26.05.10 31° μάθημα

① Πιθανογεννήτριες (Επανάληψη)

X μη-αρν. ακέραια τ.μ.

$P_X(z) = E[z^X] = \sum_{n=0}^{\infty} P(X=n) z^n$, $z \in \{s \in \mathbb{C} : |s| \leq 1\}$

1. $P(X=n) = \frac{P_X^{(n)}(0)}{n!}$ $n = 0, 1, 2 \ldots$
2. X, Y ισόνομες $\Leftrightarrow P_X(z) = P_Y(z)$
3. $\left.\begin{array}{l} X_1, X_2, \ldots, X_n \text{ ανεξ} \\ S_n = \sum_{i=1}^{n} X_i \end{array}\right\} \Rightarrow P_{S_n}(z) = P_{X_1}(z) P_{X_2}(z) \ldots P_{X_n}(z)$
4. $\left.\begin{array}{l} X_1, X_2, \ldots, X_\nu \text{ ανεξ κ' ισον} \\ N \text{ ακέραια ανεξ των } X_i \\ S_N = \sum_{i=1}^{N} X_i \end{array}\right\} \Rightarrow P_{S_N}(z) = P_N(P_X(z))$
5. $E[X(X-1)(X-2)\ldots(X-n+1)] = P_X^{(n)}(1)$

   $E[(X)_n]$ καθοδική παραγοντική ροπή n-τάξης

   Απόδειξη:

$$P_X(z) = E[z^X] \Rightarrow P_X(1) = 1$$
$$P_X'(z) = E[Xz^{X-1}] \Rightarrow P_X'(1) = E[X]$$
$$P_X''(z) = E[X(X-1)z^{X-2}] \Rightarrow P_X''(1) = E[X(X-1)]$$

② Παραδείγματα Πιθανογεννητριών

$X \sim Bernoulli(p) \Rightarrow P_X(z) = 1 - p + pz$

$X \sim Bin(n,p) \Rightarrow P_X(z) = (1-p+pz)^n$

$X \sim Geom(p) \Rightarrow P_X(z) = \frac{pz}{1-(1-p)z}$

$X \sim NegBin(n,p) \Rightarrow P_X(z) = \left(\frac{pz}{1-(1-p)z}\right)^n$

$X \sim Poisson(\lambda) \Rightarrow P_X(z) = e^{-\lambda(1-z)}$

$E[X]$, $Var[X]$, Αναγεννητικές Ιδιότητες

③ Διωνυμική Κατανομή

$X \sim Bin(n,p)$ με $P_X(z) = (1-p+pz)^n$

$E[X] = ;$

$Var[X] = ;$

1ος τρόπος : $E[X] = \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k} \ldots$

2ος τρόπος : $E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i]$, $X_i \sim Bernoulli(p)$

3ος τρόπος : $E[X] = P_X'(1) = n(1-p+pz)^{n-1} p \,\big|_{z=1} = np$

$Var[X] = E[X^2] - E^2[X] = E[X(X-1)] + E[X] - E^2[X] =$
$= P_X''(1) + P_X'(1) - (P_X'(1))^2$ ~ ισχύει για κάθε διακριτή !

$P_X''(z) = \frac{d}{dz}\left(n(1-p+pz)^{n-1} p\right) = n(n-1)(1-p+pz)^{n-2} p^2$

$P_X''(1) = n(n-1)p^2$

$Var[X] = n(n-1)p^2 + np - (np)^2 = np(1-p)$

$X \sim Bin(n,p)$

$Y \sim Bin(m,p)$

X, Y ανεξ

$Z = X + Y$

Τι κατανομή ακολουθεί η Z ;

$$\left.\begin{aligned} P_X(z) &= (1-p+pz)^n \\ P_Y(z) &= (1-p+pz)^m \end{aligned}\right\} \quad \begin{aligned} &P_Z(z) = P_X(z)P_Y(z) = (1-p+pz)^{n+m} \\ &\text{άρα } Z \sim Bin(n+m,p) \end{aligned}$$

X, Y ανεξ

④ Ροπογεννήτριες

X τ.μ.

Ροπογεννήτρια της X : $M_X(t) \overset{ορσ}{=} E[e^{tX}] = \begin{cases} \sum_x e^{tx} P(X=x), & X \text{ διακριτή} \\ \int_{-\infty}^{\infty} e^{tx} f_X(x)dx, & X \text{ συνεχής} \end{cases}$

1. X, Y ισόν $\Leftrightarrow$ $M_X(t) = M_Y(t)$

2. $\left.\begin{array}{l} X_1, X_2, \ldots, X_n \text{ ανεξ} \\ S_n = \sum_{i=1}^{n} X_i \end{array}\right\}$ $M_{S_n}(t) = M_{X_1}(t) \ldots M_{X_n}(t)$

απόδειξη : $M_{S_n}(t) = E[e^{tS_n}] = E[e^{t(X_1+X_2+\ldots+X_n)}] =$

$$= E[e^{tX_1} e^{tX_2} \ldots e^{tX_n}] =$$

$$\overset{\text{ανεξ}}{=} E[e^{tX_1}] \cdot E[e^{tX_2}] \ldots E[e^{tX_n}] =$$

$$= M_{X_1}(t) M_{X_2}(t) \ldots M_{X_n}(t)$$

3. $\left.\begin{array}{l} X_1, X_2, \ldots, X_n \text{ ανεξ + ισόν.} \\ N \text{ ακέραια ανεξ των } X_i \\ S_N = \sum_{i=1}^{N} X_i \end{array}\right\} \Rightarrow M_{S_N}(t) = P_N(M_X(t))$

απόδειξη: $M_{S_N}(t) = E[e^{tS_N}] = E[E[e^{tS_N} | N]] =$

$$= \sum_{n=0}^{\infty} P(N=n) E[e^{tS_N} | N=n] \quad (*)$$

$$E[e^{tS_N} | N=n] = E[e^{t(X_1+X_2+\ldots+X_n)} | N=n] =$$

$$= E[e^{tX_1+tX_2+\ldots+tX_n}] =$$

$$= (M_X(t))^n \quad (**)$$

Επομένως ⊛ κ' ⊛⊛ $\rightsquigarrow$ $M_{S_N}(t) = \sum_{n=0}^{\infty} P(N=n)(M_X(t))^n = P_N(M_X(t))$

⑤ Ροπογεννήτριες – Ιδιότητες (που δεν έχουν αντίστοιχες οι πιθανογεννήτριες)

1. $E[X^n] = M_X^{(n)}(0)$

   $\uparrow$ ροπή n-τάξης της X

2. X διακριτή : $M_X(t) = P_X(e^t)$

απόδειξη: 1. $M_X^{(n)}(t) = \frac{d^n}{dt^n} E[e^{tX}] = E[X^n e^{tX}]$

$\Rightarrow M_X^{(n)}(0) = E[X^n]$

2. $M_X(t) = E[e^{tX}] = E[(e^t)^X] = P_X(e^t)$

⑥ Παραδείγματα Ροπογεννητριών

1. $X \sim Bin(n,p)$

$P_X(z) = (1-p+pz)^n$

$P_Y(z) = (1-p+pe^t)^n$

$E[X] = M_X'(0) = n(1-p+pe^t)^{n-1} pe^t|_{t=0} = np$

Γενικά για διακριτές, βρίσκω πρώτα την πιθανογεννήτρια $P_X(z)$

2. $X \sim Exp(\lambda)$

σ.π.π. $f_X(x) = \lambda e^{-\lambda x}, \ x>0$

$M_X(t) = E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f_X(x)dx = \int_0^{\infty} e^{tx}\lambda e^{-\lambda x} dx = \lambda\int_0^{\infty} e^{-(\lambda-t)x} dx = \frac{\lambda}{\lambda-t}$

συγκλίνει για $t<\lambda$

$E[X] = M_X'(0) = \frac{\lambda}{(\lambda-t)^2}\Big|_{t=0} = \frac{1}{\lambda}$

$E[X^2] = M_X''(0) = \frac{2\lambda}{(\lambda-t)^3}\Big|_{t=0} = \frac{2}{\lambda^2}$

$Var[X] = E[X^2] - E^2[X] = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$

3. $Z \sim \mathcal{N}(0,1)$

$$M_Z(t) = E[e^{tZ}] = \int_{-\infty}^{\infty} e^{tz} f_Z(z)\, dz = \int_{-\infty}^{\infty} e^{tz} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz =$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2-2tz}{2}} dz = e^{t^2/2} \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(z-t)^2}{2}} dz}_{\text{σ.π.π. της } \mathcal{N}(t,1)} =$$

$$= e^{t^2/2}$$

$X \sim \mathcal{N}(\mu, \sigma^2)$

$\frac{X-\mu}{\sigma} = Z \sim \mathcal{N}(0,1) \Rightarrow X = \sigma Z + \mu$

$$M_X(t) = E[e^{tX}] = E[e^{t(\sigma Z+\mu)}] = E[e^{t\sigma Z} e^{t\mu}] = e^{t\mu} M_Z(\sigma t) = e^{t\mu + \frac{t^2\sigma^2}{2}}$$

$$E[X] = M_X'(0) = e^{t\mu + \frac{t^2\sigma^2}{2}} \left(\mu + \frac{\cancel{2}t\sigma^2}{\cancel{2}}\right) \Big|_{t=0} = \mu$$

$$E[X^2] = M_X''(0) = \ldots$$

Αναπαραγωγική ιδιότητα

$\left.\begin{array}{l} X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2) \\ X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2) \\ X_1, X_2 \text{ ανεξαρτ.} \end{array}\right\}$ $Z = X_1 + X_2$ Τι κατανομή ακολουθεί;

$X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2) \Rightarrow M_{X_1}(t) = e^{t\mu_1 + \frac{t^2\sigma_1^2}{2}}$

$X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2) \Rightarrow M_{X_2}(t) = e^{t\mu_2 + \frac{t^2\sigma_2^2}{2}}$

X_1, X_2 ανεξ $\Rightarrow M_Z(t) = M_{X_1}(t) M_{X_2}(t) = e^{t(\mu_1+\mu_2) + \frac{t^2(\sigma_1^2+\sigma_2^2)}{2}}$

$\Rightarrow Z \sim \mathcal{N}(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$