Example

Let $X_1, \dots, X_v$ r.s of $Bin(n,p)$. Find estimators of n, p using the method of moments.

Solution

We have $E[X] = \mu = np$ and $E[(X-\mu)^2] = \sigma^2 = np(1-p)$

We equate with $\bar{X} = \frac{1}{v}\sum_{i=1}^{v} X_i$, $M_2 = \frac{1}{v}\sum_{i=1}^{v}(X_i - \bar{X})^2$

So
$$\left.\begin{aligned} np &= \bar{X} \\ np(1-p) &= M_2 \end{aligned}\right\} \Rightarrow \frac{np}{np(1-p)} = \frac{\bar{X}}{M_2} \Rightarrow \hat{p} = 1 - \frac{M_2}{\bar{X}}$$

and $n\hat{p} = \bar{X} \Rightarrow n = \frac{\bar{X}}{\hat{p}} = \frac{\bar{X}}{1 - M_2/\bar{X}} = \frac{(\bar{X})^2}{\bar{X} - M_2}$

$Bin(n,p)$: $\mu > \sigma^2$

$Poisson(\lambda)$: $\mu \simeq \sigma^2$

$NegBin(n,p)$: $\mu < \sigma^2$

Example

Let $X_1, \dots, X_v$ r.s. with PDF $f(x;\theta) = \frac{3x^2}{\theta} \cdot \exp\left\{-\frac{x^3}{\theta}\right\}$, $x \geq 0$, $\theta > 0$

Find MLE of θ, check if it's unbiased/consistent.

Solution

$$L(\theta) = \prod_{i=1}^{v} f(x_i;\theta) = \prod_{i=1}^{v} \frac{3x_i^2}{\theta} \cdot \exp\left\{-\frac{x_i^3}{\theta}\right\} = \frac{3^v}{\theta^v} \cdot \prod_{i=1}^{v} x_i^2 \cdot \exp\left\{-\frac{1}{\theta}\sum_{i=1}^{v} x_i^3\right\}$$

$$\ell(\theta) = \log L(\theta) = v\log 3 - v\log\theta + \log\prod_{i=1}^{v} x_i^2 - \frac{1}{\theta}\sum_{i=1}^{v} x_i^3$$

$$\frac{\partial}{\partial\theta}\ell(\theta) = -\frac{v}{\theta} + \frac{1}{\theta^2}\cdot\sum_{i=1}^{v} x_i^3 \quad \text{so} \quad \frac{\partial}{\partial\theta}\ell(\theta) = 0 \Rightarrow \sum_{i=1}^{v} x_i^3 = v\theta \Rightarrow$$

$$\Rightarrow \hat{\theta} = \frac{1}{v}\cdot\sum_{i=1}^{v} x_i^3$$

$$\frac{\partial^2}{\partial\theta^2}\ell(\theta)\Big|_{\theta=\hat{\theta}} = -\frac{v}{\theta^2} - \frac{2}{\theta^3}\sum_{i=1}^{v} x_i^3\Big|_{\theta=\hat{\theta}} = -\frac{v\theta - 2\sum x_i^3}{\theta^3}\Big|_{\theta=\hat{\theta}} =$$

$$= -\frac{v\cdot\sum x_i^3/v - 2\sum x_i^3}{(\sum x_i^3/v)^3} = -3\cdot\frac{v^3}{(\sum x_i^3)^2} < 0 \quad \text{so } \hat{\theta} \text{ is MLE.}$$

We have $E[\hat{\theta}] = E\left[\frac{\sum x_i^3}{v}\right] = \frac{1}{v} \cdot \sum_{i=1}^{v} E[X_i^3]$.

For $Y = X^3$ we have $F_Y(y) = P(Y \leq y) = P(X^3 \leq y) = P(X \leq y^{1/3}) =$
$= F_X(y^{1/3})$, so $f_Y(y) = \frac{\partial}{\partial y} F_Y(y) = \frac{\partial}{\partial y} F_X(y^{1/3}) = \frac{1}{3} y^{-\frac{2}{3}} f_X(y^{1/3})$

$\Rightarrow f_Y(y) = \frac{1}{3} y^{-\frac{2}{3}} \cdot \frac{3y^{2/3}}{\theta} \cdot \exp\left\{-\frac{y}{\theta}\right\} = \frac{1}{\theta} \cdot e^{-y/\theta} \sim Exp\left(\frac{1}{\theta}\right)$

So $Y = X^3 \sim Exp(1/\theta)$ so $E[X^3] = \theta$

and $E[\hat{\theta}] = \frac{1}{v} \sum_{i=1}^{v} \theta = \frac{1}{v} \cdot v\theta = \theta$, so $\hat{\theta}$ is unbiased

For consistency we already have $E[\hat{\theta}] = \theta \to \theta$

and $V(\hat{\theta}) = V\left(\sum_{i=1}^{v} x_i^3 / v\right) = \frac{1}{v^2} \sum_{i=1}^{v} V(X^3) = \frac{1}{v^2} \cdot v \cdot \theta^2 = \frac{\theta^2}{v} \to 0$

so $\hat{\theta}$ is consistent.

Note (Alternative)

$Y = X^3 \sim Exp\left(\frac{1}{\theta}\right) \Rightarrow T = \sum_{i=1}^{v} X_i^3 \sim Gamma(v, \theta)$, so $E[T] = v\theta$

Confidence Intervals

$f(x;\theta) \longrightarrow \hat{\theta} \longrightarrow$ Statistic $(LB(\underline{X}), UB(\underline{X}))$

Lower Bound — Upper Bound

We want $P(LB(\underline{X}) < \theta < UB(\underline{X})) = 1 - a$

Usually, $a = 1\%, 5\%, 10\%$

Definition

$1-a$ is called **confidence level**.

Example

Let $X_1, \dots, X_v$ r.s. of $N(\mu, \sigma^2)$, where σ^2 is known.

Find CI (confidence interval) for μ.

Solution

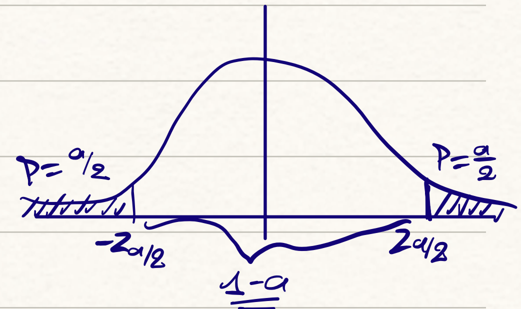
We know that $\hat{\mu} = \bar{X}$ and $\bar{X} \sim N(\mu, \frac{\sigma^2}{n}) \Rightarrow \boxed{Z = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)}$ → PIVOT

$$P(-Z_{a/2} < Z < Z_{a/2}) = 1-a \Rightarrow$$

$$\Rightarrow P\left(-Z_{a/2} < \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} < Z_{a/2}\right) = 1-a \Rightarrow$$

$$\Rightarrow P\left(-Z_{a/2} \cdot \frac{\sigma}{\sqrt{n}} < \bar{X}-\mu < Z_{a/2} \cdot \frac{\sigma}{\sqrt{n}}\right) = 1-a \Rightarrow$$

$$\Rightarrow P\left(\bar{X} - Z_{a/2} \cdot \frac{\sigma}{\sqrt{n}} < \mu < \bar{X} - Z_{a/2} \cdot \frac{\sigma}{\sqrt{n}}\right) = 1-a$$



$100\cdot(1-a)\%$ CI for μ : $\left[\bar{X} - Z_{a/2}\frac{\sigma}{\sqrt{n}}, \bar{X} + Z_{a/2}\frac{\sigma}{\sqrt{n}}\right]$, or $\bar{X} \pm Z_{a/2}\frac{\sigma}{\sqrt{n}}$

General Method

Let $T = T(\underline{X})$ estimator of θ

- We search for a r.v. $Y(T, \theta)$, function of T, θ and perhaps more parameters
- $Y(T,\theta)$ follows a known distribution, independent of the other parameters involved
- By choosing constants c_1, c_2 we can have $P(c_1 < Y < c_2) = 1-a$
- By solving for θ we get the boundaries $LB(x), UB(x)$ so that $P(LB(x) < \theta < UB(x)) = 1-a$

In the previous example $Y = Z = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$

Example

Let $X_1, \dots, X_n$ r.s. of $N(\mu, \sigma^2)$, μ, σ^2 both unknown.
Find CI for μ.

Solution

$\hat{\mu} = \bar{X} \sim N(\mu, \frac{\sigma^2}{n})$ and $Z = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$

and $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$. $\bar{X}$, S^2 are independent

So $T = \left(\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\right) \Big/ \sqrt{\frac{(n-1)S^2}{\sigma^2} / n-1} = \frac{\bar{X}-\mu}{S/\sqrt{n}} \sim t_{n-1}$

$P\left(-t_{n-1,\alpha/2} < \frac{\bar{X}-\mu}{S/\sqrt{n}} < t_{n-1,\alpha/2}\right) = 1-\alpha \implies \dots \implies$

$\implies P\left(\bar{X} - t_{n-1,\alpha/2} \cdot \frac{S}{\sqrt{n}} < \mu < \bar{X} + t_{n-1,\alpha/2}\right) = 1-\alpha$