28-11-07

Στην (26): $\eta = \min\eta$

31) $f(z) = \sum_{n=0}^{\infty} z^n \qquad |z|<1$

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}, \quad |z|<1$$

$$S_n(z) = 1+z+\dots+z^n = \frac{1-z^{n+1}}{1-z}$$

$$S_n \xrightarrow[K]{u} f \Rightarrow \sup_{z\in K} |S_n(z)-f(z)| \to 0$$

συγκλίνει ομοιόμορφα σε ένα συμπαγές

K συμπαγές $\subseteq \Delta(0,1)$

Μπορεί το K να αντικατασταθεί με το $D(0,1)$;

$$|S_n(z)-f(z)| = \left|\frac{1-z^{n+1}}{1-z} - f(z)\right| = \left|\frac{1-z^{n+1}}{1-z} - \frac{1}{1-z}\right|$$

$$= \frac{|z|^{n+1}}{|1-z|} \quad (1)$$

$n_0 \in \mathbb{N}: \quad 1-\frac{1}{n_0} \in \Delta(0,1)$

$$\left|S_{n_0}\left(1-\frac{1}{n_0}\right) - f\left(1-\frac{1}{n_0}\right)\right| \overset{(1)}{=} \frac{\left(1-\frac{1}{n_0}\right)^{n_0+1}}{\left|1-\left(1-\frac{1}{n_0}\right)\right|} =$$

$$= n_0\left(1-\frac{1}{n_0}\right)^{n_0+1} \quad (2)$$

$\longrightarrow$

$$n\left(1-\frac{1}{n}\right)^{n+1} \longrightarrow +\infty$$

$$\downarrow$$

$$\frac{1}{e}$$

Για n_0 έχουμε: $\sup\limits_{z\in D(0,1)} |S_{n_0}(z)-f(z)| \geq n_0\left(1-\frac{1}{n_0}\right)^{n_0+1}$

Άρα $n\left(1-\frac{1}{n}\right)^{n+1} \to +\infty \Rightarrow \sup\limits_{z\in D(0,1)} |S_n(z)-f(z)| \to +\infty \neq 0$

Άρα δεν έχουμε ομοιόμορφη σύγκλιση στο $\frac{1}{1-z}$ στον $D(0,$

(28) $f(z)=\dfrac{z^5}{1+z^4}$ $\qquad f^{(21)}(0)=;$

Ιδέα: $c_n=\dfrac{f^{(n)}(0)}{n!}$

$$f(z)=c_0+c_1z+\dots+c_{21}z^{21}+\dots$$

Έστω $z_0\in\mathbb{C}$, $|z_0|<1$. Τότε $|-z_0^4|<1$

$$\sum_{n=0}^{\infty}(-z_0^4)^n=\frac{1}{1-(-z_0^4)}=\frac{1}{1+z_0^4}=\sum_{n=0}^{\infty}(-1)^n z_0^{4n}$$

Αυτό ισχύει και $\forall z\in D(0,1)$ (αφού z_0 τυχαίο)

$$\sum_{n=0}^{\infty}(-1)^n z_0^{4n}=\frac{1}{1+z_0^4}\Rightarrow z_0^5\sum_{n=0}^{\infty}(-1)^n z_0^{4n}=\frac{1}{1+z_0^4}\cdot z_0^5$$

$\longrightarrow$

$$\Rightarrow \sum_{n=0}^{\infty} (-1)^n z_0^{4n+5} = f(z_0)$$

Θα ισχύει $\forall z \in \Delta(0,1)$

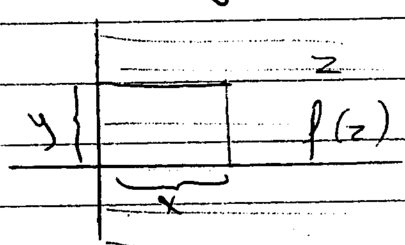
$$f(z) = \sum_{n=0}^{\infty} (-1)^n z^{4n+5}$$

$$f(z) = z^5 - z^9 + z^{13} - z^{17} + z^{21} + \dots$$

$$c_{21} = \frac{f^{(21)}(0)}{21!} \Rightarrow f^{(21)}(0) = 21!$$

Αν δεν είχαμε $n=21$ αλλά κάτι πολύ μεγαλύτερο
τότε: $x = 4n+5$ αλλιώς $f^{(\cdot)}(0) = 0$

(29) $\Omega \subseteq \mathbb{C}$ ανοιχτό
$f : \Omega \to \mathbb{C}$ ολόμορφη
$\operatorname{Re} f(z) > 0 \quad \forall z \in \Omega$
$\exists\ g : \Omega \to \mathbb{C},\ f(z) = e^{g(z)} \quad \forall z \in \Omega$

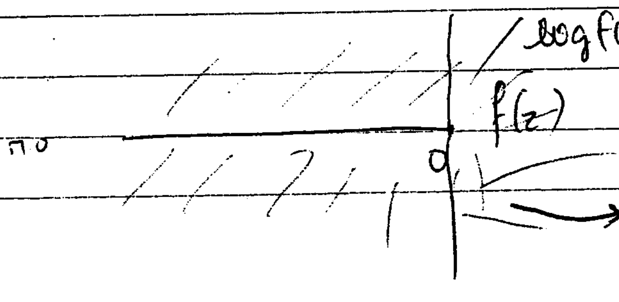


$$I_0 = \{ z \in \mathbb{C} / \operatorname{Re}(z) > 0 \}$$

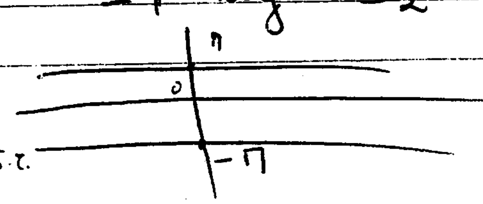
$$f(\Omega) \subseteq I_0$$

$$\log f(z) = \log e^{g(z)} = g(z)$$

$\log f(z)\ \forall z \in \Omega$ (ένας καλά ορισμένος αριθμός)



$\Omega_1 : \log \to \Omega_2$



$g: \Omega \to \mathbb{C}$, $g(z) = \log f(z) = (\log \circ f)(z)$

$e^{g(z)} = e^{\log f(z)} = f(z) \quad \forall z \in \Omega$

(30) $\displaystyle\sum_{n=1}^{\infty} \frac{(2-i)^n}{3^n \cdot n}$

$\sum z^n = \frac{1}{1-z} \qquad \sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z$, sin, cos, log, c^z, $(1+z)^\lambda$

$\frac{(2-i)^n}{3^n \cdot n} = \frac{1}{n}\left(\frac{2-i}{3}\right)^n = \frac{1}{n}(-1)^{2(n-1)}\left(\frac{2-i}{3}\right)^n = \frac{1}{n}(-1)^{n-1}(-1)^{n-1}\left(\frac{2-i}{3}\right)$

$\left(\log(1+z) = \sum_{n=0}^{\infty} \frac{1}{n}(-1)^{n-1} z^n \quad \forall z \in \Delta(0,1)\right)$

$= (-1)\frac{1}{n}(-1)^{n-1}\left((-1)\cdot\frac{2-i}{3}\right)^n$

$\left|(-1)\frac{2-i}{3}\right| = \frac{1}{3}|2-i| = \frac{1}{3}\sqrt{2^2+1^2} = \frac{\sqrt{5}}{3} < 1 \iff \sqrt{5} < 3$

$\Rightarrow 5 < 9$ που ισχύει

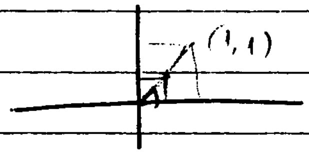
$\sum_{n=1}^{\infty} \frac{(2-i)^n}{3^n \cdot n} = \sum_{n=1}^{\infty} (-1)\frac{1}{n}(-1)^{n-1}\left(\frac{i-2}{3}\right)^n = -\sum_{n=1}^{\infty} \frac{1}{n}(-1)^{n-1}\left(\frac{i-2}{3}\right)$

όπου $\left|\frac{i-2}{3}\right| < 1$

$\log\left(1+\frac{i-2}{3}\right) = \log\left(\frac{3+i-2}{3}\right) = \log\left(\frac{1+i}{3}\right) = \rightarrow$

$$= \log\left|\frac{1+i}{3}\right| + i\arg\left(\frac{1+i}{3}\right)$$

$$\left|\frac{1+i}{3}\right| = \frac{1}{3}\left|1+i\right| = \frac{1}{3}\sqrt{1^2+1^2} = \frac{\sqrt{2}}{3}$$



$$\operatorname{Arg}\left(\frac{1+i}{3}\right) = \frac{\pi}{4}$$

Άρα $\displaystyle\sum_{n=1}^{\infty} \frac{(2-i)^n}{3^n \cdot n} = -\log\frac{\sqrt{2}}{3} - \frac{\pi}{4}$

27)

i) $|z|<1$ $\displaystyle\sum_{n=1}^{\infty} n^2 z^n$

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$$

ιν
αραγωγ.

$$\sum n z^{n-1}$$

$$\sum n(n-1) z^{n-2}$$

η παραγώγιση πάει όρο προς όρο

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \Rightarrow \left(\sum_{n=0} z^n\right)' = \left(\frac{1}{1-z}\right)' \Rightarrow \sum_{n=0} (z^n)' = \frac{1}{(1-z)^2}$$

$|z|<1$

$$\Rightarrow \sum_{n=1}^{\infty} n z^{n-1} = \frac{1}{(1-z)^2} \Rightarrow z\sum_{n=1}^{\infty} n z^{n-1} = \frac{z}{(1-z)^2}$$

$$\Rightarrow \sum_{n=1}^{\infty} n z^n = \frac{z}{(1-z)^2} \quad (1)$$

$\longrightarrow$

Ξανά παραγωγίζω $\left(\sum_{n=1}^{\infty} n z^n\right)' = \left(\frac{z}{(1-z)^2}\right)' \Rightarrow \sum_{n=1}^{\infty} (n z^n)' = \frac{z'(1-z)^2 - z((1-z)^2)'}{((1-z)^2)^2}$

$$\Rightarrow \sum_{n=1}^{\infty} n^2 z^{n-1} = \frac{(1-z)^2 - z \cdot 2(1-z)(-1)}{(1-z)^4}$$

$$= \frac{(1-z)^2 + 2z(1-z)}{(1-z)^4} = \frac{1-z+2z}{(1-z)^3} = \frac{1+z}{(1-z)^3}$$

$$z\sum_{n=1}^{\infty} n^2 z^{n-1} = \frac{z(1+z)}{(1-z)^3} \Longrightarrow$$

$$\sum_{n=1}^{\infty} n^2 z^n = \frac{z(1+z)}{(1-z)^3} \quad \forall z \in \mathbb{C} \text{ με } |z|<1$$

ii) $\sum_{n=1}^{\infty} n r^n \cos(n\theta)$, $r \in (0,1)$, $\theta \in \mathbb{R}$

$\boxed{z_0 = r e^{i\theta}}$ με $|z_0| = |r \cdot e^{i\theta}| = r < 1$

$$z_0^n = (re^{i\theta})^n = r^n e^{i\theta n} = r^n(\cos(\theta n) + i\sin(\theta n)) =$$
$$= r^n \cos(n\theta) + i r^n \sin(n\theta)$$

Παρατηρούμε ότι $Re(z_0^n) = r^n \cos(n\theta)$

Από την (1) για $z = z_0$: $\sum_{n=1}^{\infty} n z_0^n = \frac{z_0}{(1-z_0)^2}$ (2)

$$\sum_{n=1}^{\infty} n z_0^n = \sum_{n=1}^{\infty} n(r^n \cos(n\theta) + i r^n \sin(n\theta))$$
$$= \sum_{n=1}^{\infty} (n \cdot r^n \cos(n\theta) + i n r^n \sin(n\theta))$$

$$= \sum_{n=1}^{\infty} n\, r^n \cos(n\theta) + i \sum_{n=1}^{\infty} n\, r^n \sin(n\theta)$$

Άρα $\mathrm{Re}\left(\sum_{n=1}^{\infty} n\, z_0^{\,n}\right) = \sum_{n=1}^{\infty} n\, r^n \cos(n\theta)$

Από το (1) έχουμε $\Rightarrow \mathrm{Re}\left(\sum_{n=1}^{\infty} n\, z_0^{\,n}\right) = \mathrm{Re}\left(\dfrac{z_0}{(1-z_0)^2}\right)$ (3)

Από (2) και (3) $\Rightarrow \sum_{n=1}^{\infty} n \cdot {}^{n} \cos(n\theta) = \mathrm{Re}\left(\dfrac{z_0}{(1-z_0)^2}\right)$ (4)

$z_0 = r e^{i\theta}$

$\mathrm{Re}\left(\dfrac{z_0}{(1-z_0)^2}\right)$

$$\frac{z_0}{(1-z_0)^2} = \frac{z_0\left(\overline{(1-z_0)^2}\right)}{(1-z_0)^2\,\overline{(1-z_0)^2}} = \frac{z_0(1-\bar{z}_0)^2}{|(1-z_0)^2|^2} =$$

$$= \frac{1}{|1-z_0|^4}\left(z_0(1-\bar{z}_0)^2\right)$$

$\mathrm{Re}: \mathbb{R}^2 \to \mathbb{R}$

$\mathrm{Re}(x,y) = x \quad \forall (x,y) \in \mathbb{R}^2$

γραμμική ως συνάρτηση

Άρα: $\mathrm{Re}\left(\dfrac{z_0}{(1-z_0)^2}\right) = \mathrm{Re}\left(\dfrac{1}{|1-z_0|^4}\left(z_0(1-\bar{z}_0)^2\right)\right) = \dfrac{1}{|1-z_0|^4}\mathrm{Re}(z_0(1-z_0$

$$z_0(1-\bar{z}_0)^2 = re^{i\theta}(1-re^{-i\theta})^2 = re^{i\theta}(1+r^2e^{-2i\theta} - 2re^{-i\theta})$$

$$= re^{i\theta} + r^3e^{-i\theta} - 2r^2$$

→

$$Re\left(z_0(1-\bar{z}_0)^2\right) = Re\left(re^{i\theta} + r^3 e^{-i\theta} - 2r^2\right) = -2r^2 + Re\left(re^{i\theta}\right) + Re\left(r^3 e^{-i\theta}\right)$$

$$= -2r^2 + r\,Re\left(e^{i\theta}\right) + r^3\,Re\left(e^{-i\theta}\right) =$$

$$= -2r^2 + r\cos\theta + r^3\cos\theta$$

$$|1-z_0|^4$$

$$|1-z_0|^2 = |1-re^{i\theta}|^2 = |1-r(\cos\theta + i\sin\theta)|^2 = |(1-r\cos\theta) - ri\sin\theta|^2$$

$$= (1-r\cos\theta)^2 + r^2\sin^2\theta = 1 + r^2\cos^2\theta - 2r\cos\theta + r^2\sin^2\theta$$

$$= (1 + r^2 - 2r\cos\theta)^2$$

Άρα $Re\left(\dfrac{z_0(1-\bar{z}_0)^2}{|1-z_0|^4}\right) = \dfrac{r\cos\theta + r^2\cos\theta + 2r^2}{(1+r^2-2r\cos\theta)^2}.$

(33)

$$\sum_{n=1}^{\infty} \lambda(\lambda-1)(\lambda-n+1)\frac{r^n\cos(n\theta)}{n!}, \quad \lambda\in\mathbb{R} \qquad 0<r<1,\ \theta\in\mathbb{R}$$

$$(1+z)^\lambda = \sum_{n=0}^{\infty}\binom{\lambda}{n} z^n \quad \text{με} \quad \binom{\lambda}{n} = \frac{\lambda(\lambda-1)\dots(\lambda-n+1)}{n!}$$

$\forall z\in D(0,1)$
$\forall \lambda\in\mathbb{C}$ $\qquad \binom{\lambda}{0} = 1$

Θέτουμε $z_0 = re^{i\theta}$

$$z_0^n = (re^{i\theta})^n = r^n e^{in\theta} = r^n(\cos(n\theta) + i\sin(\theta n)) \Rightarrow$$

$$r^n\cos(n\theta) = Re(z_0^n)$$

$\rightarrow$

$$(1+z_0)^\lambda = \sum_{n=0}^{\infty} \binom{\lambda}{n} z_0^n \Rightarrow Re\left((1+z_0)^\lambda\right) = Re\left(\sum_{n=0}^{\infty} \binom{\lambda}{n} z_0^n\right)$$

$$= \sum_{n=0}^{\infty} \binom{\lambda}{n} Re(z_0)^n = \sum_{n=0}^{\infty} \binom{\lambda}{n} r^n \cos n\theta =$$

$$= 1 + \sum_{n=1}^{\infty} \frac{\lambda(\lambda-1)\dots(\lambda-n+1)}{n!} r^n \cos n\theta$$

$$\left(a^b = e^{b\log a}\right)$$

$$Re\left((1+z_0)^\lambda\right)$$

$$(1+z_0)^\lambda = e^{\lambda \log(1+z_0)} = e^{\lambda(\log|1+z_0| + i\arg(1+z_0))}$$

$$\left(\log(1+z_0) = \log|1+z_0| + i\arg(1+z_0)\right)$$

$$= e^{\lambda\log|1+z_0| + i\lambda\arg(1+z_0)} \qquad e^{x+yi}$$

$$Re\left((1+z_0)^\lambda\right) = e^{\lambda\log|1+z_0|} = \left(e^{\log|1+z_0|}\right)^\lambda = |1+z_0|^\lambda$$

$$|1+z_0|^2 = |1+re^{i\theta}|^2 = |1+r\cos\theta + ir\sin\theta|^2 =$$

$$= (1+r\cos\theta)^2 + (r\sin\theta)^2 = 1 + r^2\cos^2\theta + 2r\cos\theta + r^2\sin^2\theta$$

$$= (1+r^2+2r\cos\theta)$$

Άρα $|1+z_0| = \sqrt{1+r^2+2r\cos\theta}$

$$(1+z_0)^\lambda = \sum_{n=0}^{\infty} \binom{\lambda}{n} z_0^n \Rightarrow Re\left((1+z_0)^\lambda\right) = Re\left(\sum_{n=0}^{\infty} \binom{\lambda}{n} z_0^n\right)$$

$$\overset{?}{=} \sum_{n=0}^{\infty} \binom{\lambda}{n} Re(z_0)^n = \sum_{n=0}^{\infty} \binom{\lambda}{n} r^n \cos n\theta =$$

$$= 1 + \sum_{n=1}^{\infty} \frac{\lambda(\lambda-1)\cdots(\lambda-n+1)}{n!} r^n \cos n\theta$$

$$\left(a^b = e^{b\log a}\right)$$

$$Re\left((1+z_0)^\lambda\right)$$

$$(1+z_0)^\lambda = e^{\lambda \log(1+z_0)} = e^{\lambda(\log|1+z_0| + i\arg(1+z_0))}$$

$$\left(\log(1+z_0) = \log|1+z_0| + i\arg(1+z_0)\right)$$

$$= e^{\lambda\log|1+z_0| + i\lambda\arg(1+z_0)} \qquad e^{x+yi}$$

$$Re\left((1+z_0)^\lambda\right) = e^{\lambda\log|1+z_0|} = \left(e^{\log|1+z_0|}\right)^\lambda = |1+z_0|^\lambda$$

$$|1+z_0|^2 = |1+re^{i\theta}|^2 = |1+r\cos\theta + ir\sin\theta|^2 =$$

$$= (1+r\cos\theta)^2 + (r\sin\theta)^2 = 1 + r^2\cos^2\theta + 2r\cos\theta + r^2\sin^2\theta$$

$$= (1+r^2+2r\cos\theta)$$

Άρα $|1+z_0| = \sqrt{1+r^2+2r\cos\theta}$