31ο Μάθημα Πιθανοτήτων 26/5/2010

① Πιθανογεννήτριες (Επανάληψη)

X μη αρνητ. ακέρ. τ.μ.

$P_X(z) = E[z^X] = \sum_{n=0}^{\infty} P(X=n) z^n$, $z \in \{s \in \mathbb{C} : |s| \leq 1\}$

1) $P(X=n) = \frac{P_X^{(n)}(0)}{n!}$, $n = 0, 1, \ldots$

2) X, Y ισόνομες $\Leftrightarrow P_X(z) = P_Y(z)$

3) $\left.\begin{array}{l} X_1, X_2, \ldots, X_n \text{ ανεξ.} \\ S_n = \sum_{i=1}^{n} X_i \end{array}\right\} \Rightarrow P_{S_n}(z) = P_{X_1}(z) P_{X_2}(z) \ldots P_{X_n}(z)$

4) $\left.\begin{array}{l} X_1, X_2, \ldots \text{ ανεξ + ισόνομ.} \\ N \text{ ακέρ. ανεξ. των } X_i \\ S_N = \sum_{i=1}^{N} X_i \end{array}\right\} \Rightarrow P_{S_N}(z) = P_N(P_X(z))$

5) $E[X(X-1)(X-2)\ldots(X-n+1)] = P_X^{(n)}(1)$

$\qquad \shortparallel$

$E[(X)_n] \leftarrow$ καθοδική παραγοντική ροπή n-τάξης

Απόδειξη:

$P_X(z) = E[z^X] \Rightarrow P_X(1) = 1$

$P_X'(z) = E[X z^{X-1}] \Rightarrow P_X'(1) = E[X]$

$P_X''(z) = E[X(X-1) z^{X-2}] \Rightarrow P_X''(1) = E[X(X-1)]$

② Παραδείγματα Πιθανογεννήτριας

$X \sim Bernoulli(p) \Rightarrow P_X(z) = 1 - p + pz$

$X \sim Bin(n, p) \Rightarrow P_X(z) = (1 - p + pz)^n$

$X \sim Geom(p) \Rightarrow P_X(z) = \frac{pz}{1-(1-p)z}$

$X \sim NegBin(n, p) \Rightarrow P_X(z) = \left(\frac{pz}{1-(1-p)z}\right)^n$

$X \sim Poisson(\lambda) \Rightarrow P_X(z) = e^{-\lambda(1-z)}$

③ Διωνυμική Κατανομή

$X \sim Bin(n,p) \implies E[X] = ;\quad Var[X] = ;\quad P_X(z) = (1-p+pz)^n$

1^{os} τρόπος: $E[X] = \sum_{x=0}^{n} x \binom{n}{x} p^n (1-p)^{n-x}$

2^{os} τρόπος: $E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i]$, $X_i \sim Bernoulli(p)$

3^{os} τρόπος: $E[X] = P_X'(1) = n(1-p+pz)^{n-1} p \Big|_{z=1}$

Αντίστοιχα για την $Var[X]$ έχουμε:

1^{os} τρόπος: $Var[X] = E[X^2] - (E[X])^2 = E[X(X-1)] + E[X] - E[X]^2$

Ισχύει γ/ διακρ. $= \boxed{P_X''(1) + P_X'(1) - (P_X'(1))^2}$

$$P_X''(z) = \frac{d}{dz}\left(n(1-p+pz)^{n-1}p\right) = n(n-1)(1-p+pz)^{n-2}p^2$$

$$Var[X] = n(n-1)p^2 + np - (np)^2 = np(1-p)$$

Τώρα:

$$\left.\begin{array}{l} X \sim Bin(n,p) \\ Y \sim Bin(m,p) \\ X, Y \text{ ανεξ.} \end{array}\right\} \quad Z = X+Y. \text{ Τι κατανομή ακολουθεί η } Z$$

$\Downarrow$

$$\left.\begin{array}{l} P_X(z) = (1-p+pz)^n \\ P_Y(z) = (1-p+pz)^m \\ X, Y \text{ ανεξ.} \end{array}\right. \implies P_Z(z) = P_X(z)P_Y(z) = (1-p+pz)^{n+m} \implies \underline{\underline{Z \sim Bin(m+n,p)}}$$

④ Ροπογεννήτριες

X τ.μ. Ροπογεννήτρια της X: $M_X(t) \overset{\text{ορσ}}{=} E[e^{tX}] = \begin{cases} \sum_x e^{tx} P(X=x), & X \text{ διακριτή} \\ \int_{-\infty}^{\infty} e^{tx} f_X(x)\,dx, & X \text{ συνεχ.} \end{cases}$

ΙΔΙΟΤΗΤΕΣ

1) X, Y ισόνομες $\Leftrightarrow M_X(t) = M_Y(t)$

2) $X_1, X_2, \dots X_n$ ανεξ. , $S_n = \sum_{i=1}^{n} X_i$ $\Rightarrow M_{S_n}(t) = M_{X_1}(t) \dots M_{X_n}(t)$

Απόδειξη:

$$M_{S_n}(t) = E[e^{tS_n}] = E[e^{t(X_1+X_2+\dots+X_n)}] = E[e^{tX_1+tX_2+\dots+tX_n}] \underset{\text{ανεξ}}{=}$$
$$= E[e^{tX_1}] \cdot E[e^{tX_2}] \dots E[e^{tX_n}] = M_{X_1}(t) M_{X_2}(t) \dots M_{X_n}(t)$$

3) $X_1, X_2, \dots$ ανεξ.
N ακερ. ανεξ. των X_i
$S_N = \sum_{i=1}^{N} X_i$
$\Rightarrow M_{S_N}(t) = P_N(M_X(t))$

Απόδειξη

$$M_{S_N}(t) = E[e^{tS_N}] = E[E[e^{tS_N} / N]]$$
$$= \sum_{n=0}^{\infty} P(N=n) E[e^{tS_N} / N=n] \quad (*)$$

$$E[e^{tS_N}/N=n] = E[e^{t(X_1+X_2+\dots+X_N)} / N=n] = E[e^{tX_1+tX_2+\dots+tX_N}] = (M_X(t))^n \quad (**)$$

Επομένως: $(*), (**) \Rightarrow M_{S_N}(t) = \sum_{n=0}^{\infty} P(N=n) \cdot (M_X(t))^n = P_N(M_X(t))$

5) Ροπογεννήτριες - Ιδιότητες (επιπλέον)

- $E[X^n] = M_X^{(n)}(0) \quad (1)$

  ↳ ροπή n τάξης της X

- X διακριτή $M_X(t) = P_X(e^t) \quad (2)$

Απόδειξη: (της (1))

$$M_X^{(n)}(t) = \frac{d^n}{dt^n} E[e^{tX}] = E[X^n e^{tX}] \Rightarrow M_X^{(n)}(0) = E[X^n]$$

Απόδειξη: (της (2))

$$M_X(t) = E[e^{tX}] = E[(e^t)^X] = P_X(e^t)$$

⑥ Παράδειγμα Ροπογεννήτριας

1) $X \sim Bin(n,p)$

$$P_X(z) = (1-p+pz)^n$$

$$M_X(t) = (1-p+pe^t)^n$$

$$E[X] = M_X'(0) = n(1-p+pz)^{n-1} pe^t \Big|_{t=0} = np$$

Γενικά για διακριτές βρίσκω πρώτα την $P_X(z)$.

2) $X \sim Exp(\lambda)$

$$f_X(x) = \lambda \cdot e^{-\lambda x}, \quad x>0 \quad \sigma.\pi.\pi.$$

$$M_X(t) = E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f_X(x)\,dx = \int_0^{\infty} e^{tx} \lambda \cdot e^{-\lambda x}\,dx =$$

$$= \lambda \int_0^{\infty} e^{-(\lambda-t)x}\,dx = \frac{\lambda}{\lambda-t} \qquad t<\lambda$$

$$E[X] = M_X'(0) = \frac{\lambda}{(\lambda-t)^2}\Big|_{t=0} = \frac{1}{\lambda}$$

$$E[X^2] = M_X''(0) = \frac{2\lambda}{(\lambda-t)^3}\Big|_{t=0} = \frac{2}{\lambda^2}$$

$$\Rightarrow Var[X] = E[X^2]-(E[X])^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

3) $Z \sim N(0,1)$

$$M_Z(t) = E[e^{tZ}] = \int_{-\infty}^{\infty} e^{tz} f_Z(z)\,dz = \int_{-\infty}^{\infty} e^{tz} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\,dz$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2-2tz}{2}}\,dz = e^{t^2/2} \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(z-t)^2}{2}}\,dz}_{\Downarrow \ \sigma\pi\pi \text{ της } N(t,1)} = e^{t^2/2}$$

$X \sim N(\mu, \sigma^2)$

$$\frac{X-\mu}{\sigma} = Z \sim N(0,1) \Rightarrow X = \sigma Z + \mu$$

$$M_X(t) = E[e^{tX}] = E[e^{t(\sigma Z+\mu)}] = E[e^{t\sigma Z} \cdot e^{t\mu}] = e^{t\mu} \cdot M_Z(\sigma t)$$

$$= e^{t\mu + \frac{t^2\sigma^2}{2}}$$

$$E[X] = M_X'(0) = e^{t\mu+\frac{t^2\sigma^2}{2}} \left(\mu + \frac{2t\sigma^2}{2}\right)\Big|_{t=0} = \mu.$$

Αναγεννητική Ιδιότητα

$$\left.\begin{array}{l} X_1 \sim N(\mu_1, \sigma_1^2) \\ X_2 \sim N(\mu_2, \sigma_2^2) \\ X_1, X_2 \text{ ανεξ.} \end{array}\right\} \Rightarrow \begin{array}{l} Z = X_1 + X_2 \\ \text{Τι κατανομή ακολουθεί} \end{array}$$

$$X_1 \sim N(\mu_1, \sigma_1^2) \Rightarrow M_{X_1}(t) = e^{t\mu_1 + \frac{t^2\sigma_1^2}{2}}$$

$$X_2 \sim N(\mu_2, \sigma_2^2) \Rightarrow M_{X_2}(t) = e^{t\mu_2 + \frac{t^2\sigma_2^2}{2}}$$

$$X_1, X_2 \text{ ανεξ.} \Rightarrow M_Z(t) = M_{X_1}(t)\, M_{X_2}(t) = e^{t(\mu_1+\mu_2) + \frac{t^2(\sigma_1^2+\sigma_2^2)}{2}}$$

$$\Rightarrow Z \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$$