πχ) $X \sim Bin(\nu, p)$, ν-γνωστό p-άγνωστο

$$f(x;p) = \binom{\nu}{x} p^x \cdot (1-p)^{\nu - x} = \binom{\nu}{x} \left(\frac{p}{1-p}\right)^x \cdot (1-p)^\nu =$$

$$= \binom{\nu}{x} \exp\left\{ x \log \frac{p}{1-p} + \nu \log(1-p) \right\}$$

$$= h(x) \exp\left\{ T(x) \cdot \eta(p) - B(p) \right\}$$ όπου:

$h(x) = \binom{\nu}{x}$, $T(x) = x$, $\eta(p) = \log \frac{p}{1-p}$

και $B(p) = -\nu \log(1-p)$.

και το στήριγμα ανεξ. του p $\longrightarrow$ άρα ΕοΚ.

Κανονική μορφή: $\eta(p) = \log \frac{p}{1-p} = \eta \Rightarrow p = \frac{e^\eta}{1+e^\eta}$

$$A(\eta) = -\nu \log\left(1 - \frac{e^\eta}{1+e^\eta}\right) = -\nu \log \frac{1}{1+e^\eta} =$$

$$= \nu \log(1 + e^\eta)$$

$\hookrightarrow E(T(X)) = E(X) = \frac{d}{d\eta} A(\eta) = \frac{\nu}{1+e^\eta} \cdot e^\eta = \nu \frac{e^\eta}{1+e^\eta} = \nu p$

$\hookrightarrow V(T(X)) = V(X) = \frac{d^2}{d\eta^2} A(\eta) =$

$$= \frac{\nu e^{\eta}}{1+e^{\eta}} - \frac{\nu e^{\eta}}{(1+e^{\eta})^2} \cdot e^{\eta} = \nu p - \nu p^2 = \nu p(1-p)$$

$$\hookrightarrow M_T(u) = \exp\{A(u+\eta) - A(\eta)\} =$$

$$= \exp\{\nu \log(1+e^{u+\eta}) - \nu \log(1+e^{\eta})\} =$$

$$= \exp\left\{\log\left(\frac{1+e^{u+\eta}}{1+e^{\eta}}\right)^{\nu}\right\} = \left(\frac{1+e^{u+\eta}}{1+e^{\eta}}\right)^{\nu} \xRightarrow{e^{\eta}=p/q}$$

$$\Rightarrow M_T(u) = \left(\frac{1+\frac{p}{q}\cdot e^{u}}{1+\frac{p}{q}}\right)^{\nu} = \left(\frac{q+pe^{u}}{q+p}\right)^{\nu} = \left(q+pe^{u}\right)^{\nu} \quad \blacksquare$$

P2

Τετάρτη, 6 Οκτωβρίου 2021 1:40 μμ

Πχ) (α) Αν X α.τ. των ΕΟΚ με σ.π.

$$f(x,\theta) = e^{\eta(\theta)\cdot T(x) - B(\theta)} \cdot h(x)$$

ΝΔΟ: $E(T(x)) = \dfrac{B'(\theta)}{\eta'(\theta)}$ κι $V(T(x)) = \left[\dfrac{B'(\theta)}{\eta'(\theta)}\right]' \dfrac{1}{\eta'(\theta)}$

(β) Έστω $X_1, X_2, \dots, X_\nu$ α.δ. από σ.π.π.

$$f(x,\theta) = k x^{k-1} \theta e^{-\theta x^k}, \quad x>0,\ \theta>0,\ k>0$$

Ανήκει στην ΕΟΚ; Να βρεθούν $E(X^k)$, $V(X^k)$.

(α) $\int \exp\{\eta(\theta)T(x) - B(\theta)\} \cdot h(x)\,dx = 1 \Rightarrow$

$\Rightarrow \int \exp\{\eta(\theta)T(x)\} \cdot h(x)\,dx = \exp\{B(\theta)\}$

Παραγωγίζω ως προς θ και εναλλάσσω τη σειρά ολοκλ./παραγ. (*)

$$\int \exp\{\eta(\theta)T(x)\}T(x)\eta'(\theta) \cdot h(x)\,dx = \exp\{B(\theta)\} \cdot B'(\theta)$$

$$\Rightarrow \eta'(\theta) \cdot \underbrace{\int \exp\{\eta(\theta)T(x) - B(\theta)\} \cdot T(x) \cdot h(x)\,dx}_{E(T(x))} = B'(\theta)$$

$$\Rightarrow \boxed{E(T(x)) = \frac{B'(\theta)}{\eta'(\theta)}}$$

Από ⊛ $\xrightarrow[\text{ως προς } \theta]{\text{παραγ.}}$

$$\int \exp\{\eta(\theta)T(x)\}\, T(x)^2 \cdot \eta'(\theta)^2 \cdot h(x)\,dx \; +$$

$$+ \int \exp\{\eta(\theta)\cdot T(x)\}\, T(x)\, \eta''(\theta)\, h(x)\,dx =$$

$$= \exp\{B(\theta)\}\cdot B'(\theta)^2 + \exp\{B(\theta)\}\cdot B''(\theta) \Longrightarrow$$

$$\Longrightarrow \eta'(\theta)^2 \cdot \int T(x)^2 \exp\{\eta(\theta)T(x) - B(\theta)\}\cdot h(x)\,dx \; +$$

$$+ \eta''(\theta) \cdot \int T(x) \exp\{\eta(\theta)\cdot T(x) - B(\theta)\}\cdot h(x)\,dx =$$

$$= B'(\theta)^2 + B''(\theta) \Longrightarrow$$

$$\Longrightarrow \eta'(\theta)^2 \cdot E\left(T(x)^2\right) + \eta''(\theta)\, E\left(T(x)\right) = B'(\theta)^2 + B''(\theta)$$

$$\Longrightarrow \eta'(\theta)^2\, E\left(T(x)^2\right) = B'(\theta)^2 + B''(\theta) - \eta''(\theta)\,\frac{B'(\theta)}{\eta'(\theta)}$$

Οπότε: $V(T(x)) = E\left(T(x)^2\right) - E\left(T(x)\right)^2 =$

$$= \frac{B'(\theta)^2 \cancel{\eta'(\theta)} + B''(\theta)\eta'(\theta) - \eta''(\theta)B'(\theta) - B'(\theta)^2 \cancel{\eta'(\theta)}}{\eta'(\theta)^3}$$

$$= \frac{1}{\eta'(\theta)} \cdot \left[\frac{B''(\theta)\eta'(\theta) - \eta''(\theta)\cdot B''(\theta)}{\eta'(\theta)^2}\right] = \frac{1}{\eta'(\theta)} \cdot \left[\frac{B'(\theta)}{\eta'(\theta)}\right]'$$

(B) $f(x;\theta) = k x^{k-1} \cdot \theta e^{-\theta x^k}$

$$= kx^{k-1} \cdot \exp\{\log\theta - \theta x^k\}$$

$$= h(x) \cdot \exp\{\eta(\theta) \cdot T(x) - B(\theta)\}$$

<u>όπου</u>: $h(x) = kx^{k-1}$, $\eta(\theta) = -\theta$, $T(x) = x^k$

και $B(\theta) = -\log\theta$.

$$E(T(x)) = E(x^k) = \frac{B'(\theta)}{\eta'(\theta)} = \frac{1}{\theta}$$

$$V(T(x)) = V(x^k) = \frac{1}{\eta'(\theta)} \cdot \left[\frac{B'(\theta)}{\eta'(\theta)}\right]' = \frac{1}{(-\theta)'} \cdot \left[\frac{1}{\theta}\right]'$$

$$= \frac{1}{\theta^2}$$

P3

Πχ) Έστω $X_1, X_2, \ldots, X_\nu$ τ.δ. από $N(\mu, \sigma^2)$ με μ, σ^2 - άγνωστα, τότε ΝΔΟ η από κοινού σ.π.π. ανήκει στην ν-διάστατη 2-παραμετρική ΕΟΚ.

Να υπολογίσετε: $E\left[\sum_{j=1}^{\nu} X_j^2\right]$ κ' $E\left[\sum_{j=1}^{\nu} X_j\right]$

και $Cov\left(\sum_{j=1}^{\nu} X_j^2, \sum_{j=1}^{\nu} X_j\right)$

Είναι $X_1, X_2, \ldots, X_\nu$ τ.δ. μσ

$$f(x_i; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi}} \cdot \exp\left\{-\frac{x_i^2}{2\sigma^2} + \frac{x_i \mu}{\sigma^2}\right\} \cdot \exp\left\{-\frac{\mu}{2\sigma^2} - \frac{1}{2}\log\sigma^2\right\}$$

$$= h(x_i) \cdot \exp\left\{\eta_1(\mu, \sigma^2) \cdot T_1(x_i) + \eta_2(\mu, \sigma^2) \cdot T_2(x_i)\right\} \cdot b(\mu, \sigma^2)$$

όπου: $h(x_i) = \frac{1}{\sqrt{2\pi}}$, $\eta_1(\mu, \sigma^2) = -\frac{1}{2\sigma^2}$, $T_1(x_i) = x_i^2$

$\eta_2(\mu, \sigma^2) = \frac{\mu}{\sigma^2}$, $T_2(x_i) = x_i$

κ' $b(\mu, \sigma^2) = \exp\left\{-\frac{\mu^2}{2\sigma^2} - \frac{1}{2}\log\sigma^2\right\}$

Κανονική μορφή:

$$f(x_i, \eta_1, \eta_2) = h(x) \exp\left\{\eta_1 T_1(x_i) + \eta_2 T_2(x_i) - A(\eta_1, \eta_2)\right\} =$$

$$= \frac{1}{-} \cdot \exp\left\{\eta_1 x_i^2 + \eta_2 x_i + \frac{\eta_2^2}{4\eta_1} + \frac{1}{2}\log(-2\eta_1)\right\}$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \exp\left\{ \eta_1 x_i^2 + \eta_2 x_i + \underbrace{\frac{\eta_2}{4\eta_1} + \frac{1}{2}\log(-2\eta_1)}\right\}$$

Από κοινού σ.π.π.

$$f_{\underset{\sim}{X}}(\underset{\sim}{x}; \mu, \sigma^2) = b^*(\mu, \sigma^2)\, \exp\left\{ \eta_1(\mu, \sigma^2) T_1^*(\underset{\sim}{x}) + \eta_2(\mu, \sigma^2) T_2^*(\underset{\sim}{x}) \right\} \cdot h^*(\underset{\sim}{x})$$

<u>όπου</u>: $b^*(\mu, \sigma^2) = [b(\mu, \sigma^2)]^{\nu}$

$$= \left[\exp\left\{ -\frac{\mu^2}{2\sigma^2} - \frac{1}{2}\log \sigma^2 \right\}\right]^{\nu} =$$

$$= \exp\left\{ -\frac{\nu \mu^2}{2\sigma^2} - \frac{\nu}{2}\log\sigma^2 \right\}$$

$$T_1^*(\underset{\sim}{x}) = \sum_{j=1}^{\nu} T_1(x_j) = \sum_{j=1}^{\nu} x_j^2$$

$$T_2^*(\underset{\sim}{x}) = \sum_{j=1}^{\nu} T_2(x_j) = \sum_{j=1}^{\nu} x_j$$

$$h(\underset{\sim}{x}) = \prod_{j=1}^{\nu} h(x_j) = \prod_{j=1}^{\nu} \frac{1}{\sqrt{2\pi}} = (2\pi)^{-\nu/2}$$

<u>Συνεπώς</u>

$$f_{\underset{\sim}{X}}(\underset{\sim}{x}; \mu, \sigma^2) = (2\pi)^{-\nu/2} \cdot \exp\left\{ -\frac{1}{2\sigma^2} \sum_{j=1}^{\nu} x_j^2 + \frac{\mu}{\sigma^2} \sum_{j=1}^{\nu} x_j - \frac{\nu\mu^2}{2\sigma^2} - \frac{\nu}{2}\log\sigma^2 \right\}$$

<u>κανονική μορφή</u>:

$$\eta_1(\mu, \sigma^2) = -\frac{1}{2\sigma^2} = \eta_1 \Rightarrow \sigma^2 = -\frac{1}{2\eta_1}$$

$$\eta_2(\mu,\sigma^2) = \frac{\mu}{\sigma^2} = \eta_2 \Rightarrow \mu = \eta_2\sigma^2 = -\frac{\eta_2}{2\eta_1}$$

$$A^*(\eta_1,\eta_2) = \frac{\nu\mu^2}{2\sigma^2} + \frac{\nu}{2}\log\sigma^2 = \frac{\nu\left(\eta_2^2/4\eta_1^2\right)}{2\left(-1/2\eta_1\right)} + \frac{\nu}{2}\log\left(-\frac{1}{2\eta_1}\right)$$

$$= -\frac{\nu\eta_2^2}{4\eta_1} - \frac{\nu}{2}\log(-2\eta_1) .$$

Άρα: $f_{\underset{\sim}{x}}(\underset{\sim}{x};\eta_1,\eta_2) = \exp\{\eta_1 T_1^*(\underset{\sim}{x}) + \eta_2 T_2^*(\underset{\sim}{x}) - A^*(\eta_1,\eta_2)\} \cdot h^*(\underset{\sim}{x})$

Οπότε:

$$E(T_1^*(\underset{\sim}{x})) = \frac{\partial}{\partial\eta_1}A^*(\eta_1,\eta_2) = \frac{\partial}{\partial\eta_1}\left[-\frac{\nu\eta_2^2}{4\eta_1} - \frac{\nu}{2}\log(-2\eta_1)\right]$$

$$= \frac{\nu\eta_2^2}{16\eta_1^2}\cdot 4 - \frac{\nu}{2}\,\frac{1}{-2\eta_1}\cdot(-2) = \frac{\nu\eta_2^2}{4\eta_1^2} - \frac{\nu}{2\eta_1} = E\left[\sum_{j=1}^{\nu} X_j^2\right]$$

$$E(T_2^*(\underset{\sim}{x})) = \frac{\partial}{\partial\eta_2}A^*(\eta_1,\eta_2) = \frac{\partial}{\partial\eta_2}\left[-\frac{\nu\eta_2^2}{4\eta_1} - \frac{\nu}{2}\log(-2\eta_1)\right]$$

$$= -\frac{\nu\eta_2}{2\eta_1} = E\left[\sum_{j=1}^{\nu} X_j\right].$$

$$\mathrm{Cov}(T_1^*(\underset{\sim}{x}), T_2^*(\underset{\sim}{x})) = \frac{\partial^2}{\partial\eta_1\partial\eta_2}A^*(\eta_1,\eta_2) = \frac{\partial}{\partial\eta_1}\left[-\frac{\nu\eta_2}{2\eta_1}\right]$$

$$= \frac{\nu\eta_2}{2\eta_1^2}$$