

27/04/2015

Ασκήσεις (... βωέχεια ...)

Διάλεξη -
28/04/15 12:00 Φωκιάς (Ακαδημαϊκός)
Αμφιθέατρο Αριστοτέλη
(Φυσικό)

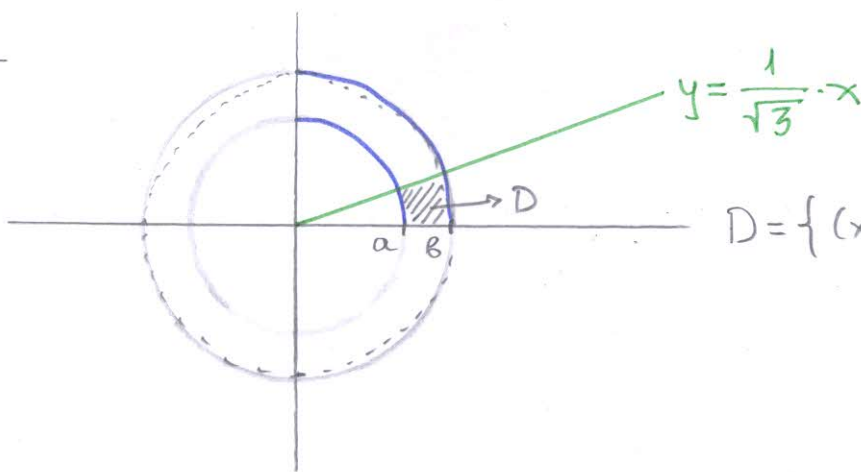
7) i) $I = \iint_D \eta f(x^2+y^2) dx dy$, D το τεταρτημόριο του $x-y$

φράσσεται από τις $x^2+y^2=a^2$, $x^2+y^2=b^2$ ($0 < a < b$)
και $y=0$, $x=\sqrt{3}y$

ii) $J = \iint_D \lambda u(x^2+y^2) dx dy$, D το τεταρτημόριο του $x-y$

φράσσεται από τις $x^2+y^2=a^2$, $x^2+y^2=b^2$ ($0 < a < b$)

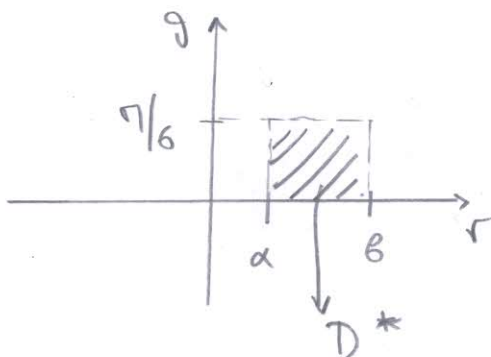
Λύση



$$D = \left\{ (x,y) : a^2 \leq x^2+y^2 \leq b^2, \right. \\ \left. 0 \leq y \leq \frac{1}{\sqrt{3}} x \right\}$$

$$x = r \cos \vartheta \\ y = r \sin \vartheta$$

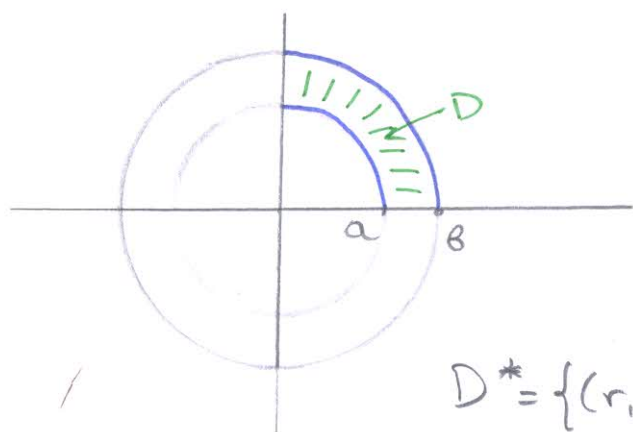
$$D^* = \left\{ (r, \vartheta) : a \leq r \leq b, 0 \leq \vartheta \leq \frac{\pi}{6} \right\}$$



$$\left(\begin{aligned} r \sin \vartheta &= \frac{1}{\sqrt{3}} r \cos \vartheta \\ \varepsilon \vartheta &= \frac{1}{\sqrt{3}}, \vartheta = \frac{\pi}{6} \end{aligned} \right)$$

$$I = \int_a^b \int_0^{\frac{\pi}{6}} \frac{\pi}{6} \rho(r^2) \cdot r d\theta dr = \frac{\pi}{6} \left[-\frac{6\omega r^2}{2} \right]_a^b = \frac{\pi}{12} (6\omega(a^2) - 6\omega(b^2))$$

ii)



$$D = \{(x, y) : a^2 \leq x^2 + y^2 \leq b^2\}$$

$$x, y \geq 0$$

$$(x, y \geq 0, \omega\theta, \pi\theta \geq 0)$$

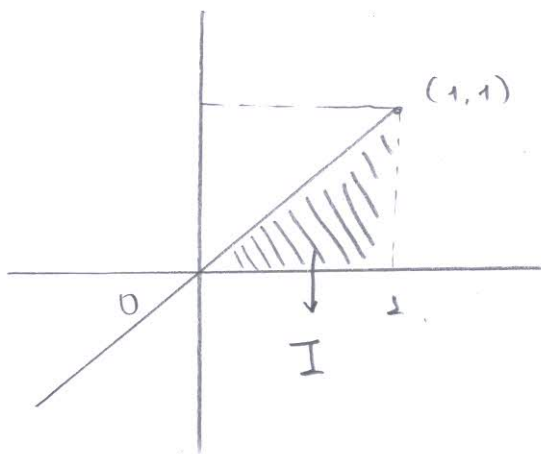
$$D^* = \{(r, \theta) : a \leq r \leq b, 0 \leq \theta \leq \frac{\pi}{2}\}$$

$$J = \int_a^b \int_0^{\frac{\pi}{2}} \frac{\pi}{2} \ln(r^2) \cdot r d\theta dr = \frac{\pi}{4} [b^2 \ln b^2 - a^2 \ln a^2 - (b^2 - a^2)]$$

8) $I = \iint_D dx dy$, $D = \{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq x, x \leq 1\}$ Απρίλιος '14

Να υπολογιστεί το I με καρτεσιανές και πολικές συντεταγμένες.

Λύση



Καρτεσιανές Συντεταγμένες

$$I = \int_0^1 \left(\int_0^x dy \right) dx = \int_0^1 x dx$$

$$= \frac{x^2}{2} \Big|_0^1 = \frac{1}{2}$$

Πολικές Συντεταγμένες

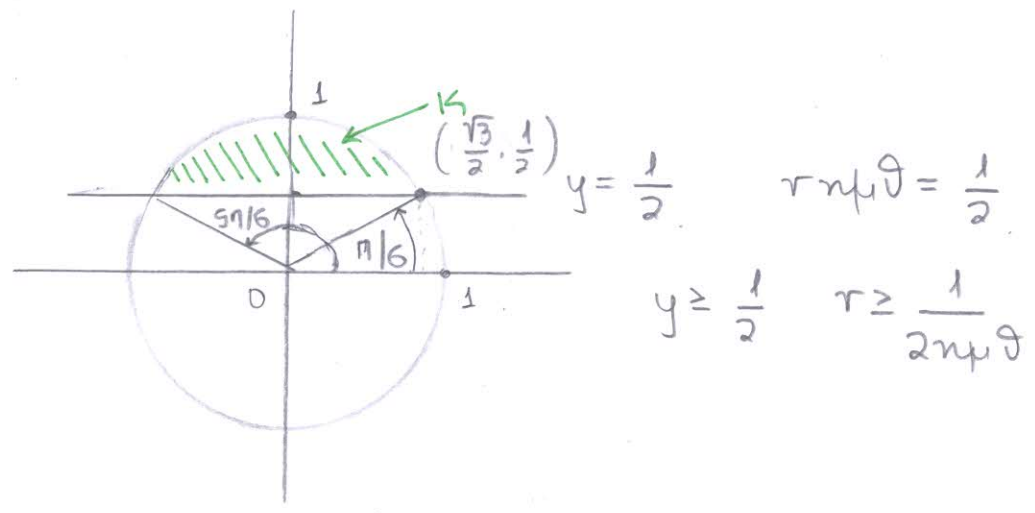
$$D^* = \{(r, \theta) : 0 \leq \theta \leq \frac{\pi}{4}, 0 < r \leq \frac{1}{\cos\theta}\}$$

$$x=1, r \cos\theta = 1, r = \frac{1}{\cos\theta}$$

$$J = \int_0^{\frac{\pi}{4}} \left(\int_0^{\frac{1}{\cos\theta}} r dr \right) d\theta = \frac{1}{2} \int_0^{\frac{\pi}{4}} \frac{1}{\cos^2\theta} d\theta = \frac{1}{2} (\tan\theta) \Big|_0^{\frac{\pi}{4}} = \frac{1}{2} (1-0) = \frac{1}{2}$$

9) $I = \iint_K \frac{y^3}{(x^2+y^2)^{3/2}}$, $K = \{(x,y) : x^2+y^2 \leq 1, 1 \leq 2y \leq 2\}$

Ліва



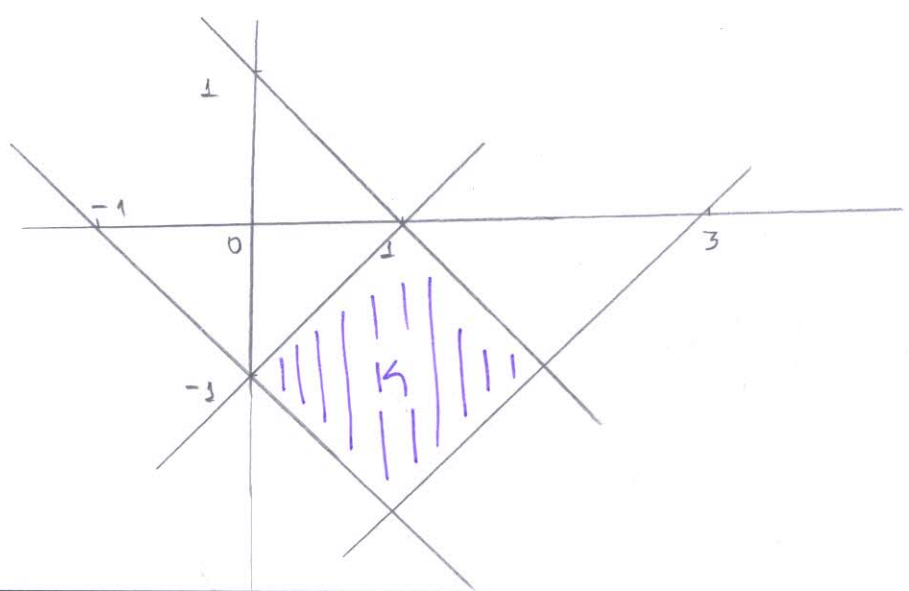
$K^* = \left\{ (r, \vartheta) : \frac{\pi}{6} \leq \vartheta \leq \frac{5\pi}{6}, \frac{1}{2\cos\vartheta} \leq r \leq 1 \right\}$

$I = \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \int_{\frac{1}{2\cos\vartheta}}^1 \frac{(r\cos\vartheta)^3}{(r^2)^{3/2}} \cdot r dr d\vartheta = \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{1}{2} \left(1 - \frac{1}{4\cos^2\vartheta} \right) \cos^3\vartheta d\vartheta = \dots$

10) i) $I = \iint_K \frac{(x+y)^4}{(x-y)^{5/2}} dx dy$, $K = \{(x,y) : -1 \leq x+y \leq 1, 1 \leq x-y \leq 3\}$

ii) $J = \iint_K \frac{(x+2y)^3}{(2x+5y^2+2xy)^{3/2}} dx dy$, $K = \{(x,y) : 2x^2+5y^2+2xy \leq 1, \frac{1}{2} \leq x+2y \leq 1\}$

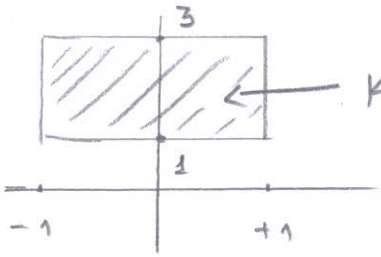
Ліва i)



$$\begin{aligned}
 u = x+y & \quad \left| \quad x = \frac{u+v}{2} \right. \\
 v = x-y & \quad \left. \quad y = \frac{u-v}{2} \right.
 \end{aligned}
 \Rightarrow \vec{T}(u,v) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \left(\frac{u+v}{2}, \frac{u-v}{2} \right)$$

(αριθτερόστροφο)

$$\det \vec{T}(u,v) = \det \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} = -\frac{1}{2} < 0$$



$K^* = \{ (u,v) : -1 \leq u \leq 1, 1 \leq v \leq 3 \}$

$$\begin{aligned}
 I &= \iint_K \frac{(x+y)^4}{(x-y)^{5/2}} dx dy = \int_{-1}^{+1} \left(\int_1^3 \frac{u^4}{v^{5/2}} \cdot \left| -\frac{1}{2} \right| dv \right) du = \\
 &= \frac{1}{2} \left(\int_{-1}^1 u^4 \left(\int_{+1}^3 \frac{1}{v^{5/2}} dv \right) du \right) = \dots
 \end{aligned}$$

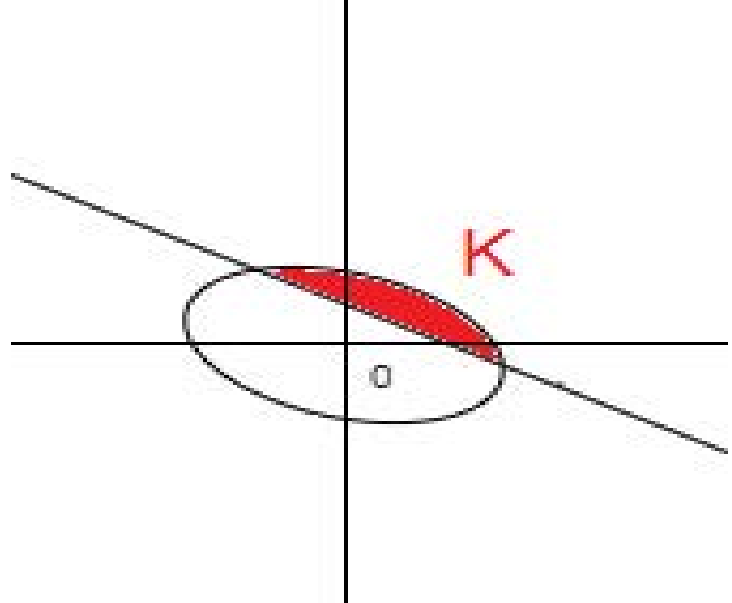
ii) $K = \left\{ (x,y) : 2x^2 + 5y^2 + 2xy \leq 1, \frac{1}{2} \leq x+2y \leq 1 \right\}$

$$\begin{aligned}
 u = x+2y & \quad \left| \quad u^2 + v^2 = 2x^2 + 5y^2 + 2xy \right. \\
 v = x-y & \quad \left. \quad \left. \begin{aligned} x &= \frac{u+2v}{3} \\ y &= \frac{u-v}{3} \end{aligned} \right. \right.
 \end{aligned}$$

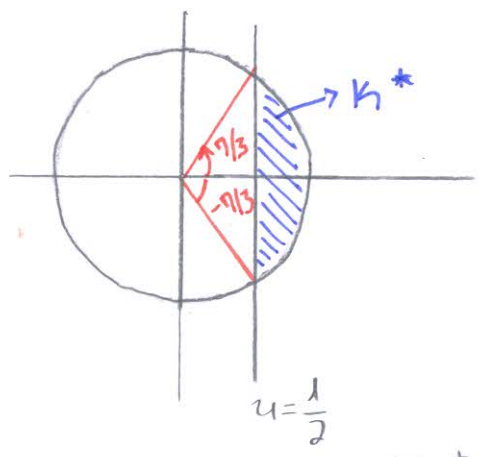
$$\begin{aligned}
 \vec{T}(u,v) &= \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \quad \det \vec{T}(u,v) = -\frac{1}{9} - \frac{2}{9} = \\
 &= -\frac{3}{9} = -\frac{1}{3} < 0
 \end{aligned}$$

(αριθτερόστροφο)

$$\left. \begin{aligned}
 &2x^2 + 5y^2 + 2xy = 1 \quad (*) \quad (x,y) \begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 1 \\
 &\text{Ιδιοτιμές του πίνακα } \begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix} : \lambda_1, \lambda_2 > 0 \Rightarrow \\
 &\Rightarrow \text{Η εξίσωση } (*) \text{ παριστάνει έλλειψη.}
 \end{aligned} \right\}$$



$$K^* = \left\{ (u,v) : u^2 + v^2 \leq 1, \frac{1}{2} \leq u \leq 1 \right\}$$



$$J = \frac{1}{3} \iint \frac{u^3}{(u^2+v^2)^{3/2}} du dv$$

Πολικός Μετασχηματισμός
 $u = r \cos \theta$ ($u = \frac{1}{2}, r = \frac{1}{2 \cos \theta}$)
 $v = r \sin \theta$

$$K^{**} = \left\{ (r, \theta) : -\frac{\pi}{3} \leq \theta \leq \frac{\pi}{3}, \frac{1}{2 \cos \theta} \leq r \leq 1 \right\}$$

Άρα $J = \frac{1}{3} \int_{-\pi/3}^{+\pi/3} \left(\int_{\frac{1}{2 \cos \theta}}^1 \frac{(r \cos \theta)^3}{(r^2)^{3/2}} dr \right) d\theta = \frac{1}{6} \int_{-\pi/3}^{+\pi/3} 6 \cos^3 \theta \left(1 - \frac{1}{4 \cos^2 \theta} \right) d\theta$

Σημείωση Ασκ 10

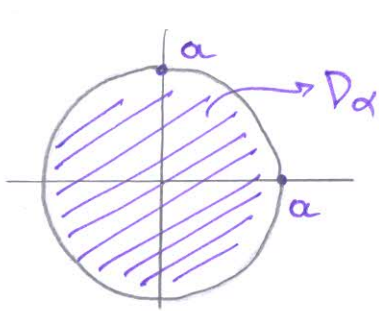
$$i) \overline{T}(u,v) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \quad \begin{matrix} = \dots \\ \text{του } (u,v) \end{matrix} \quad \begin{matrix} \left(\frac{1}{2}, \frac{1}{2} \right) \longrightarrow \left(\frac{1}{4}, 0 \right) \text{ του } (x,y) \\ \left(\frac{1}{2}, -\frac{1}{2} \right) \longrightarrow \left(0, \frac{1}{4} \right) \end{matrix}$$

Δηλαδή, η ορθογώνια βάση του (u,v) , $\frac{1}{2}(1,1), \frac{1}{2}(1,-1)$
 $\overline{T} \rightarrow \frac{1}{4}(1,0), \frac{1}{4}(0,1)$ ορθογώνια βάση.

$$i) \vec{T}(u,v) = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

H βάση $(1,1)$, $(2,-1)$, $(u,v) \rightarrow$ όχι ορθογώνια βάση
 \downarrow \downarrow
 $(1,0)$ $(0,1) \rightarrow$ ορθογώνια (βλ. Γραμ. Αλγ. Ι)

$$ii) I_a = \iint_{D_a} e^{-(x^2+y^2)} dx dy, D_a = \{(x,y) : x^2+y^2 \leq a^2\} \quad (a > 0)$$



Λύση:

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$I_a = \int_0^{2\pi} \left(\int_0^a e^{-r^2} r dr \right) d\theta = \pi(1 - e^{-a^2})$$

$$ii) \int_{-\infty}^{+\infty} e^{-t^2} dt = \sqrt{\pi} \quad \left(\lim_{x \rightarrow +\infty} \int_{-x}^{+x} e^{-t^2} dt = \int_{-\infty}^{+\infty} e^{-t^2} dt \right)$$

Λύση $x > 0$

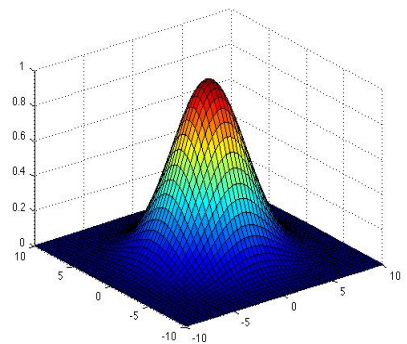
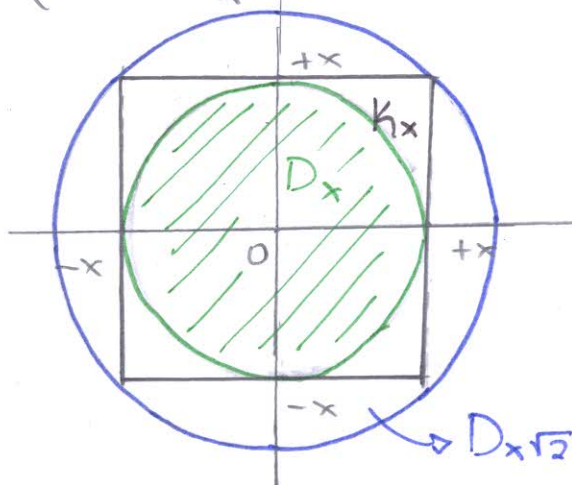
$$K_x = [-x, x] \times [-x, x]$$

$$\iint_{K_x} e^{-(t_1^2+t_2^2)} dt_1 dt_2 =$$

K_x

$$= \int_{-x}^{+x} \left(\int_{-x}^{+x} e^{-t_1^2-t_2^2} dt_2 \right) dt_1 = \left(\int_{-x}^{+x} e^{-t_1^2} dt_1 \right) \left(\int_{-x}^{+x} e^{-t_2^2} dt_2 \right) =$$

$$= \left(\int_{-x}^{+x} e^{-t^2} dt \right)^2$$



$$D_x = \{ (t_1, t_2) : \sqrt{t_1^2 + t_2^2} \leq x \}$$

$$D_{x\sqrt{2}} = \{ (t_1, t_2) : \sqrt{t_1^2 + t_2^2} \leq x\sqrt{2} \}$$

$$\begin{cases} e^{-(t_1^2 + t_2^2)} > 0 \\ D_x \subseteq K_x \subseteq D_{x\sqrt{2}} \end{cases}$$

$$\iint_{D_x} e^{-(t_1^2 + t_2^2)} \leq \iint_{K_x} e^{-(t_1^2 + t_2^2)} \leq \iint_{D_{x\sqrt{2}}} e^{-(t_1^2 + t_2^2)}$$

Apa

$$\pi(1 - e^{-x^2}) \leq \iint_{K_x} e^{-(t_1^2 + t_2^2)} = \left(\int_{-x}^{+x} e^{-t^2} dt \right)^2 \leq \pi(1 - e^{-2x^2})$$

\Downarrow \Downarrow
 Area K_x Area $D_{x\sqrt{2}}$

$$\Rightarrow \lim_{x \rightarrow +\infty} \left(\int_{-\infty}^{+\infty} e^{-t^2} dt \right)^2 = \pi \quad \text{Apa} \quad \int_{-\infty}^{+\infty} e^{-t^2} dt = \sqrt{\pi} \quad (e^{-t^2} > 0)$$

