Proof (Continuation from last lecture)

$$\sum_{i=1}^{\nu} \frac{\partial}{\partial\theta} \log f(x_i;\theta) = K(\theta)[\hat{\theta}-\theta] \xRightarrow{\partial/\partial\theta}$$

$$\Rightarrow \sum_{i=1}^{\nu} \frac{\partial^2}{\partial\theta^2} \log f(x_i;\theta) = K'(\theta)[\hat{\theta}-\theta] + K(\theta)(-1) \Rightarrow$$

$$\Rightarrow E\left[\sum_{i=1}^{\nu} \frac{\partial^2}{\partial\theta^2} \log f(x_i;\theta)\right] = K'(\theta)\cdot(\cancel{E[\hat{\theta}]}^{\theta} - \theta) - K(\theta) \Rightarrow$$

$$\Rightarrow K(\theta) = -E\left[\sum_{i=1}^{\nu} \frac{\partial^2}{\partial\theta^2} f(x_i;\theta)\right] = I_{\underline{X}}(\theta) = \frac{1}{CR-LB} \Rightarrow$$

$$\Rightarrow -K(\theta) < 0$$

Example

Let $X_1,\dots,X_\nu$ r.s of $U(a,b)$. Find MLE of (a,b)

Solution

$$f(x;a,b) = \begin{cases} \frac{1}{b-a}, & a<x<b \\ 0, & \text{otherwise} \end{cases} = \frac{1}{b-a} I(a<x<b)$$

$$L(a,b) = \prod_{i=1}^{\nu} f(x_i;a,b) = \prod_{i=1}^{\nu} \frac{1}{b-a}\cdot I(a<x_i<b) = \frac{1}{(b-a)^\nu}\cdot\prod_{i=1}^{\nu} I(a<x_i<b)$$

We can't calculate $\log L(a,b)$ so we maximize $L(a,b)$.

We have $\prod_{i=1}^{\nu} I(a<x_i<b) = I(a<x_{(1)}<\infty)\cdot I(0<x_{(\nu)}<b)$ so it is clear that the likelihood is maximum if $\hat{a} = x_{(1)} = \min_i\{x_i\}$ and $\hat{b} = x_{(\nu)} = \max_i\{x_i\}$ so the MLE is $(\hat{a},\hat{b}) = (x_{(1)}, x_{(\nu)})$

Theorem

Let $X_1,\dots,X_\nu$ r.s. with PDF $f(x;\theta)$, $\theta\in\Theta\subseteq\mathbb{R}$ and $\varphi(\cdot)$ function defined at Θ and "1-1".

If $\hat{\theta}$ is MLE of θ, then $\varphi(\hat{\theta})$ is MLE of $\varphi(\theta)$.

Theorem

Let $X_1,\dots,X_\nu$ r.s with PDF $f(x;\theta)$ and $T=T(\underline{X})$ sufficient

statistic for θ.
If $\hat{\theta}$ is MLE of θ, then it is unique (it is the only MLE) and it is a function of $T(\underline{x})$.

Proof

$L(\theta)=\prod_{i=1}^{v} f(x;\theta)= g(T(\underline{x});\theta)\cdot h(\underline{x}) \Rightarrow$

$\Rightarrow \max_{\theta} L(\theta) = h(\underline{x})\cdot \max_{\theta} g(T(\underline{x});\theta)$, so if there exists a unique MLE then it is a function of $T(\underline{x})$

Example

Let $X_1,\dots,X_v$ r.s. with CDF $F(x;\theta)=1-\frac{\theta^3}{x^3}$, $\theta>0$, $x\geq\theta$

a) Find MLE of θ and its PDF.

b) Find $a\in\mathbb{R}$ so that $\delta=aT=aX_{(1)}$ is an u.e. of θ

c) Prove that δ is consistent

Solution

a) $f(x;\theta)=\frac{\partial}{\partial x}F(x;\theta)=\frac{3\theta^3}{x^4}$, $x\geq\theta$ $=\begin{cases}\frac{3\theta^3}{x^4}, & x\geq\theta\\ 0, & \text{otherwise}\end{cases}$

or $f(x;\theta)=\frac{3\theta^3}{x^4}\cdot I(\theta\leq x<\infty)$

$L(\theta)=\prod_{i=1}^{v} f(x_i;\theta)=\prod_{i=1}^{v}\frac{3\theta^3}{x_i^4}\cdot I(\theta<x_i<\infty)=\frac{3^v\theta^{3v}}{\prod_{i=1}^{v} x_i^4}\cdot\prod_{i=1}^{v} I(\theta\leq x_i<\infty)$

We want $3^v\theta^{3v}$ to be maximum, but it is $\theta<x_{(1)}<\dots<x_{(v)}$ so $\hat{\theta}=x_{(1)}$.

Distribution of $T=X_{(1)}$: $F_T(t)=P(T\leq t)=P(X_{(1)}\leq t)=$

$=1-P(X_{(1)}>t)=1-P(X_1,\dots,X_v>t)=1-\prod_{i=1}^{v}P(X_i>t)=$

$=1-(P(X>t))^v=1-\frac{\theta^{3v}}{t^{3v}}$

so $f(t)=\frac{\partial}{\partial t}F_T(t)=3v\frac{\theta^{3v}}{t^{3v+1}}=3v\,\theta^{3v}\cdot t^{-3v-1}$, $t>\theta$

b) $E[T] = E[X_{(1)}] = \int_\theta^\infty t f_T(t)\,dt = \int_\theta^\infty t \cdot 3v\theta^{3v} \cdot t^{-3v-1}\,dt =$
$= 3v\theta^{3v} \frac{t^{-3v+1}}{-3v+1}\Big|_\theta^\infty = \frac{3v}{3v-1} \cdot \theta$

We want $E[\delta] = E[aT] = \theta \Rightarrow aE[T] = \theta \Rightarrow a = \frac{3v-1}{3v}$

c) $V(\delta) = V(aT) = a^2 V(T) = \left(\frac{3v-1}{3v}\right)^2 V(T) = \left(\frac{3v-1}{3v}\right)^2 \left(E[T^2] - (E[T])^2\right)$

We have $E[T^2] = \int_\theta^\infty t^2 f_T(t)\,dt = \dots = \frac{3v}{3v-2}\theta^2$

$V(\delta) = \dots = \frac{1}{3v(3v-2)}\theta^2 \xrightarrow{v\to\infty} 0$ so δ is consistent.

Method of Moments

If we want to estimate s parameters we set the s first moments (at zero) of the sample equal to those of the actual population.

$$M_1 = \frac{1}{v}\sum_{i=1}^{v} X_i = \bar{X} = \mu_1' = E[X]$$
$$M_2 = \frac{1}{v}\sum_{i=1}^{v} X_i^2 = \mu_2' = E[X^2]$$
$$\vdots$$
$$M_s = \frac{1}{v}\sum_{i=1}^{v} X_i^s \underset{\text{demanded}}{=} \mu_3 = E[X^s]$$

Example

Let $X_1, \dots, X_v$ r.s. of Bernoulli(p). Find an estimator of p using the method of moments.

Solution

$f(x;p) = p^x(1-p)^{1-x}$, $x = 0,1$, $p \in (0,1)$

$\mu = E[X] = p$ and $\bar{X} = \frac{1}{v}\sum_{i=1}^{v} X_i$, so $\hat{p} = \bar{X}$

Example

Let $X_1, \dots, X_n$ r.s of Poisson(λ). In the same way as before the method of moments results in $\hat{\lambda} = \bar{X}$

Note

If we want to estimate 2 parameters, we often set $E[X]$ equal to $\bar{X}$ and as a second equation we set $E[(X-\mu)^2] = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$ (second moment at μ)

So: $E[X] = \bar{X}$ and $E[X^2] = \frac{1}{n}\sum_{i=1}^{n} X_i^2$

<u>or</u> $E[X] = \bar{X}$ and $E[(X-\mu)^2] = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$