Example

Let $X_1, \dots, X_\nu$ r.s. of $U(\theta_1, \theta_2)$, θ_1, θ_2 unknown. Find a sufficient statistic for $\underline{\theta} = (\theta_1, \theta_2)$

Solution

$f(x\,;\,\theta_1,\theta_2) = \frac{1}{\theta_2-\theta_1} \cdot I(\theta_1 < x < \theta_2)$,

where $I(\theta_1 < x < \theta_2) = \begin{cases} 1, & \theta_1 < x < \theta_2 \\ 0, & \text{otherwise} \end{cases}$

The joint PDF is: $f(\underline{x}\,;\,\theta_1,\theta_2) = \prod_{i=1}^{\nu} f(x_i\,;\,\theta_1,\theta_2) =$

$= \prod_{i=1}^{\nu} \frac{1}{\theta_2-\theta_1} I(\theta_1 < x_i < \theta_2) = (\theta_2-\theta_1)^{-\nu} \prod_{i=1}^{\nu} I(\theta_1 < x < \theta_2)$

We demand: $\prod_{i=1}^{\nu} I(\theta_1 < x_i < \theta_2) = 1 \Leftrightarrow x_i \in (\theta_1, \theta_2)\ \forall i = 1,\dots,\nu$

$\Leftrightarrow \theta_1 < x_{(1)} \leq x_{(2)} \leq \dots \leq x_{(\nu)} < \theta_2 \Leftrightarrow \theta_1 < x_{(1)}$ και $x_{(\nu)} < \theta_2$

Hence: $f(\underline{x}\,;\,\underline{\theta}) = (\theta_2-\theta_1)^{-\nu} I(\theta_1 < x_{(1)} < \infty) \cdot I(-\infty < x_{(\nu)} < \theta_2)$

$= g(\underline{T}(\underline{x})\,;\,\underline{\theta}) \cdot h(\underline{x})$,

where $T(\underline{x}) = (x_{(1)}, x_{(\nu)})$, $g(\underline{T}(x)\,;\,\theta) = f(\underline{x}\,;\,\theta)$, $h(\underline{x}) = 1$

so $T(\underline{x})$ is sufficient for $\underline{\theta}$.

Completeness

Definition (General)

Let X random variable with PDF $f(x\,;\,\theta)$ $\xrightarrow{} s=1$, $\theta \in \Theta \subseteq \mathbb{R}$ and let $g: \mathbb{R} \to \mathbb{R}$ such that $g(X)$ is a random variable. We suppose that $E[g(X)]$ exists $\forall \theta \in \Theta$ and define the family of distributions $\mathcal{F} = \{f(x\,;\,\theta),\ \theta \in \Theta\}$.

The r.v. X (or the family $\mathcal{F}$) is called complete if $\forall g$ we have: $E[g(X)]=0 \ \forall \theta \in \Theta \Rightarrow g(x)=0$ for every x except for a set N for which $P(X \in N)=0, \ \forall \theta \in \Theta$.

Note

For our purposes: when we have a r.s. $X_1, \dots, X_n$ with PDF $f(x;\theta)$ and a complete statistic $T=T(\underline{X})$ with PDF $h_T(t;\theta)$ and $\mathcal{F}=\{h(t;\theta) \mid \theta \in \Theta\}$ family of distributions, then:

$E[g(T)]=0 \ \forall \theta \in \Theta \Rightarrow g(t)=0 \ \forall t$

Example

Let $X_1, \dots, X_n$ r.s. of Poisson(λ). Find a sufficient and complete statistic for λ.

Solution

We know (from factorization criterion) that $T=T(\underline{X})=\sum_{i=1}^{n} X_i$ sufficient for λ. We also know that $T \sim \text{Poisson}(n\lambda)$.

Let $g(T)$ function of T, then $E[g(T)]=\sum_{t=0}^{\infty} g(t)h(t\lambda)=0 \ \forall \lambda>0$

$\Rightarrow E[g(T)]=\sum_{t=0}^{\infty} g(t)e^{-n\lambda} \cdot \frac{(n\lambda)^t}{t!}=0 \Rightarrow$

$\Rightarrow e^{-n\lambda}\left[g(0)+g(1)n\lambda+g(2)\frac{(n\lambda)^2}{2!}+\dots\right]=0 \quad \forall \lambda>0$

We have a series that converges to $0 \ \forall \lambda>0$, so all the coefficients must be 0. Hence:

$g(0)=0, \ g(1)n\lambda=0 \Rightarrow g(1)=0, \ \dots$

So $g(t)=0 \ \forall t=0,1,\dots$ and hence T is complete.

Example

Let $X_1,\dots,X_\nu$ r.s of Bernoulli(ϱ). Find sufficient and complete statistic.

Solution

We can show that $T=T(\underline{X})=\sum_{i=1}^{\nu}X_i$ sufficient for ϱ.
Also (because $X_1,\dots,X_\nu$ independent) we know $T\sim Bin(\nu,\varrho)$
Let $g(T)$ function of T. Then

$$E[g(T)]=\sum_{t=0}^{\nu}g(t)h(t,\varrho)=\sum_{t=0}^{\nu}g(t)\binom{\nu}{t}\varrho^t(1-\varrho)^{\nu-t} \underset{\overline{\overline{\quad\quad}}}{\text{θέτω } z=\varrho/1-\varrho,\ z>0}$$

$$=(1-\varrho)^{\nu-t}\sum_{t=0}^{\nu}g(t)\binom{\nu}{t}z^t$$

so $E[g(T)]=0\quad\forall z \Rightarrow \sum_{t=0}^{\nu}g(t)\binom{\nu}{t}z^t=0\quad\forall z$

So we have a polynomial equal to 0 $\forall z>0$ and hence,

$g(t)\binom{\nu}{t}=0\quad\forall t=0,\dots,\nu \Rightarrow g(t)=0\quad\forall t=0,\dots,\nu.$

Finally T is sufficient and complete for ϱ.

Example

Examine whether the family $\mathcal{F}=\{f(x;\theta)\mid\theta\in\Theta\}$ is complete or not in the following cases:

a) $f(x;\theta)=\frac{1}{\theta-a}I(a<x<\theta)$, $a\in\mathbb{R}$, $\theta\in(a,\infty)$ $\rightsquigarrow U(a,\theta)$

b) $f(x;\theta)=\frac{1}{\beta-\theta}I(\theta<x<\beta)$, $\beta\in\mathbb{R}$, $\theta\in(-\infty,\beta)$ $\rightsquigarrow U(\theta,\beta)$

c) $f(x;\theta)=\frac{1}{2\theta}I(-\theta<x<\theta)$, $\theta\in(0,+\infty)$ $\rightsquigarrow U(-\theta,\theta)$

Solution

a) Let $g(X)$ function of X with PDF in $\mathcal{F}$. We want
$E[g(X)]=0\quad\forall\theta \Rightarrow g(x)=0\quad\forall x.$
It is: $E[g(X)]=\int_a^\theta g(x)\frac{1}{\theta-a}dx=0\quad\forall\theta>a \xRightarrow{\frac{\partial}{\partial\theta}}$

$\Rightarrow g(\theta)\frac{1}{\theta-a} = 0 \quad \forall \theta > a \quad \Rightarrow \quad g(\theta) = 0 \quad \forall \theta \in (a, +\infty)$

$\Rightarrow g(x) = 0 \quad \forall x \in (a, \theta)$

So $\mathcal{F}$ is complete

b) Same as a)

c) We will show that $\tilde{\mathcal{F}}$ is not complete. Suppose $g(X) = X$.

Then $E[g(X)] = E[X] = \int_{-\theta}^{\theta} x \frac{1}{2\theta} \, dx = \frac{1}{2\theta}\left(\frac{\theta^2}{2} - \frac{\theta^2}{2}\right) = 0$

and it is not true that $g(x) = 0 \ \forall x$, so $\tilde{\mathcal{F}}$ is incomplete.