1 Ροπογεννήτριες

$$M_X(u) = \begin{cases} \sum_x e^{ux} P(X=x), & \text{Διακριτή;} \\ \int_x e^{ux} f(x)dx, & \text{Συνεχής.} \end{cases}$$

1. Έστω $X \sim Poisson(\lambda)$:

$$\begin{aligned} M_X(u) &= \sum_x = 0^{\infty} e^{ux} P(X=x) \\ &= \sum_x = 0^{\infty} e^{ux} e^{-\lambda} \frac{\lambda^x}{x!} \\ &= e^{-\lambda} \sum_x = 0^{\infty} \frac{(e^u \lambda)^x}{x!} \\ &= e^{-\lambda} e^{e^u \lambda} \\ &= e^{\lambda(e^u - 1)}. \end{aligned}$$

2. Έστω $X \sim Bin(n,p)$:

$$\begin{aligned} M_X(u) &= \sum_x = 0^{\infty} e^{ux} P(X=x) \\ &= \sum_x = 0^{\infty} e^{ux} \binom{n}{x} p^x q^{n-x} \\ &= \sum_x = 0^{\infty} \binom{n}{x} (e^u p)^x q^{n-x} \\ &= (pe^u + q)^n \end{aligned}$$

3. Έστω $X \sim N(\mu, \sigma^2)$:

$$M_X(u) = e^{\mu u + \frac{\sigma^2 u^2}{2}}.$$

4. Έστω $X_1, X_2, ..., X_n$ με $X_i \sim Bin(N,p)$. Τότε $\sum X_i \sim Bin(nN,p)$:

$$\begin{aligned} M_{\sum X_i}(u) &= E\left(e^{u\sum X_i}\right) \\ &= E\left(e^{uX_1}e^{uX_2}...e^{uX_n}\right) \\ &= E\left(e^{uX_1}\right)E\left(e^{uX_2}\right)...E\left(e^{uX_n}\right) \\ &= (pe^u+q)^N(pe^u+q)^N...(pe^u+q)^N \\ &= (pe^u+q)^{nN} \end{aligned}$$

5. Έστω $X_1, X_2, ..., X_n$ με $X_i \sim N(\mu, \sigma^2)$. Τότε $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$:

$$\begin{aligned} M_{\bar{X}}(u) &= E\left(e^{u\bar{X}}\right) \\ &= E\left(e^{u\frac{1}{n}\sum X_i}\right) \\ &= E\left(e^{\frac{u}{n}X_1}e^{\frac{u}{n}X_2}...e^{\frac{u}{n}X_n}\right) \\ &= M_{X_1}\left(e^{\frac{u}{n}}\right)M_{X_2}\left(e^{\frac{u}{n}}\right)...M_{X_n}\left(e^{\frac{u}{n}}\right) \\ &= \prod_{i=1}^{n} M_{X_i}\left(e^{\frac{u}{n}}\right) \\ &= \prod_{i=1}^{n} e^{\mu\frac{u}{n}+\frac{\sigma^2u^2}{2n^2}} \\ &= e^{n\mu\frac{u}{n}+n\frac{\sigma^2u^2}{2n^2}} \\ &= e^{\mu u+\frac{\sigma^2u^2}{2n}}. \end{aligned}$$