Πολικές Συντεταγμένες - Τελεστής Laplace

$x = r\cos\theta \quad , \; r = (x^2+y^2)^{1/2} \quad , \; \tilde{u}(r,\theta) = u(r\cos\theta, r\sin\theta) = u(x,y)$

$y = r\sin\theta \quad , \; \theta = \mathrm{Arctan}\,\frac{y}{x} \quad , \; r_x = \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}}\, 2x = \frac{x}{r} = \cos\theta, \; r_y = \sin\theta$

$$\theta_x = \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{\partial}{\partial x}\left(\frac{y}{x}\right) = \frac{\cancel{x^2}}{x^2+y^2}\left(-\frac{y}{\cancel{x^2}}\right) = -\frac{r\sin\theta}{r^2} = -\frac{\sin\theta}{r}$$

$$\theta_y = \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{\partial}{\partial y}\left(\frac{y}{x}\right) = \frac{x^{\cancel{2}}}{x^2+y^2}\frac{1}{\cancel{x}} = \frac{r\cos\theta}{r^2} = \frac{\cos\theta}{r}$$

- $|\nabla u|^2 = \tilde{u}_r^2 + \frac{1}{r^2}\tilde{u}_\theta^2$

$$\Big(\; u_x = \frac{\partial}{\partial x}(\tilde{u}(r,\theta)) = \tilde{u}_r r_x + \tilde{u}_\theta \theta_x = \tilde{u}_r\cos\theta + \tilde{u}_\theta\left(-\frac{\sin\theta}{r}\right)$$

$$u_y = \frac{\partial}{\partial y}(\tilde{u}(r,\theta)) = \tilde{u}_r r_y + \tilde{u}_\theta \theta_y = \tilde{u}_r\sin\theta + \tilde{u}_\theta\left(\frac{\cos\theta}{r}\right)$$

$$u_x^2 + u_y^2 = \left(\tilde{u}_r\cos\theta + \tilde{u}_\theta\left(-\frac{\sin\theta}{r}\right)\right)^2 + \left(\tilde{u}_r\sin\theta + \tilde{u}_\theta\left(\frac{\cos\theta}{r}\right)\right)^2$$

$$= \tilde{u}_r^2\cos^2\theta + \tilde{u}_\theta^2\frac{\sin^2\theta}{r^2} + \tilde{u}_r^2\sin^2\theta + \tilde{u}_\theta^2\frac{\cos^2\theta}{r^2}$$

$$= \tilde{u}_r^2 + \frac{1}{r^2}\tilde{u}_\theta^2 . \; \Big).$$

- $\Delta u := u_{xx} + u_{yy} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$

$$\Big(\; u_{xx} = \frac{\partial}{\partial x}\left(\tilde{u}_r\cos\theta + \tilde{u}_\theta\left(-\frac{\sin\theta}{r}\right)\right)$$

$$= \frac{\partial}{\partial r}\left(\tilde{u}_r\cos\theta + \tilde{u}_\theta\left(-\frac{\sin\theta}{r}\right)\right) r_x + \frac{\partial}{\partial \theta}\left(\tilde{u}_r\cos\theta + \tilde{u}_\theta\left(-\frac{\sin\theta}{r}\right)\right)\theta_x$$

$$= \left(\tilde{u}_{rr}\cos\theta + \tilde{u}_{\theta r}\left(-\frac{\sin\theta}{r}\right) + \tilde{u}_\theta\frac{\sin\theta}{r^2}\right)\cos\theta$$

$$+ \left(\tilde{u}_{r\theta}\cos\theta + \tilde{u}_r(-\sin\theta) + \tilde{u}_{\theta\theta}\left(-\frac{\sin\theta}{r}\right) - \tilde{u}_\theta\frac{\cos\theta}{r}\right)\left(-\frac{\sin\theta}{r}\right)$$

$$= \underbrace{\tilde{u}_{rr}\cos^2\theta}_{} + \quad - \tilde{u}_{\theta r}\frac{\sin\theta\cos\theta}{r} + \tilde{u}_{\theta}\frac{\sin\theta\cos\theta}{r^2}$$

$$+ \tilde{u}_{r\theta}\frac{\sin\theta\cos\theta}{r} + \tilde{u}_{r}\frac{\sin^2\theta}{r^2} + \tilde{u}_{\theta\theta}\frac{\sin^2\theta}{r^2} + \tilde{u}_{\theta}\frac{\sin\theta\cos\theta}{r^2}$$

$$u_{yy} = \frac{\partial}{\partial y}\left(\tilde{u}_r\sin\theta + \tilde{u}_\theta\left(\frac{\cos\theta}{r}\right)\right) = \frac{\partial}{\partial r}\left(\tilde{u}_r\sin\theta + \tilde{u}_\theta\left(\frac{\cos\theta}{r}\right)\right) r_y$$

$$+ \frac{\partial}{\partial\theta}\left(\tilde{u}_r\sin\theta + \tilde{u}_\theta\left(\frac{\cos\theta}{r}\right)\right)\theta_y$$

$$= \left(\tilde{u}_{rr}\sin\theta + \tilde{u}_{\theta r}\left(\frac{\cos\theta}{r}\right) - \tilde{u}_\theta\frac{\cos\theta}{r^2}\right)\sin\theta$$

$$+ \left(\tilde{u}_{r\theta}\sin\theta + \tilde{u}_r\cos\theta + \tilde{u}_{\theta\theta}\left(\frac{\cos\theta}{r}\right) - \frac{\tilde{u}_\theta\sin\theta}{r}\right)\frac{\cos\theta}{r}$$

$$= \underbrace{\tilde{u}_{rr}\sin^2\theta}_{} + \tilde{u}_{\theta r}\frac{\sin\theta\cos\theta}{r} - \tilde{u}_r\frac{\sin\theta\cos\theta}{r^2}$$

$$+ \tilde{u}_{r\theta}\frac{\sin\theta\cos\theta}{r} + \tilde{u}_r\frac{\cos^2\theta}{r} + \tilde{u}_{\theta\theta}\frac{\cos^2\theta}{r^2} - \tilde{u}_\theta\frac{\sin\theta\cos\theta}{r^2}$$

$$= \tilde{u}_{rr} + \frac{1}{r^2}\tilde{u}_{\theta\theta} + \frac{1}{r}\tilde{u}_r .$$

//