot < a(\delta_1), \delta_y >, \end{aligned}$$

where the second equality follows from (i) above.

(iii) If $a(\delta_1) = 0$, then the equality of (ii) above implies that the inner product $< a(\xi), \delta_g >$ vanishes for all vectors $\xi \in \ell^2 G$ and all group elements $g \in G$. It follows readily from this that $a = 0$. □

[1] In fact, this property characterizes the operators in $\mathcal{N}G$; cf. Exercise 5.3.

We note that the linear functional $r_1 : \mathbf{C}G \longrightarrow \mathbf{C}$, which maps an element $a \in \mathbf{C}G$ onto the coefficient of $1 \in G$ in a, extends to a linear functional

$$\tau : \mathcal{N}G \longrightarrow \mathbf{C},$$

by letting $\tau(a) = < a(\delta_1), \delta_1 >$ for all $a \in \mathcal{N}G$.

Remark 2.4 Let G be a group and τ the linear functional defined above. Then, the assertion of Lemma 2.3(i) implies that $\tau(a) = < a(\delta_g), \delta_g >$ for all $a \in \mathcal{N}G$ and $g \in G$.

Proposition 2.5 *Let G be a group and τ the linear functional defined above. Then:*

(i) τ is a WOT-continuous trace.

(ii) τ is positive and faithful, i.e. $\tau(a^*a) \geq 0$ for all $a \in \mathcal{N}G$, whereas $\tau(a^*a) = 0$ if and only if $a = 0$.

(iii) τ is normalized, i.e. $\tau(1) = 1$, where $1 \in \mathcal{N}G$ is the identity operator.

The trace τ will be referred to as the canonical trace on the von Neumann algebra $\mathcal{N}G$.

Proof. (i) It is clear that τ is WOT-continuous. In order to show that τ is a trace, we fix an operator $a \in \mathcal{N}G$ and note that for any $g \in G$ we have

$$< aL_g(\delta_{g^{-1}}), \delta_{g^{-1}} > = < a(\delta_1), \delta_{g^{-1}} > = < a(\delta_1), L_g^*(\delta_1) > = < L_g a(\delta_1), \delta_1 >,$$

where the second equality follows since $L_g^* = L_{g^{-1}}$. Invoking Remark 2.4, we conclude that $\tau(aL_g) = \tau(L_g a)$. This being the case for all $g \in G$, it follows that $\tau(aa') = \tau(a'a)$ for all $a' \in L(\mathbf{C}G)$. Since multiplication in $\mathcal{B}(\ell^2 G)$ is separately WOT-continuous (cf. Remark 1.1(ii)), the WOT-continuity of τ implies that $\tau(aa') = \tau(a'a)$ for all $a' \in \mathcal{N}G$.

(ii) For any $a \in \mathcal{N}G$ we have

$$\tau(a^*a) = < a^*a(\delta_1), \delta_1 > = < a(\delta_1), a(\delta_1) > = \| a(\delta_1) \|^2 \geq 0.$$

In particular, $\tau(a^*a) = 0$ if and only if $a(\delta_1) = 0$; this proves the final assertion, in view of Lemma 2.3(iii).

(iii) We compute $\tau(1) = < \delta_1, \delta_1 > = \| \delta_1 \|^2 = 1$. □

3 The center of $\mathcal{N}G$

Let us consider the subset $G_f \subseteq G$, which consists of all elements $g \in G$ that have finitely many conjugates. Since the cardinality of the conjugacy class $[g]$ of any element $g \in G$ is equal to the index of the centralizer C_g of g in G, it follows that $G_f = \{g \in G : [G : C_g] < \infty\}$. We shall denote by $\mathcal{C}(G)$ the set of conjugacy classes of the elements of G and let $\mathcal{C}_f(G)$ be the subset of $\mathcal{C}(G)$ that consists of those conjugacy classes $[g]$, for which $g \in G_f$.

Lemma 3.1 *Let G_f and $\mathcal{C}_f(G)$ be the sets defined above. Then:*

(i) G_f is a characteristic (and hence normal) subgroup of G.

(ii) For any commutative ring k the center $Z(kG)$ of the group algebra kG is a free k-module with basis consisting of the elements $\zeta_{[g]} = \sum\{x : x \in [g]\}$, $[g] \in \mathcal{C}_f(G)$.

Proof. (i) It is clear that G_f is non-empty, since $1 \in G_f$. We note that for any two elements $g_1, g_2 \in G$ the intersection $C_{g_1} \cap C_{g_2}$ is contained in the centralizer of the product $g_1 g_2$. In particular, if $g_1, g_2 \in G_f$ then

$$[G : C_{g_1 g_2}] \leq [G : C_{g_1} \cap C_{g_2}] \leq [G : C_{g_1}]\,[G : C_{g_2}] < \infty$$

and hence $g_1 g_2 \in G_f$. For any element $g \in G$ we have $C_g = C_{g^{-1}}$; therefore, $g^{-1} \in G_f$ if $g \in G_f$. We have proved that G_f is a subgroup of G. In order to prove that G_f is characteristic in G, let us consider an automorphism $\sigma : G \longrightarrow G$. Then, σ restricts to a bijection between the conjugacy classes $[g]$ and $[\sigma(g)]$ for any element $g \in G$. In particular, $g \in G_f$ if and only if $\sigma(g) \in G_f$.

(ii) It is clear that the subset $\{\zeta_{[g]} : [g] \in \mathcal{C}_f(G)\} \subseteq kG$ is linearly independent over k. Moreover, $x\zeta_{[g]}x^{-1} = \zeta_{[g]}$ for all $x \in G$ and hence $\zeta_{[g]} \in Z(kG)$ for all $[g] \in \mathcal{C}_f(G)$. In order to show that the $\zeta_{[g]}$'s form a basis of $Z(kG)$, let us consider a central element $a = \sum_{g \in G} a_g g \in kG$, where $a_g \in k$ for all $g \in G$. Then, $a = xax^{-1}$ for all $x \in G$ and hence $a_g = a_{x^{-1}gx}$ for all $g, x \in G$. Therefore, the function $g \mapsto a_g$, $g \in G$, is constant on conjugacy classes. Since its support is finite, that function must vanish on the infinite conjugacy classes. It follows that a is a linear combination of the $\zeta_{[g]}$'s, as needed. □

Let $\mathcal{Z}G$ be the center of the von Neumann algebra $\mathcal{N}G$; it is clear that $\mathcal{Z}G = \mathcal{N}G \cap (\mathcal{N}G)'$, being WOT-closed, is itself a von Neumann algebra of operators on $\ell^2 G$. Our next goal is to identify $\mathcal{Z}G$ with the WOT-closure of the self-adjoint subalgebra $Z(L(\mathbf{C}G)) \subseteq \mathcal{B}(\ell^2 G)$. We note that

$$Z(L(\mathbf{C}G)) = L(\mathbf{C}G) \cap (L(\mathbf{C}G))' \subseteq L(\mathbf{C}G)'' \cap (L(\mathbf{C}G))''' = \mathcal{N}G \cap (\mathcal{N}G)'.$$

Hence, $\mathcal{Z}G$ being WOT-closed, we have $\overline{Z(L(\mathbf{C}G))}^{WOT} \subseteq \mathcal{Z}G$. In order to prove the reverse inclusion, we shall need a couple of auxiliary results.

Lemma 3.2 *Let $a \in \mathcal{Z}G$ be an operator in the center of $\mathcal{N}G$. Then:*

(i) For all $g, h \in G$ we have $<a(\delta_1), \delta_{g^{-1}hg}> = <a(\delta_1), \delta_h>$.

(ii) The inner product $<a(\delta_1), \delta_g>$ depends only upon the conjugacy class $[g] \in \mathcal{C}(G)$ and vanishes if $g \notin G_f$.

(iii) For any $g \in G$ we have

$$a(\delta_g) = \sum\nolimits_{[x] \in \mathcal{C}_f(G)} <a(\delta_1), \delta_x> L_{\zeta_{[x]}}(\delta_g) \in \ell^2 G.$$

Proof. (i) We fix the elements $g, h \in G$ and compute

$$\begin{aligned} <a(\delta_1), \delta_{g^{-1}hg}> &= <a(\delta_1), L_{g^{-1}}(\delta_{hg})> \\ &= <L^*_{g^{-1}} a(\delta_1), \delta_{hg}> \\ &= <L_g a(\delta_1), \delta_{hg}> \\ &= <a L_g(\delta_1), \delta_{hg}> \\ &= <a(\delta_g), \delta_{hg}> \\ &= <a(\delta_1), \delta_h> \end{aligned}$$

In the above chain of equalities, the third one follows since $L^*_{g^{-1}} = L_g$, the fourth one since a commutes with L_g, whereas the last one was established in Lemma 2.3(i).

(ii) It follows from (i) that the function $g \mapsto <a(\delta_1), \delta_g>$, $g \in G$, is constant on conjugacy classes. Being square-summable, that function must vanish on those elements $g \in G$ with infinitely many conjugates.

(iii) It follows from (i) and (ii) above that

$$\begin{aligned} a(\delta_1) &= \sum\nolimits_{[x] \in \mathcal{C}_f(G)} <a(\delta_1), \delta_x> \sum \{\delta_{x'} : x' \in [x]\}) \\ &= \sum\nolimits_{[x] \in \mathcal{C}_f(G)} <a(\delta_1), \delta_x> L_{\zeta_{[x]}}(\delta_1). \end{aligned} \qquad (1)$$

On the other hand, for any $g \in G$ the operator L_g commutes with a (since $a \in \mathcal{Z}G$) and $L_{\zeta_{[x]}}$ for any $x \in G_f$ (since the $L_{\zeta_{[x]}}$'s are central in $L(\mathbf{C}G)$; cf. Lemma 3.1(ii)). Therefore, we have

$$\begin{aligned} a(\delta_g) &= aL_g(\delta_1) \\ &= L_g a(\delta_1) \\ &= \sum\nolimits_{[x]\in\mathcal{C}_f(G)} <a(\delta_1),\delta_x> L_g L_{\zeta_{[x]}}(\delta_1) \\ &= \sum\nolimits_{[x]\in\mathcal{C}_f(G)} <a(\delta_1),\delta_x> L_{\zeta_{[x]}} L_g(\delta_1) \\ &= \sum\nolimits_{[x]\in\mathcal{C}_f(G)} <a(\delta_1),\delta_x> L_{\zeta_{[x]}}(\delta_g). \end{aligned}$$

In the above chain of equalities, the third one follows from Eq.(1), in view of the continuity of L_g. □

Corollary 3.3 *Let $a \in \mathcal{Z}G$ be an operator in the center of $\mathcal{N}G$ and $b \in (Z(L(\mathbf{C}G)))'$ an operator in the commutant of $Z(L(\mathbf{C}G))$ in $\mathcal{B}(\ell^2G)$. Then, for any two elements $g,h \in G$ the family of complex numbers $(<a(\delta_1),\delta_x> \cdot <b(\delta_g),\delta_{x^{-1}h}>)_{x\in G}$ is summable and*

$$\sum\nolimits_{x\in G} <a(\delta_1),\delta_x> \cdot <b(\delta_g),\delta_{x^{-1}h}> = <ba(\delta_g),\delta_h> .$$

Proof. In view of the continuity of b, Lemma 3.2(iii) implies that

$$\begin{aligned} ba(\delta_g) &= \sum\nolimits_{[x]\in\mathcal{C}_f(G)} <a(\delta_1),\delta_x> bL_{\zeta_{[x]}}(\delta_g) \\ &= \sum\nolimits_{[x]\in\mathcal{C}_f(G)} <a(\delta_1),\delta_x> L_{\zeta_{[x]}} b(\delta_g) \\ &= \sum\nolimits_{x\in G} <a(\delta_1),\delta_x> L_x b(\delta_g). \end{aligned}$$

In the above chain of equalities, the second one follows since b commutes with $L_{\zeta_{[x]}} \in Z(L(\mathbf{C}G))$ for all $[x] \in \mathcal{C}_f(G)$ (cf. Lemma 3.1(ii)), whereas the last one is a consequence of Lemma 3.2(ii). Therefore, we have

$$\begin{aligned} <ba(\delta_g),\delta_h> &= \sum\nolimits_{x\in G} <a(\delta_1),\delta_x> \cdot <L_x b(\delta_g),\delta_h> \\ &= \sum\nolimits_{x\in G} <a(\delta_1),\delta_x> \cdot <b(\delta_g),L_x^*(\delta_h)> \\ &= \sum\nolimits_{x\in G} <a(\delta_1),\delta_x> \cdot <b(\delta_g),L_{x^{-1}}(\delta_h)> \\ &= \sum\nolimits_{x\in G} <a(\delta_1),\delta_x> \cdot <b(\delta_g),\delta_{x^{-1}h}>, \end{aligned}$$

where the first equality follows from the continuity of the inner product $<\_,\delta_h>$ and the third one from the equalities $L_x^* = L_{x^{-1}}$, $x \in G$. □

We are now ready to prove the following result, describing the center of the von Neumann algebra $\mathcal{N}G$.

Proposition 3.4 *The center $\mathcal{Z}G$ of the von Neumann algebra $\mathcal{N}G$ is the WOT-closure in $\mathcal{B}(\ell^2G)$ of the center $Z(L(\mathbf{C}G))$ of the algebra $L(\mathbf{C}G)$.*

Proof. As we have already noted, the von Neumann algebra $\mathcal{Z}G$ contains the WOT-closure of $Z(L(\mathbf{C}G))$. On the other hand, the WOT-closure of the $*$-algebra $Z(L(\mathbf{C}G))$ coincides with its bicommutant in $\mathcal{B}(\ell^2G)$ (cf. Theorem 1.6). Hence, it only remains to show that $\mathcal{Z}G \subseteq (Z(L(\mathbf{C}G)))''$, i.e. that any $a \in \mathcal{Z}G$ commutes with any $b \in (Z(L(\mathbf{C}G)))'$. Let us fix such a pair of operators a,b. Since $a \in \mathcal{Z}G \subseteq \mathcal{N}G$, we have

$$<a(\xi),\delta_h> = \sum\nolimits_{x\in G} <a(\delta_1),\delta_x> \cdot <\xi,\delta_{x^{-1}h}>$$

for all $\xi \in \ell^2 G$ and $h \in G$ (cf. Lemma 2.3(ii)). In particular, we have

$$<ab(\delta_g), \delta_h> = \sum_{x \in G} <a(\delta_1), \delta_x> \cdot <b(\delta_g), \delta_{x^{-1}h}>$$

for all $g, h \in G$. Therefore, Corollary 3.3 implies that

$$<ab(\delta_g), \delta_h> = <ba(\delta_g), \delta_h>$$

for all $g, h \in G$ and hence $ab = ba$, as needed. □

Remark 3.5 Let $\mathcal{H}$ be a Hilbert space, $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ a unital self-adjoint subalgebra and $\mathcal{N}$ its WOT-closure. Even though the center $Z(\mathcal{N})$ of $\mathcal{N}$ always contains the WOT-closure of the center $Z(\mathcal{A})$ of $\mathcal{A}$, the inclusion $\overline{Z(\mathcal{A})}^{WOT} \subseteq Z(\mathcal{N})$ may be proper (in contrast to the situation described in Proposition 3.4); cf. Exercise 5.4.

4 The center-valued trace on $\mathcal{N}G$

Our goal is to construct a trace

$$t = t_G : \mathcal{N}G \longrightarrow \mathcal{Z}G,$$

which is WOT-continuous on bounded sets, maps $\mathcal{Z}G$ identically onto itself and is closely related to the canonical trace τ.

I. The trace on $\mathbf{C}G$. We shall begin by defining t on the group algebra $\mathbf{C}G$. More precisely, we define the linear map

$$t_0 : \mathbf{C}G \longrightarrow Z(\mathbf{C}G),$$

by letting $t_0(g) = 0$ if $g \notin G_f$ and $t_0(g) = \frac{1}{[G:C_g]}\zeta_{[g]}$ if $g \in G_f$.[2]

Proposition 4.1 *Let $t_0 : \mathbf{C}G \longrightarrow Z(\mathbf{C}G)$ be the $\mathbf{C}$-linear map defined above. Then:*
(i) t_0 is a trace with values in $Z(\mathbf{C}G)$,
(ii) $t_0(a) = a$ for all $a \in Z(\mathbf{C}G)$,
(iii) $t_0(aa') = at_0(a')$ for all $a \in Z(\mathbf{C}G)$ and $a' \in \mathbf{C}G$ (i.e. t_0 is $Z(\mathbf{C}G)$-linear),
(iv) $t_0(a^*) = t_0(a)^*$ for all $a \in \mathbf{C}G$ and
(v) the trace functional r_1 on $\mathbf{C}G$ factors as the composition

$$\mathbf{C}G \xrightarrow{t_0} Z(\mathbf{C}G) \xrightarrow{r'_1} \mathbf{C},$$

where r'_1 is the restriction of r_1 to the center $Z(\mathbf{C}G)$.

Proof. (i) Since t_0 is $\mathbf{C}$-linear, it suffices to show that $t_0(g) = t_0(g')$ whenever $[g] = [g'] \in \mathcal{C}(G)$. But this is an immediate consequence of the definition of t_0.

(ii) We consider an element $g \in G_f$ with $[G : C_g] = n$ and let $[g] = \{g_1, \ldots, g_n\}$. Then, $t_0(g_i) = t_0(g)$ for all $i = 1, \ldots, n$ and hence

$$t_0(\zeta_{[g]}) = t_0\Big(\sum_{i=1}^n g_i\Big) = \sum_{i=1}^n t_0(g_i) = nt_0(g) = \zeta_{[g]}.$$

[2]This definition is imposed by the requirement that t_0 extends to a trace on the von Neumann algebra $\mathcal{N}G$ with values in $\mathcal{Z}G$, which is WOT-continuous on bounded sets and maps $\mathcal{Z}G$ identically onto itself; cf. Exercise 5.5.

Since t_0 is $\mathbf{C}$-linear, the proof is finished by invoking Lemma 3.1(ii).

(iii) We consider an element $g \in G_f$ with $[G : C_g] = n$ and let $[g] = \{g_1, \ldots, g_n\}$; then, $g_i \in G_f$ for all $i = 1, \ldots, n$. If $g' \in G$ is an element with $g' \notin G_f$, then (G_f being a subgroup of G, in view of Lemma 3.1(i)) $g_i g' \notin G_f$ for all $i = 1, \ldots, n$. In particular,

$$t_0(\zeta_{[g]} g') = t_0\Big(\sum\nolimits_{i=1}^n g_i g'\Big) = \sum\nolimits_{i=1}^n t_0(g_i g') = 0 = \zeta_{[g]} t_0(g').$$

We now assume that $g' \in G_f$ and consider the conjugacy class $[g'] = \{g'_1, \ldots, g'_m\}$, where $m = [G : C_{g'}]$. Then, for any $j = 1, \ldots, m$ there exists an element $x_j \in G$, such that $g'_j = x_j g' x_j^{-1}$. Since $\zeta_{[g]}$ is central in $\mathbf{C}G$, we have $\zeta_{[g]} g'_j = x_j \zeta_{[g]} g' x_j^{-1}$ and hence (t_0 being a trace, in view of (i) above) $t_0\Big(\zeta_{[g]} g'_j\Big) = t_0\Big(\zeta_{[g]} g'\Big)$ for all $j = i, \ldots, m$. It follows that

$$\zeta_{[g]}\zeta_{[g']} = t_0\Big(\zeta_{[g]}\zeta_{[g']}\Big) = t_0\Big(\sum\nolimits_{j=1}^m \zeta_{[g]} g'_j\Big) = \sum\nolimits_{j=1}^m t_0\Big(\zeta_{[g]} g'_j\Big) = m t_0\Big(\zeta_{[g]} g'\Big),$$

where the first equality is a consequence of (ii) above, since the element $\zeta_{[g]}\zeta_{[g']}$ is central in $\mathbf{C}G$. We conclude that

$$t_0\Big(\zeta_{[g]} g'\Big) = \frac{1}{m}\zeta_{[g]}\zeta_{[g']} = \zeta_{[g]} t_0(g')$$

in this case as well. Therefore, we have proved that $t_0\Big(\zeta_{[g]} g'\Big) = \zeta_{[g]} t_0(g')$ for all $g' \in G$. Since this is the case for any $g \in G_f$, the linearity of t_0, combined with Lemma 3.1(ii), finishes the proof.

(iv) Since both sides of the equality to be proved are conjugate linear in a, it suffices to consider the case where $a = g$, for some element $g \in G$. In that case, we have $a^* = g^{-1}$. If $g \in G_f$ and $[g] = \{g_1, \ldots, g_n\}$, then $g^{-1} \in G_f$ and $[g^{-1}] = \Big\{g_1^{-1}, \ldots, g_n^{-1}\Big\}$. Therefore, we have

$$t_0(a^*) = t_0\Big(g^{-1}\Big) = \sum\nolimits_{i=1}^n g_i^{-1} = \Big(\sum\nolimits_{i=1}^n g_i\Big)^* = t_0(g)^* = t_0(a)^*.$$

If g is not contained in G_f, which is a subgroup of G, then g^{-1} is not contained in G_f either and hence both $t_0(a)^* = t_0(g)^*$ and $t_0(a^*) = t_0(g^{-1})$ vanish.

(v) It suffices to verify that the linear functionals $r'_1 \circ t_0$ and r_1 have the same value on g for all $g \in G$. But this follows immediately from the definitions. $\square$

II. A factorization of the trace on $\mathbf{C}G$. In order to extend the trace t_0 defined above to the von Neumann algebra $\mathcal{N}G$, we shall consider the linear maps

$$\Delta : \mathbf{C}G \longrightarrow \mathbf{C}G_f \quad \text{and} \quad c : \mathbf{C}G_f \longrightarrow Z(\mathbf{C}G),$$

which are defined by letting Δ map any group element $g \in G$ onto g (resp. onto 0) if $g \in G_f$ (resp. if $g \notin G_f$) and c map any $g \in G_f$ onto $\frac{1}{[G:C_g]}\zeta_{[g]}$. Then, t_0 can be expressed as the composition

$$\mathbf{C}G \xrightarrow{\Delta} \mathbf{C}G_f \xrightarrow{c} Z(\mathbf{C}G).$$

Viewing the algebras above as algebras of operators acting on $\ell^2 G$ by left translations, we shall study the continuity properties of Δ and c and show that both of them extend to the respective WOT-closures.

III. The map Δ. We begin by considering a (possibly infinite) family $(\mathcal{H}_s)_{s \in S}$ of Hilbert spaces and define $\mathcal{H}$ to be the corresponding Hilbert space direct sum. Then, $\mathcal{H} = \bigoplus_{s \in S} \mathcal{H}_s$

consists of those elements $\xi = (\xi_s)_s \in \prod_{s\in S} \mathcal{H}_s$, for which the series $\sum_{s\in S} \| \xi_s \|_s^2$ is convergent. (Here, we denote for any $s \in S$ by $\|\cdot\|_s$ the norm of the Hilbert space $\mathcal{H}_s$.) The inner product on $\mathcal{H}$ is defined by letting $<\xi, \eta> = \sum_{s\in S} <\xi_s, \eta_s>_s$ for any two vectors $\xi = (\xi_s)_s$ and $\eta = (\eta_s)_s$ of $\mathcal{H}$, where $<\_, \_>_s$ denotes the inner product of $\mathcal{H}_s$ for all $s \in S$. The Hilbert spaces $\mathcal{H}_s$, $s \in S$, admit isometric embeddings as closed orthogonal subspaces of $\mathcal{H}$ by means of the operators $\iota_s : \mathcal{H}_s \longrightarrow \mathcal{H}$, which map an element $\xi_s \in \mathcal{H}_s$ onto the element $\iota_s(\xi_s) = (\eta_{s'})_{s'} \in \mathcal{H}$ with $\eta_s = \xi_s$ and $\eta_{s'} = 0$ for $s' \neq s$. Then, the Hilbert space $\mathcal{H}$ is the closed linear span of the subspaces $\iota_s(\mathcal{H}_s)$, $s \in S$. For any index $s \in S$ we shall also consider the projection $P_s : \mathcal{H} \longrightarrow \mathcal{H}_s$, which maps an element $\xi = (\xi_{s'})_{s'} \in \mathcal{H}$ onto $\xi_s \in \mathcal{H}_s$. It is clear that P_s is a continuous linear map with $\| P_s \| \leq 1$ for all $s \in S$. Moreover, for any vectors $\xi \in \mathcal{H}$ and $\eta_s \in \mathcal{H}_s$ we have $< P_s(\xi), \eta_s >_s = < \xi, \iota_s(\eta_s) >$; therefore, $P_s = \iota_s^*$ is the adjoint of ι_s for all $s \in S$.

Let us consider a bounded operator $a \in \mathcal{B}(\mathcal{H})$ and a vector $\xi = (\xi_s)_s \in \mathcal{H}$. Then, the family $(P_s a \iota_s(\xi_s))_s \in \prod_{s\in S} \mathcal{H}_s$ is also a vector in $\mathcal{H}$, since

$$\begin{aligned} \sum_{s\in S} \| P_s a \iota_s(\xi_s) \|_s^2 &\leq \sum_{s\in S} \| a \iota_s(\xi_s) \|^2 \\ &\leq \| a \|^2 \sum_{s\in S} \| \iota_s(\xi_s) \|^2 \\ &= \| a \|^2 \sum_{s\in S} \| \xi_s \|_s^2 \\ &= \| a \|^2 \| \xi \|^2 . \end{aligned} \tag{2}$$

This is the case for any $\xi \in \mathcal{H}$ and hence we may consider the map

$$\Delta(a) : \mathcal{H} \longrightarrow \mathcal{H},$$

which maps an element $\xi = (\xi_s)_s \in \mathcal{H}$ onto $\Delta(a)(\xi) = (P_s a \iota_s(\xi_s))_s \in \mathcal{H}$. Itcis clear that the map $\Delta(a)$ is linear. Moreover, it follows from (2) that $\Delta(a)$ is a bounded operator; in fact, we have $\| \Delta(a) \| \leq \| a \|$. Therefore, we may consider the map

$$\Delta : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H}),$$

which is given by $a \mapsto \Delta(a)$, $a \in \mathcal{B}(\mathcal{H})$. The map Δ is linear and continuous with respect to the operator norm topology on $\mathcal{B}(\mathcal{H})$; in fact, $\| \Delta \| \leq 1$.[3] It is easily seen that

$$\Delta(a)\iota_s = \iota_s P_s a \iota_s \tag{3}$$

for all $a \in \mathcal{B}(\mathcal{H})$ and all indices $s \in S$. Since Δ is a contraction, it induces by restriction to the r-ball a map

$$\Delta_r : (\mathcal{B}(\mathcal{H}))_r \longrightarrow (\mathcal{B}(\mathcal{H}))_r$$

for any radius r. Of course, Δ_r is continuous with respect to the operator norm topology on $(\mathcal{B}(\mathcal{H}))_r$.

Lemma 4.2 *The map Δ_r defined above is WOT-continuous for any r.*

Proof. Let $(a_\lambda)_\lambda$ be a bounded net of operators in $\mathcal{B}(\mathcal{H})$, which is WOT-convergent to 0. In order to show that the net $(\Delta(a_\lambda))_\lambda$ of operators in $\mathcal{B}(\mathcal{H})$ is WOT-convergent to 0 as well, it suffices, in view of Proposition 1.2, to show that $\lim_\lambda < \Delta(a_\lambda)(\xi), \eta > = 0$, whenever

[3] The decomposition $\mathcal{H} = \bigoplus_{s\in S} \mathcal{H}_s$ identifies the algebra $\mathcal{B}(\mathcal{H})$ with a certain algebra of $S \times S$ matrices whose (s, s')-entry consists of bounded operators from $\mathcal{H}_{s'}$ to $\mathcal{H}_s$ for all $s, s' \in S$. Under this identification, the linear map Δ maps any $a = (a_{ss'})_{s,s'\in S}$ onto the diagonal matrix $\text{diag}\{a_{ss} : s \in S\}$.

there are two indices $s, s' \in S$ and vectors $\xi_s \in \mathcal{H}_s$ and $\eta_{s'} \in \mathcal{H}_{s'}$, such that $\xi = \iota_s(\xi_s)$ and $\eta = \iota_{s'}(\eta_{s'})$. Since

$$\Delta(a_\lambda)(\xi) = \Delta(a_\lambda)\iota_s(\xi_s) = \iota_s P_s a_\lambda \iota_s(\xi_s)$$

(cf. Eq.(3)), the inner product $< \Delta(a_\lambda)(\xi), \eta > = < \Delta(a_\lambda)(\xi), \iota_{s'}(\eta_{s'}) >$ vanishes if $s \neq s'$. On the other hand, if $s = s'$ we have

$$\begin{aligned} < \Delta(a_\lambda)(\xi), \eta > &= < \iota_s P_s a_\lambda \iota_s(\xi_s), \iota_s(\eta_s) > \\ &= < P_s a_\lambda \iota_s(\xi_s), \eta_s >_s \\ &= < a_\lambda \iota_s(\xi_s), \iota_s(\eta_s) >, \end{aligned}$$

where the last equality follows since $P_s = \iota_s^*$. Since WOT-$\lim_\lambda a_\lambda = 0$, we conclude that $\lim_\lambda < \Delta(a_\lambda)(\xi), \eta > = 0$ in this case as well. □

In order to apply the conclusion of Lemma 4.2, we consider the group G and a subgroup $H \leq G$. If S is a set of representatives of the left cosets of H in G, then the decomposition of G into the disjoint union of the cosets Hs, $s \in S$, induces a Hilbert space decomposition $\ell^2 G = \bigoplus_{s \in S} \ell^2(Hs)$. We consider the operator

$$\Delta : \mathcal{B}(\ell^2 G) \longrightarrow \mathcal{B}(\ell^2 G),$$

which is associated with that decomposition as above. In particular, let us fix an element $g \in G$ and try to identify the operator $\Delta(L_g) \in \mathcal{B}(\ell^2 G)$. For any $x \in G$ there is a unique $s = s(x) \in S$, such that $x \in Hs$. Then,

$$\Delta(L_g)(\delta_x) = \Delta(L_g)\iota_s(\delta_x) = \iota_s P_s L_g \iota_s(\delta_x) = \iota_s P_s L_g(\delta_x) = \iota_s P_s(\delta_{gx}),$$

where the second equality follows from Eq.(3). We note that $gx \in Hs$ if and only if $g \in H$ and hence $\Delta(L_g)(\delta_x)$ is equal to $\iota_s(\delta_{gx}) = \delta_{gx}$ if $g \in H$ and vanishes if $g \notin H$. Since this is the case for all $x \in G$, we conclude that

$$\Delta(L_g) = \begin{cases} L_g & \text{if } g \in H \\ 0 & \text{if } g \notin H \end{cases}$$

In particular, $\Delta(L_g)$ is an element of the subalgebra $L(\mathbf{C}H) \subseteq \mathcal{B}(\ell^2 G)$. (We note that here $L(\mathbf{C}H)$ is viewed as an algebra of operators acting on $\ell^2 G$.) Hence, Δ restricts to a linear map

$$\Delta : L(\mathbf{C}G) \longrightarrow L(\mathbf{C}H) \subseteq \mathcal{B}(\ell^2 G).$$

Corollary 4.3 *Let H be a subgroup of G and consider the linear operator*

$$\Delta : L(\mathbf{C}G) \longrightarrow L(\mathbf{C}H) \subseteq \mathcal{B}(\ell^2 G),$$

which is defined above. Then:
(i) The operator Δ is a contraction.
(ii) The map

$$\Delta_r : (L(\mathbf{C}G))_r \longrightarrow (L(\mathbf{C}H))_r \subseteq (\mathcal{B}(\ell^2 G))_r,$$

induced from Δ by restriction to the respective r-balls, is WOT-continuous for any r. □

IV. The map c. We shall begin by considering a group N together with an automorphism $\phi : N \longrightarrow N$. Then, ϕ extends by linearity to an automorphism of the complex group algebra $\mathbf{C}N$, which will be still denoted (by an obvious abuse of notation) by ϕ. We shall also consider the associated automorphism L_ϕ of the algebra of operators $L(\mathbf{C}N) \subseteq \mathcal{B}(\ell^2 N)$, which is defined by letting $L_\phi(L_a) = L_{\phi(a)}$ for all $a \in \mathbf{C}N$. On the other hand, there is a unitary operator $\Phi \in \mathcal{B}(\ell^2 N)$, such that $\Phi(\delta_x) = \delta_{\phi(x)}$ for all $x \in N$; here, we denote by $(\delta_x)_{x \in N}$ the canonical orthonormal basis of $\ell^2 N$.

Lemma 4.4 *Let N be a group and ϕ an automorphism of N.*

(i) The associated isometry Φ of the Hilbert space $\ell^2 N$ is such that $L_{\phi(a)} \circ \Phi = \Phi \circ L_a \in \mathcal{B}(\ell^2 N)$ for all $a \in \mathbf{C}N$.

(ii) The automorphism L_ϕ of $L(\mathbf{C}N)$ is norm-preserving and WOT-continuous.

Proof. (i) By linearity, it suffices to verify that $L_{\phi(x)} \circ \Phi = \Phi \circ L_x$ for all $x \in N$. For any element $y \in N$ we have

$$(L_{\phi(x)} \circ \Phi)(\delta_y) = L_{\phi(x)}(\delta_{\phi(y)}) = \delta_{\phi(x)\phi(y)} = \delta_{\phi(xy)} = \Phi(\delta_{xy}) = (\Phi \circ L_x)(\delta_y).$$

Since the bounded operators $L_{\phi(x)} \circ \Phi$ and $\Phi \circ L_x$ agree on the orthonormal basis $\{\delta_y : y \in N\}$ of the Hilbert space $\ell^2 N$, they are equal.

(ii) For any $a \in \mathbf{C}N$ we have $L_{\phi(a)} = \Phi \circ L_a \circ \Phi^{-1}$, in view of (i) above. Since Φ is unitary, it follows that $\| L_{\phi(a)} \| = \| L_a \|$ for all $a \in \mathbf{C}N$ and hence L_ϕ is norm-preserving. On the other hand, the map $b \mapsto \Phi \circ b \circ \Phi^{-1}$, $b \in \mathcal{B}(\ell^2 N)$, is WOT-continuous (cf. Remark 1.1(ii)). Being a restriction of it, L_ϕ is WOT-continuous as well. □

We now assume that N is a group on which the group G acts by automorphisms. Then, for any $g \in G$ we are given an automorphism $\phi_g : N \longrightarrow N$, in such a way that $\phi_g \circ \phi_{g'} = \phi_{gg'}$ for all $g, g' \in G$. There is an induced action of G by automorphisms $(\phi_g)_g$ on the complex group algebra $\mathbf{C}N$ and a corresponding action of G by automorphisms $(L_{\phi_g})_g$ on the algebra of operators $L(\mathbf{C}N) \subseteq \mathcal{B}(\ell^2 N)$. More precisely, for any $g \in G$ the automorphism $\phi_g : \mathbf{C}N \longrightarrow \mathbf{C}N$ is the linear extension of $\phi_g \in \mathrm{Aut}(N)$, whereas $L_{\phi_g} : L(\mathbf{C}N) \longrightarrow L(\mathbf{C}N)$ maps L_a onto $L_{\phi_g(a)}$ for all $a \in \mathbf{C}N$.

If the G-action on N is such that all orbits are finite (equivalently, if for any element $x \in N$ the stabilizer subgroup Stab_x has finite index in G), then we define the linear map

$$c : L(\mathbf{C}N) \longrightarrow L(\mathbf{C}N),$$

as follows: For any $x \in N$ with G-orbit $\{x_1, \ldots, x_m\} \subseteq N$, where $m = m(x) = [G : \mathrm{Stab}_x]$, we let $c(L_x) = \frac{1}{m} \sum_{i=1}^{m} L_{x_i} \in L(\mathbf{C}N)$.

Lemma 4.5 *Assume that G acts on a group N by automorphisms, in such a way that all orbits are finite, and consider the linear operator c on $L(\mathbf{C}N)$ defined above.*

(i) Let x be an element of N and $H \leq G$ a subgroup of finite index with $H \subseteq Stab_x$. If $[G : H] = k$ and $\{g_1, \ldots, g_k\}$ is a set of representatives of the right H-cosets $\{gH : g \in G\}$, then $c(L_x) = \frac{1}{k} \sum_{i=1}^{k} L_{\phi_{g_i}(x)}$.

(ii) The operator c is a contraction.

(iii) The map

$$c_r : (L(\mathbf{C}N))_r \longrightarrow (L(\mathbf{C}N))_r,$$

induced from c by restriction to the r-balls, is WOT-continuous for any r.

Proof. (i) Since H is contained in the stabilizer Stab_x, we have $\phi_g(x) = \phi_{g'}(x) \in N$ if $gH = g'H$. Therefore, the right hand side of the equality to be proved doesn't depend upon the choice of the set of representatives of the cosets $\{gH : g \in G\}$. Let $\{s_1, \ldots, s_m\}$ be a set of representatives of the cosets $\{g\,\mathrm{Stab}_x : g \in G\}$, where $m = m(x) = [G : \mathrm{Stab}_x]$. Then, the G-orbit of x is the set $\{\phi_{s_1}(x), \ldots, \phi_{s_m}(x)\}$ and hence

$$c(L_x) = \frac{1}{m}\sum\nolimits_{i=1}^{m} L_{\phi_{s_i}(x)}.$$

We now let $\{u_1, \ldots, u_l\}$ be a set of representatives of the cosets $\{gH : g \in \mathrm{Stab}_x\}$, where $l = [\,\mathrm{Stab}_x : H]$. Then, the set $\{s_iu_j : 1 \le i \le m\ ,\ 1 \le j \le l\}$ is a set of representatives of the cosets $\{gH : g \in G\}$. In particular, $k = [G : H] = [G : \mathrm{Stab}_x] \cdot [\,\mathrm{Stab}_x : H] = ml$. Since the u_j's stabilize x, we have $\phi_{s_iu_j}(x) = \phi_{s_i}(x)$ for all i, j and hence

$$c(L_x) = \frac{1}{m}\sum\nolimits_{i=1}^{m} L_{\phi_{s_i}(x)} = \frac{l}{k}\sum\nolimits_{i=1}^{m} L_{\phi_{s_i}(x)} = \frac{1}{k}\sum\nolimits_{i=1}^{m}\sum\nolimits_{j=1}^{l} L_{\phi_{s_iu_j}(x)},$$

as needed.

(ii) Let $a = \sum_{i=1}^{r} a_ix_i \in \mathbf{C}N$, where $a_i \in \mathbf{C}$ and $x_i \in N$ for all $i = 1, \ldots, r$. We consider the subgroup $H = \bigcap_{i=1}^{r} \mathrm{Stab}_{x_i}$, which has finite index in G, and fix a set of representatives $\{g_1, \ldots, g_k\}$ of the cosets $\{gH : g \in G\}$. We note that $L_a = \sum_{i=1}^{r} a_iL_{x_i}$, whereas $L_{\phi_{g_j}(a)} = \sum_{i=1}^{r} a_iL_{\phi_{g_j}(x_i)}$ for all $j = 1, \ldots, k$. Hence, it follows from (i) above that

$$c(L_a) = \sum\nolimits_{i=1}^{r} a_ic(L_{x_i}) = \sum\nolimits_{i=1}^{r} a_i\frac{1}{k}\sum\nolimits_{j=1}^{k} L_{\phi_{g_j}(x_i)} = \frac{1}{k}\sum\nolimits_{j=1}^{k} L_{\phi_{g_j}(a)}.$$

Since $\parallel L_{\phi_{g_j}(a)} \parallel = \parallel L_a \parallel$ for all $j = 1, \ldots, k$ (cf. Lemma 4.4(ii)), we may conclude that $\| c(L_a) \| \le \| L_a \|$ and hence c is a contraction.

(iii) Let $(a_\lambda)_\lambda$ be a net of elements in the group algebra $\mathbf{C}N$, such that the net of operators $(L_{a_\lambda})_\lambda$ is bounded and WOT-convergent to $0 \in \mathcal{B}(\ell^2N)$. For any index λ we write $a_\lambda = \sum_{x \in N} a_{\lambda,x}x$, where the $a_{\lambda,x}$'s are complex numbers, and note that

$$< L_{a_\lambda}(\delta_1), \delta_x > = < \sum\nolimits_{x' \in N} a_{\lambda,x'}\delta_{x'}, \delta_x > = a_{\lambda,x}$$

for all $x \in N$; in particular, it follows that $\lim_\lambda a_{\lambda,x} = 0$ for all $x \in N$. In order to show that the bounded net $(c(L_{a_\lambda}))_\lambda$ of operators in $L(\mathbf{C}N) \subseteq \mathcal{B}(\ell^2N)$ is WOT-convergent to 0 as well, it suffices to show that

$$\lim\nolimits_\lambda < c(L_{a_\lambda})(\delta_y), \delta_z > = 0$$

for all $y, z \in N$ (cf. Proposition 1.2). For any pair of elements $x, x' \in N$ we write $x \sim x'$ if and only if x and x' are in the same orbit under the G-action, whereas $m(x)$ denotes the cardinality of the G-orbit of x. Then,

$$\begin{aligned} c(L_{a_\lambda}) &= \sum\nolimits_{x \in N} a_{\lambda,x}c(L_x) \\ &= \sum\nolimits_{x \in N} a_{\lambda,x}\tfrac{1}{m(x)}\sum\{L_{x'} : x' \sim x\} \\ &= \sum\nolimits_{x' \in N}\sum\{a_{\lambda,x}\tfrac{1}{m(x)} : x \sim x'\}L_{x'} \end{aligned}$$

and hence

$$< c(L_{a_\lambda})(\delta_y), \delta_z > = \sum\{a_{\lambda,x}\tfrac{1}{m(x)} : x \sim zy^{-1}\}.$$

Since $\lim_\lambda a_{\lambda,x} = 0$ for each one of the finitely many x's in the G-orbit of zy^{-1}, we conclude that $\lim_\lambda < c(L_{a_\lambda})(\delta_y), \delta_z > = 0$. □

Let $\mathcal{H}$ be a Hilbert space, S a non-empty index set and $\mathcal{H}^{(S)}$ the Hilbert space direct sum of the constant family of Hilbert spaces $(\mathcal{H}_s)_{s\in S}$ with $\mathcal{H}_s = \mathcal{H}$ for all $s \in S$. For any bounded operator $a \in \mathcal{B}(\mathcal{H})$ there is an associated linear operator $a^{(S)} : \mathcal{H}^{(S)} \longrightarrow \mathcal{H}^{(S)}$, which maps an element $(\xi_s)_s \in \mathcal{H}^{(S)}$ onto $(a(\xi_s))_s$. The map $a^{(S)}$ is well-defined, since for any $(\xi_s)_s \in \mathcal{H}^{(S)}$ we have

$$\sum_{s\in S} \| a(\xi_s) \|^2 \leq \sum_{s\in S} \| a \|^2 \| \xi_s \|^2 = \| a \|^2 \sum_{s\in S} \| \xi_s \|^2 < \infty.$$

It follows that the operator $a^{(S)}$ is bounded and $\| a^{(S)} \| \leq \| a \|$. In fact, we may fix an index $s \in S$ and consider the restriction of $a^{(S)}$ on the subspace $\iota_s(\mathcal{H})$, in order to conclude that $\| a^{(S)} \| = \| a \|$. Hence, the linear map

$$\nu : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}\left(\mathcal{H}^{(S)}\right),$$

which is given by $a \mapsto a^{(S)}$, $a \in \mathcal{B}(\mathcal{H})$, is an isometry and we may consider its restriction to the r-balls

$$\nu_r : (\mathcal{B}(\mathcal{H}))_r \longrightarrow \left(\mathcal{B}\left(\mathcal{H}^{(S)}\right)\right)_r.$$

Then, a net $(a_\lambda)_\lambda$ in $(\mathcal{B}(\mathcal{H}))_r$ is WOT-convergent to 0 if and only if this is the case for the associated net $\left(a_\lambda^{(S)}\right)_\lambda$ of operators on $\mathcal{H}^{(S)}$. Indeed, if WOT-$\lim_\lambda a_\lambda^{(S)} = 0$, then we may consider the restriction of the $a_\lambda^{(S)}$'s on the subspace $\iota_s(\mathcal{H}) \subseteq \mathcal{H}^{(S)}$, for some index $s \in S$, in order to conclude that WOT-$\lim_\lambda a_\lambda = 0$. Conversely, assume that the bounded net $(a_\lambda)_\lambda$ of operators in $\mathcal{B}(\mathcal{H})$ is WOT-convergent to 0. Then, for any pair of indices $s, s' \in S$ and any vectors $\xi, \xi' \in \mathcal{H}$, we have

$$< a_\lambda^{(S)} \iota_s(\xi), \iota_{s'}(\xi') > = < \iota_s a_\lambda(\xi), \iota_{s'}(\xi') > = \begin{cases} < a_\lambda(\xi), \xi' > & \text{if } s = s' \\ 0 & \text{if } s \neq s' \end{cases}$$

where the first equality follows since $a^{(S)}\iota_s = \iota_s a$ for any $a \in \mathcal{B}(\mathcal{H})$. In any case, we conclude that $\lim_\lambda < a_\lambda^{(S)} \iota_s(\xi), \iota_{s'}(\xi') > = 0$ and hence the bounded net $\left(a_\lambda^{(S)}\right)_\lambda$ is WOT-convergent to 0 (cf. Proposition 1.2).

Corollary 4.6 *Assume that G acts on a group N by automorphisms, in such a way that all orbits are finite. We consider a group N' containing N as a subgroup and let c be the linear operator on $L(\mathbf{C}N) \subseteq L(\mathbf{C}N') \subseteq \mathcal{B}(\ell^2 N')$, which is defined as in the paragraph before Lemma 4.5. Then:*

(i) The operator c is a contraction.

(ii) The map

$$c_r : (L(\mathbf{C}N))_r \longrightarrow (L(\mathbf{C}N))_r,$$

induced from c by restriction to the r-balls, is continuous with respect to the weak operator topology on $(L(\mathbf{C}N))_r \subseteq (\mathcal{B}(\ell^2 N'))_r$ for any r.

Proof. For any element $a \in \mathbf{C}N$ we shall denote by L_a (resp. L'_a) the left translation induced by a on the Hilbert space $\ell^2 N$ (resp. $\ell^2 N'$). If $S \subseteq N'$ is a set of representatives of the cosets $\{Nx : x \in N'\}$, then the Hilbert space $\ell^2 N' = \bigoplus_{s\in S} \ell^2(Ns)$ is naturally identified with $(\ell^2 N)^{(S)}$, in such a way that L'_a is identified with $L_a^{(S)}$ for all $a \in \mathbf{C}N$. Therefore, assertions (i) and (ii) are immediate consequences of Lemma 4.5(ii),(iii), in view of the discussion above. □

V. The WOT-continuity of the trace on $L(\mathbf{C}G)$. Since $L : \mathbf{C}G \longrightarrow L(\mathbf{C}G)$ is an algebra isomorphism, it follows that the center $Z(L(\mathbf{C}G))$ of $L(\mathbf{C}G)$ coincides with $L(Z(\mathbf{C}G))$, where $Z(\mathbf{C}G)$ is the center of $\mathbf{C}G$. Hence, the linear map $t_0 : \mathbf{C}G \longrightarrow Z(\mathbf{C}G)$ of Proposition 4.1 induces a linear map

$$t : L(\mathbf{C}G) \longrightarrow Z(L(\mathbf{C}G)),$$

by letting $t(L_a) = L_{t_0(a)}$ for any $a \in \mathbf{C}G$. Using the results obtained above, we can now establish certain key continuity properties of t.

Proposition 4.7 *Let $t : L(\mathbf{C}G) \longrightarrow Z(L(\mathbf{C}G))$ be the linear map defined above. Then:*
(i) t is a contraction and its restriction

$$t_r : (L(\mathbf{C}G))_r \longrightarrow (Z(L(\mathbf{C}G)))_r$$

to the respective r-balls is WOT-continuous for any r,
(ii) t is a trace with values in $Z(L(\mathbf{C}G))$,
(iii) $t(L_a) = L_a$ for all $L_a \in Z(L(\mathbf{C}G))$,
(iv) $t(L_a L_{a'}) = L_a t(L_{a'})$ for all $L_a \in Z(L(\mathbf{C}G))$ and $L_{a'} \in L(\mathbf{C}G)$ (i.e. t is $Z(L(\mathbf{C}G))$-linear),
(v) $t(L_a^*) = t(L_a)^*$ for all $L_a \in L(\mathbf{C}G)$ and
(vi) the canonical trace functional τ on $L(\mathbf{C}G)$ factors as the composition

$$L(\mathbf{C}G) \stackrel{t}{\longrightarrow} Z(L(\mathbf{C}G)) \stackrel{\tau'}{\longrightarrow} \mathbf{C},$$

where τ' is the restriction of τ to the center $Z(L(\mathbf{C}G))$.

Proof. (i) Let $G_f \trianglelefteq G$ be the normal subgroup consisting of those elements $g \in G$ that have finitely many conjugates and consider the linear map

$$\Delta : L(\mathbf{C}G) \longrightarrow L(\mathbf{C}G_f),$$

which is defined on the set of generators L_g, $g \in G$, by letting $\Delta(L_g) = L_g$ if $g \in G_f$ and $\Delta(L_g) = 0$ if $g \notin G_f$. The orbit of an element $g \in G_f$ under the conjugation action of G is the conjugacy class $[g] \in \mathcal{C}(G)$, a finite set with $[G : C_g]$ elements. We consider the linear map

$$c : L(\mathbf{C}G_f) \longrightarrow L(\mathbf{C}G_f),$$

which maps L_g onto $\frac{1}{[G:C_g]} \sum\{L_x : x \in [g]\}$ for all $g \in G_f$. It is clear that the composition

$$L(\mathbf{C}G) \stackrel{\Delta}{\longrightarrow} L(\mathbf{C}G_f) \stackrel{c}{\longrightarrow} L(\mathbf{C}G_f)$$

coincides with the composition

$$L(\mathbf{C}G) \stackrel{t}{\longrightarrow} Z(L(\mathbf{C}G)) \hookrightarrow L(\mathbf{C}G_f).$$

Therefore, (i) is a consequence of Corollaries 4.3 and 4.6. The proof of assertions (ii), (iii), (iv), (v) and (vi) follows readily from Proposition 4.1. □

VI. The construction of t on $\mathcal{N}G$. Using the results obtained above, we shall now construct the center-valued trace t on the von Neumann algebra $\mathcal{N}G$ of the countable group G. We note that the countability of G implies that the Hilbert space ℓ^2G is separable. For any radius r we consider the r-ball $(\mathcal{B}(\ell^2G))_r$ of the algebra of bounded operators on ℓ^2G. Then, the space $((\mathcal{B}(\ell^2G))_r, \text{WOT})$ is compact and metrizable; in fact, we can choose for any r a metric d_r on $((\mathcal{B}(\ell^2G))_r, \text{WOT})$, in such a way that

$$d_r(a,a') = d_{2r}(a'-a,0) \tag{4}$$

for all $a,a' \in (\mathcal{B}(\ell^2G))_r$ (cf. Theorem 1.3 and its proof). In view of Kaplansky's density theorem (Theorem 1.8), the r-ball $(\mathcal{N}G)_r$ is the WOT-closure of the r-ball $(L(\mathbf{C}G))_r$. It follows that $((\mathcal{N}G)_r, \text{WOT})$ is also a compact metric space; in particular, it is a complete metric space. In fact, $((\mathcal{N}G)_r, \text{WOT})$ can be identified with the completion of its dense subspace $((L(\mathbf{C}G))_r, \text{WOT})$. As an immediate consequence of the discussion above, we note that any operator in $\mathcal{N}G$ is the WOT-limit of a bounded sequence of operators in $L(\mathbf{C}G)$. Using a similar argument, combined with Proposition 3.4, we may identify the complete metric space $((\mathcal{Z}G)_r, \text{WOT})$ with the completion of its dense subspace $((Z(L(\mathbf{C}G)))_r, \text{WOT})$. It follows that any operator in $\mathcal{Z}G$ is the WOT-limit of a bounded sequence of operators in $Z(L(\mathbf{C}G))$.

We now consider the linear map $t : L(\mathbf{C}G) \longrightarrow Z(L(\mathbf{C}G))$ of Proposition 4.7. We know that t is a contraction, whereas its restriction t_r to the respective r-balls is WOT-continuous for all r. Having fixed the radius r, we note that the continuity of t_{2r} at 0 implies that for any $\varepsilon > 0$ there is $\delta = \delta(r,\varepsilon) > 0$, such that

$$d_{2r}(a,0) < \delta \Longrightarrow d_{2r}(t(a),0) < \varepsilon$$

for all $a \in (L(\mathbf{C}G))_{2r}$. Taking into account the linearity of t and Eq.(4), it follows that

$$d_r(a,a') < \delta \Longrightarrow d_r(t(a),t(a')) < \varepsilon$$

for all $a,a' \in (L(\mathbf{C}G))_r$. Therefore, the map

$$t_r : ((L(\mathbf{C}G))_r, \text{WOT}) \longrightarrow ((Z(L(\mathbf{C}G)))_r, \text{WOT})$$

is uniformly continuous and hence admits a unique extension to a continuous map between the completions

$$t_r : ((\mathcal{N}G)_r, \text{WOT}) \longrightarrow ((\mathcal{Z}G)_r, \text{WOT}). \tag{5}$$

Taking into account the uniqueness of these extensions, it follows that there is a well-defined map

$$t : \mathcal{N}G \longrightarrow \mathcal{Z}G,$$

which is contractive, extends $t : L(\mathbf{C}G) \longrightarrow Z(L(\mathbf{C}G))$ and its restriction to the respective r-balls is the WOT-continuous map t_r of (5) for all r.

Theorem 4.8 *Let $\mathcal{Z}G$ be the center of the von Neumann algebra $\mathcal{N}G$ and $t : \mathcal{N}G \longrightarrow \mathcal{Z}G$ the map defined above. Then:*

(i) t extends the trace $t_0 : \mathbf{C}G \longrightarrow Z(\mathbf{C}G)$, in the sense that the following diagram is commutative

$$\begin{array}{ccc} \mathbf{C}G & \xrightarrow{t_0} & Z(\mathbf{C}G) \\ {\scriptstyle L}\downarrow & & \downarrow{\scriptstyle L} \\ \mathcal{N}G & \xrightarrow{t} & \mathcal{Z}G \end{array}$$

(ii) t is a contraction and its restriction to bounded sets is WOT-continuous,
(iii) t is $\mathbf{C}$-linear,
(iv) t is a trace with values in $\mathcal{Z}G$,
(v) $t(a) = a$ for all $a \in \mathcal{Z}G$,
(vi) $t(aa') = at(a')$ for all $a \in \mathcal{Z}G$ and $a' \in \mathcal{N}G$ (i.e. t is $\mathcal{Z}G$-linear),
(vii) $t(a^*) = t(a)^*$ for all $a \in \mathcal{N}G$,
(viii) the canonical trace functional τ on $\mathcal{N}G$ factors as the composition

$$\mathcal{N}G \xrightarrow{t} \mathcal{Z}G \xrightarrow{\tau'} \mathbf{C},$$

where τ' is the restriction of τ on $\mathcal{Z}G$.
(ix) $t(a^*a)$ is non-zero and self-adjoint for all $a \in \mathcal{N}G \setminus \{0\}$.
The trace t is referred to as the center-valued trace on $\mathcal{N}G$.

Proof. Assertions (i) and (ii) follow from the construction of t.

(iii) As we have already noted, for any $a, a' \in \mathcal{N}G$ there are bounded sequences $(a_n)_n$ and $(a'_n)_n$ in $L(\mathbf{C}G)$, such that WOT-$\lim_n a_n = a$ and WOT-$\lim_n a'_n = a'$. Then, for any $\lambda, \lambda' \in \mathbf{C}$ the sequence $(\lambda a_n + \lambda' a'_n)_n$ is bounded and WOT-convergent to $\lambda a + \lambda' a'$. In view of the linearity of t on $L(\mathbf{C}G)$, we have $t(\lambda a_n + \lambda' a'_n) = \lambda t(a_n) + \lambda' t(a'_n)$ for all n. Since t is WOT-continuous on bounded sets, it follows that $t(\lambda a + \lambda' a') = \lambda t(a) + \lambda' t(a')$.

(iv) We recall that multiplication in $\mathcal{B}(\ell^2 G)$ is separately continuous in the weak operator topology (cf. Remark 1.1(ii)). For any element $a \in L(\mathbf{C}G)$ the map $a' \mapsto t(aa') - t(a'a)$, $a' \in \mathcal{N}G$, is WOT-continuous on bounded sets and vanishes on $L(\mathbf{C}G)$, in view of Proposition 4.7(ii). Therefore, approximating any operator of $\mathcal{N}G$ by a bounded sequence in $L(\mathbf{C}G)$, we conclude that $t(aa') = t(a'a)$ for all $a' \in \mathcal{N}G$. We now fix $a' \in \mathcal{N}G$ and consider the map $a \mapsto t(aa') - t(a'a)$, $a \in \mathcal{N}G$. This latter map is WOT-continuous on bounded sets and vanishes on $L(\mathbf{C}G)$, as we have just proved. Hence, using the same argument as above, we conclude that $t(aa') = t(a'a)$ for all $a \in \mathcal{N}G$.

(v) We know that any operator $a \in \mathcal{Z}G$ is the WOT-limit of a bounded sequence of operators in $Z(L(\mathbf{C}G))$; therefore, the equality $t(a) = a$ is an immediate consequence of Proposition 4.7(iii), in view of the WOT-continuity of t on bounded sets.

(vi) We fix an operator $a \in Z(L(\mathbf{C}G))$ and consider the map $a' \mapsto t(aa') - at(a')$, $a' \in \mathcal{N}G$. This map is WOT-continuous on bounded sets and vanishes on $L(\mathbf{C}G)$ (cf. Proposition 4.7(iv)). Approximating any operator of $\mathcal{N}G$ by a bounded sequence in $L(\mathbf{C}G)$, we conclude that $t(aa') = at(a')$ for all $a' \in \mathcal{N}G$. We now fix an element $a' \in \mathcal{N}G$ and consider the map $a \mapsto t(aa') - at(a')$, $a \in \mathcal{Z}G$. This map is WOT-continuous on bounded sets and vanishes on $Z(L(\mathbf{C}G))$, as we have just proved. Hence, approximating any operator of $\mathcal{Z}G$ by a bounded sequence in $Z(L(\mathbf{C}G))$, it follows that $t(aa') = at(a')$ for all $a \in \mathcal{Z}G$.

(vii) We know that any operator $a \in \mathcal{N}G$ is the WOT-limit of a bounded sequence of operators in $L(\mathbf{C}G)$, whereas the adjoint operator is WOT-continuous on $\mathcal{B}(\mathcal{H})$ (cf. Remark 1.1(iii)). Therefore, the equality $t(a^*) = t(a)^*$ is an immediate consequence of Proposition 4.1(v), in view of the WOT-continuity of t on bounded sets.

(viii) Since the trace τ is WOT-continuous, the equality $\tau = \tau' \circ t$ follows from the WOT-continuity of t on bounded sets, combined with Proposition 4.7(vi), by approximating any operator $a \in \mathcal{N}G$ by a bounded sequence of operators in $L(\mathbf{C}G)$.

(ix) In view of (vii) above, the operator $t(a^*a) \in \mathcal{Z}G$ is self-adjoint for all $a \in \mathcal{N}G$. Since $\tau(a^*a) = \tau(t(a^*a))$ (cf. (viii)), we may invoke Proposition 2.5(ii) in order to conclude that $t(a^*a) = 0$ only if $a = 0$. □

5 Exercises

1. Let $\ell^2\mathbf{N}$ be the Hilbert space of square summable sequences of complex numbers and consider the operators $a, b \in \mathcal{B}(\ell^2\mathbf{N})$, which are defined by letting $a(\xi_0, \xi_1, \xi_2, \ldots) = (\xi_1, \xi_2, \ldots)$ and $b(\xi_0, \xi_1, \xi_2, \ldots) = (0, \xi_0, \xi_1, \xi_2, \ldots)$ for all $(\xi_0, \xi_1, \xi_2, \ldots) \in \ell^2\mathbf{N}$.

   (i) Show that $\| a^n \| = \| b^n \| = 1$ for all $n \geq 1$.

   (ii) Show that the sequence $(a^n)_n$ is SOT-convergent to 0, but not norm-convergent to 0. In particular, the sequence $(a^n)_n$ is WOT-convergent to 0.

   (iii) Show that the sequence $(b^n)_n$ is WOT-convergent to 0, but not SOT-convergent to 0.

   (iv) Show that the sequence $(a^n b^n)_n$ is not WOT-convergent to 0. In particular, multiplication in $\mathcal{B}(\ell^2\mathbf{N})$ is not jointly WOT-continuous.

2. Let R be a ring, n a positive integer and $\mathbf{M}_n(R)$ the corresponding matrix ring. For any subset $X \subseteq R$ we consider the subset $\mathbf{M}_n(X)$ (resp. $X \cdot I_n$) of $\mathbf{M}_n(R)$, which consists of all $n \times n$ matrices with entries in X (resp. of all matrices of the form xI_n, $x \in X$). Show that:

   (i) The commutant $(\mathbf{M}_n(X))'$ of $\mathbf{M}_n(X)$ in $\mathbf{M}_n(R)$ is equal to $X' \cdot I_n$, where X' is the commutant of X in R. In particular, the center $Z(\mathbf{M}_n(R))$ of $\mathbf{M}_n(R)$ is equal to $Z(R) \cdot I_n$, where $Z(R)$ is the center of R.

   (ii) The commutant $(X \cdot I_n)'$ of $X \cdot I_n$ in $\mathbf{M}_n(R)$ is equal to $\mathbf{M}_n(X')$.

3. Let G be a group. The goal of this Exercise is to show that the property of Lemma 2.3(i) characterizes the operators in the von Neumann algebra $\mathcal{N}G$. To that end, let us fix an operator $a \in \mathcal{B}(\ell^2 G)$, for which $< a(\delta_g), \delta_{hg} > = < a(\delta_1), \delta_h >$ for all $g, h \in G$.

   (i) Show that for any operator $b \in \mathcal{B}(\ell^2 G)$ and any elements $g, h \in G$ the families of complex numbers $(< a(\delta_1), \delta_x > \cdot < b(\delta_g), \delta_{x^{-1}h} >)_x$ and $(< a(\delta_1), \delta_x > \cdot < b(\delta_{xg}), \delta_h >)_x$ are summable with sum $< ab(\delta_g), \delta_h >$ and $< ba(\delta_g), \delta_h >$ respectively.

   (ii) Assume that $b \in \mathcal{B}(\ell^2 G)$ is an operator in the commutant $L(\mathbf{C}G)'$ of the subalgebra $L(\mathbf{C}G) \subseteq \mathcal{B}(\ell^2 G)$. Then, show that $ab = ba$. In particular, conclude that $a \in L(\mathbf{C}G)'' = \mathcal{N}G$.

4. Let $\mathcal{H}$ be a Hilbert space, $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ a unital self-adjoint subalgebra and $\mathcal{N} = \mathcal{A}''$ its WOT-closure. Let $Z(\mathcal{A})$ be the center of $\mathcal{A}$ and $Z(\mathcal{N})$ the center of $\mathcal{N}$.

   (i) Show that $Z(\mathcal{N})$ contains the WOT-closure of $Z(\mathcal{A})$.

   In contrast to the situation described in Proposition 3.4, the inclusion $Z(\mathcal{A})'' \subseteq Z(\mathcal{N})$ may be proper. It is the goal of this Exercise to provide an example, which was communicated to me by E. Katsoulis, where $Z(\mathcal{A})'' \neq Z(\mathcal{N})$. To that end, we let $\mathcal{H}_0$ be an infinite dimensional Hilbert space and consider the Hilbert space $\mathcal{H} = \mathcal{H}_0 \oplus \mathbf{C}$.

   (ii) For any $a \in \mathcal{B}(\mathcal{H}_0)$ and any scalar $\lambda \in \mathbf{C}$ we consider the linear map $T(a, \lambda) : \mathcal{H} \longrightarrow \mathcal{H}$, which maps any element $(v, z) \in \mathcal{H}$ onto $(a(v) + \lambda v, \lambda z)$. Show that $T(a, \lambda) \in \mathcal{B}(\mathcal{H})$.

   (iii) Consider the ideal $\mathcal{F} \subseteq \mathcal{B}(\mathcal{H}_0)$ of finite rank operators and let

$$\mathcal{A} = \{T(a, \lambda) : a \in \mathcal{F}, \lambda \in \mathbf{C}\},$$

in the notation of (ii) above. Show that $\mathcal{A}$ is a unital self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$, whose center $Z(\mathcal{A})$ consists of the scalar multiples of the identity.

(iv) Let $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ be the subalgebra defined in (iii) above. Show that the center $Z(\mathcal{A}'')$ of the bicommutant $\mathcal{A}''$ is 2-dimensional and conclude that the inclusion $Z(\mathcal{A})'' \subseteq Z(\mathcal{A}'')$ is proper.

5. Let G be a countable group, $\mathcal{N}G$ the associated von Neumann algebra and $\mathcal{Z}G$ its center. We consider a $\mathbf{C}$-linear trace $t' : \mathcal{N}G \longrightarrow \mathcal{Z}G$, which is WOT-continuous on bounded sets and maps $\mathcal{Z}G$ identically onto itself. The goal of this Exercise is to show that t' coincides with the center-valued trace t constructed in Theorem 4.8.

   (i) Let $g \in G$ be an element with finitely many conjugates and C_g its centralizer in G. Show that $t'(L_g) = \frac{1}{[G:C_g]} L_{\zeta_{[g]}} \in \mathcal{Z}G$.

   (ii) Let $(g_n)_n$ be a sequence of distinct elements of G. Show that the sequence of operators $(L_{g_n})_n$ in $\mathcal{B}(\ell^2 G)$ is WOT-convergent to 0.

   (iii) Let $g \in G$ be an element with infinitely many conjugates. Show that $t'(L_g) = 0$.

   (iv) Show that $t' = t$.

6. (i) Let $R = \mathbf{M}_n(\mathbf{C})$ be the algebra of $n \times n$ matrices with entries in $\mathbf{C}$. Show that there is a unique $\mathbf{C}$-linear trace $t : R \longrightarrow Z(R)$, which is the identity on $Z(R)$. The trace t is given by letting $t(A) = \frac{tr(A)}{n} I_n$ for all matrices $A \in R$. (Here, we denote by tr the usual trace of a matrix.)

   (ii) Let G be a finite group with r mutually non-isomorphic irreducible complex representations $V_1, \ldots, V_r$ and consider the corresponding characters $\chi_1, \ldots, \chi_r$ and the dimensions $n_i = \dim V_i = \chi_i(1)$, $i = 1, \ldots, r$. Show that the Wedderburn decomposition

$$\mathbf{C}G \simeq \prod_{i=1}^{r} \mathbf{M}_{n_i}(\mathbf{C})$$

   identifies the the center-valued trace $t : \mathcal{N}G \longrightarrow \mathcal{Z}G$ with the map

$$t : \mathbf{C}G \longrightarrow \prod_{i=1}^{r} \mathbf{C} \cdot I_{n_i},$$

   which is defined by letting $t(a) = \left(\frac{\chi_1(a)}{n_1} I_{n_1}, \ldots, \frac{\chi_r(a)}{n_r} I_{n_r} \right)$ for all $a \in \mathbf{C}G$.