Theorem

Let $X_1, \dots, X_v$ r.s. with PDFs $f(x;\theta)$, $\theta \in \Theta \subseteq \mathbb{R}$ and $T = T(\underline{X})$ sufficient statistic for θ.
Let $g(\dots;\theta)$ the PDF of $T(\underline{X})$ and $V = V(\underline{X})$ (or any other statistic) for which we suppose that it is independent of $T(\underline{X})$.
Then $V = V(\underline{X})$ is an ancillary statistic, meaning that its distribution is independent of θ.

Theorem (Basu)

Let $X_1, \dots, X_v$ r.s with PDFs $f(x;\underline{\theta})$, $\underline{\theta} \in \Theta \subseteq \mathbb{R}^s$ and $\underline{T} = \underline{T}(\underline{x})$ sufficient for $\underline{\theta}$.
Let $g(\dots;\theta)$ the PDF of $T(\underline{X})$ and suppose that $G = \{g(\dots;\theta) \mid \theta \in \Theta\}$ is complete.
Let $\underline{V} = \underline{V}(\underline{X})$ a statistic other than $\underline{T}$.
If the distribution of $\underline{V}$ is independent of $\underline{\theta}$ then $\underline{V}(\underline{x})$ is independent of $\underline{T}(\underline{x})$

Example

Let $Z \sim \text{Gamma}(v, \theta)$ with $f_Z(x) = \frac{1}{\Gamma(v)\theta^v} \cdot Z^{v-1} \cdot e^{-Z/\theta}$, $x \geq 0$, $v, \theta > 0$, then $T = {}^{2Z}/_{\theta} \sim \chi^2_{2v}$.

Proof

$$F_T(t) = P(T \leq t) = P\left(\frac{2Z}{\theta} \leq t\right) = P\left(Z \leq \frac{\theta t}{2}\right) = F_Z\left(\frac{\theta t}{2}\right)$$

$$f_T(t) = \frac{\partial}{\partial t} F_T(t) = \frac{\partial}{\partial t} F_Z\left(\frac{\theta t}{2}\right) = \frac{\theta}{2} \cdot f_Z\left(\frac{\theta t}{2}\right) = \frac{\theta}{2} \cdot \frac{1}{\Gamma(v) \cdot \theta^v} \cdot \frac{\theta^{v-1} t^{v-1}}{2^{v-1}} \cdot e^{-t/2}$$

$$= \frac{1}{\Gamma(\frac{2v}{2}) \cdot 2^v} \cdot t^{v-1} \cdot e^{-t/2}. \quad \text{So} \quad T \sim Gamma\left(\frac{2v}{2}, 2\right) = \chi^2_{2v}$$

(If we had $f_Z(z) = \frac{\theta^v}{\Gamma(v)} \cdot z^{v-1} \cdot e^{-\theta z}$ then $T = 2\theta z \sim \chi^2_{2v}$

⋮

Example

Let $X_1, \ldots, X_v$ r.s. of $Exp(\theta)$ with $f(x;\theta) = \theta e^{-\theta x}$, $x>0, \theta>0$

Find sufficient and complete statistic

Solution

a) Factorization criterion: $f(\underline{x};\theta) = \prod_{i=1}^{v} f(x_i;\theta) = \prod_{i=1}^{v} \theta e^{-\theta x_i} = \theta^v e^{-\theta \sum_{i=1}^{v} x_i} =$

$= g(T(\underline{x}), \theta) \cdot h(\underline{x})$, for $h(\underline{x}) = 1$, $g(T(\underline{x});\theta) = \theta^v \cdot e^{-\theta \sum x_i}$

hence $T(\underline{X}) = \sum_{i=1}^{v} X_i$ is sufficient for θ.

b) We find the PDF of T:

It is $T = \sum_{i=1}^{v} X_i \sim Gamma(v, \theta)$ so $f_T(t) = \frac{\theta^v}{\Gamma(v)} \cdot t^{v-1} \cdot e^{-\theta t}$, $t>0$

c) Let $g(T)$ a function of T. Then:

$E[g(T)] = \int_0^\infty g(t) \cdot f_T(t) dt = \int_0^\infty g(t) \frac{\theta^v}{\Gamma(v)} \cdot t^{v-1} e^{-\theta t} dt$

and $E[g(T)] = 0 \Rightarrow \int_0^\infty g(t) \cdot \frac{t^{v-1}}{\Gamma(v)} \cdot e^{-\theta t} dt = 0$

(*) The Laplace transform of a function $f(t)$ is defined as $\mathcal{L}(f(t)) = \int_0^\infty f(t) \cdot e^{-st} dt = F(s)$, for $t \geq 0$

In our case it is $f(t) = g(t) \frac{t^{v-1}}{\Gamma(v)}$

Beause the Laplace transform is unique we have:

$g(t) \frac{t^{v-1}}{\Gamma(v)} = 0 \quad \forall t>0 \quad \Rightarrow g(t) = 0 \quad \forall t>0$

Hence, $T = \sum_{i=1}^{v} X_i$ is complete.

Example

Let $X_1, \dots, X_\nu$ r.s. of $N(\mu, 1)$. Find sufficient and complete statistic for μ.

Solution

a) We have shown previously that $T(\underline{X}) = \sum_{i=1}^{\nu} X_i$ is sufficient for μ.

b) $T = \sum_{i=1}^{\nu} X_i \sim N(\nu\mu, \nu)$ so $f_T(t) = \frac{1}{\sqrt{2\pi\nu}} \exp\{-\frac{(t-\nu\mu)^2}{2\nu}\}$

c) Let $g(T)$ a function of T. Then

$E[g(T)] = 0 \quad \forall \mu \in \mathbb{R} \Rightarrow$

$\Rightarrow \frac{1}{\sqrt{2\pi\nu}} \int_{-\infty}^{\infty} g(t) \exp\{-\frac{1}{2\nu} \cdot (t-\nu\mu)^2\} dt = 0 \quad \forall \mu \in \mathbb{R} \Rightarrow$

$\Rightarrow \int_{-\infty}^{+\infty} g(t)\, e^{-\frac{t^2}{2\nu}} \cdot e^{t\mu} \cdot e^{-\nu\mu^2/2} dt = 0 \quad \forall \mu \in \mathbb{R} \Rightarrow$

$\Rightarrow \int_{-\infty}^{+\infty} g(t)\, e^{-t^2/2\nu} \cdot e^{\mu t} dt = 0 \quad \forall \mu \in \mathbb{R}$

So the Laplace transform of $g(t) \cdot e^{-\frac{t^2}{2\nu}}$ is zero $\forall t \in \mathbb{R}$.

Hence, since the transform is unique, we have:

$g(t) e^{-t^2/2\nu} = 0 \quad \forall t \in \mathbb{R} \Rightarrow g(t) = 0 \quad \forall t \in \mathbb{R}$ and T is complete.

Example

Let $X_1, \dots, X_\nu$ r.s of $N(0, \sigma^2)$. Find sufficient and complete statistic for σ^2.

Solution

a) From the factorization criterion we derive that $T = \sum_{i=1}^{\nu} X_i^2$ is sufficient for σ^2.

b) It is: $X_i \sim N(0, \sigma^2) \Rightarrow \frac{X_i}{\sigma} \sim N(0,1) \Rightarrow$

$\Rightarrow \left(\frac{X_1}{\sigma}\right)^2 = \frac{X_1^2}{\sigma^2} \sim \chi_1^2$

and $Y = \sum_{i=1}^{\nu} \frac{X_i^2}{\sigma^2} = \frac{1}{\sigma^2} \cdot \sum_{i=1}^{\nu} X_i^2 = \frac{T}{\sigma^2} \sim \chi_\nu^2$

so $T = \sigma^2 Y$ and we are looking for its PDF.

$F_T(t) = P(T \le t) = P(\sigma^2 Y \le t) = P(Y \le t/\sigma^2) = F_Y(t/\sigma^2) \overset{d/dt}{\Longrightarrow}$

$f_T(t) = \frac{1}{\sigma^2} f_Y(t/\sigma^2) = \frac{1}{\sigma^2} \frac{1}{2^{\nu/2} \cdot \Gamma(\nu/2)} \cdot \left(\frac{t}{\sigma^2}\right)^{\frac{\nu}{2}-1} \cdot e^{-t/2\sigma^2} =$

$= \frac{1}{(2\sigma^2)^{\nu/2} \Gamma(\frac{\nu}{2})} \cdot t^{\frac{\nu}{2}-1} \cdot e^{-t/2\sigma^2}$

c) Let $g(T)$ function of T. $E[g(T)] = 0 \quad \forall \sigma^2 > 0 \Rightarrow$

$\Rightarrow \int_0^\infty g(t) \cdot t^{\frac{\nu}{2}-1} \cdot e^{-t/2\sigma^2} dt = 0 \quad \forall \sigma^2 > 0$

So, since the Laplace transform is unique, it is:

$g(t) \cdot t^{\frac{\nu}{2}-1} = 0 \quad \forall t > 0 \Rightarrow g(t) = 0 \quad \forall t > 0$

and hence T is complete.

Proposition

If X belongs to the EFD (expon. family of distributions) with 1 parameter θ and $X_1, \dots, X_\nu$ r.s of the same distribution as X, specifically:

$f(x;\theta) = \exp\{q(\theta) \cdot T(x) - B(\theta)\} \cdot h(x)$

then the statistic $Y = \sum_{i=1}^{\nu} T(X_i)$ is sufficient and complete for θ.