Μάθημα 15 (16/11/2020)

Αλλαγή Μεταβλητής/τών

$\mathbb{R}^d$ (Οι συναρτήσεις είναι συνεχείς)

$f : [a, b] \longrightarrow \mathbb{R}$

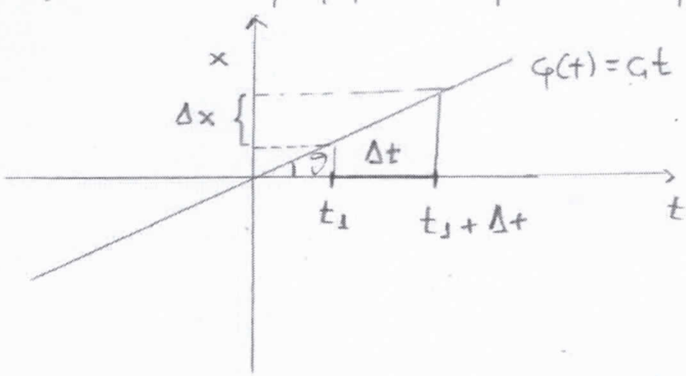
$\varphi : [\gamma, \delta] \longrightarrow [a, b]$, C^1, $\varphi'(t) \neq 0$, $t \in [\gamma, \delta]$, επί

$$\int_a^b f(x)\,dx = \int_\gamma^\delta f(\varphi(t)) \cdot |\varphi'(t)|\,dt$$

- Τι κρύβεται στο $dx = |\varphi'(t)|\,dt$;

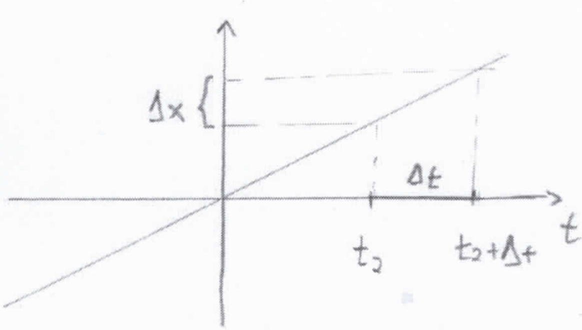
- $\varphi(t) = ct$, $c \neq 0$ Γραμμικός μετασχηματισμός (Αλλαγή Κλίμακας $c \neq 1$)

$c > 0$



$\Delta x = c\,\Delta t$, $\Delta x = \varphi'(t)\,\Delta t$

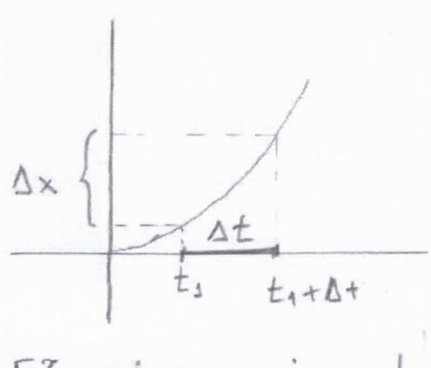
$c = \varepsilon\varphi\theta$

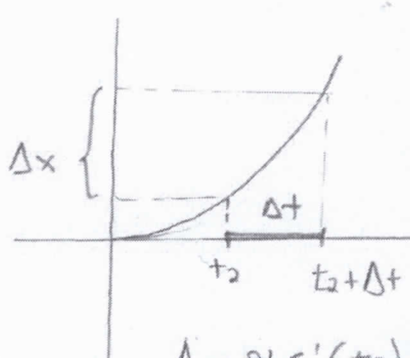
- 

Δεν εξαρτάται από το t_1 το Δx!

- $\varphi(t) = t^2$, $t \in (0, +\infty)$

$\Delta x \cong \varphi'(t_1)\,\Delta t$

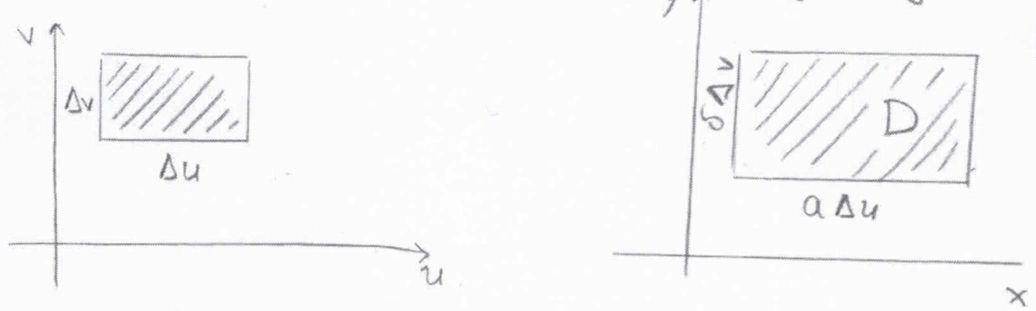




$d=2 \quad \vec{T}(u,v) = \begin{pmatrix} a & b \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = (au+bv, \gamma u+\delta v)$

$\det \begin{pmatrix} a & b \\ \gamma & \delta \end{pmatrix} \neq 0$

πχ. $\vec{T}(u,v) = \begin{pmatrix} a & 0 \\ 0 & \delta \end{pmatrix}$, $a\delta \neq 0$ πχ $a\delta > 0$ (Διατηρούνται οι γωνίες / Ορθογώνιος Μετασχηματισμός)



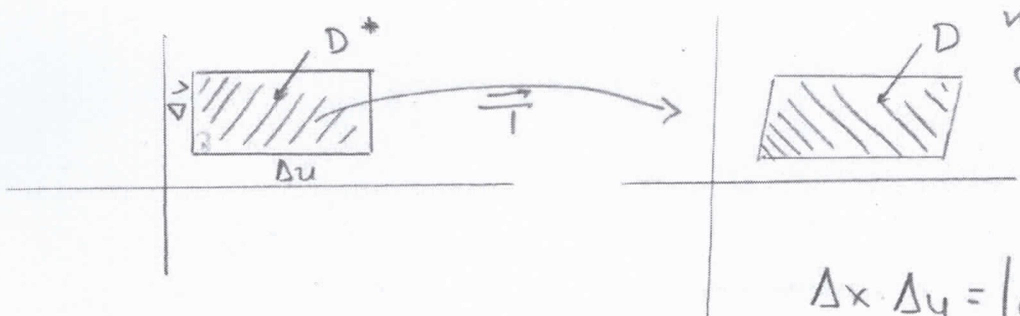
$\Delta x \cdot \Delta y = (a\delta)\, \Delta u \cdot \Delta v =$

$= \left|\det \begin{pmatrix} a & b \\ \delta & \delta \end{pmatrix}\right| \Delta u \cdot \Delta v$

$= \left|\det J_{\vec{T}}(u,v)\right| \Delta u \cdot \Delta v$

$\vec{T}(u,v) = \begin{pmatrix} a & b \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$, $\det J_{\vec{T}}(u,v) \neq 0$

Διατηρείται η παραλληλία, αλλάζουν οι γωνίες

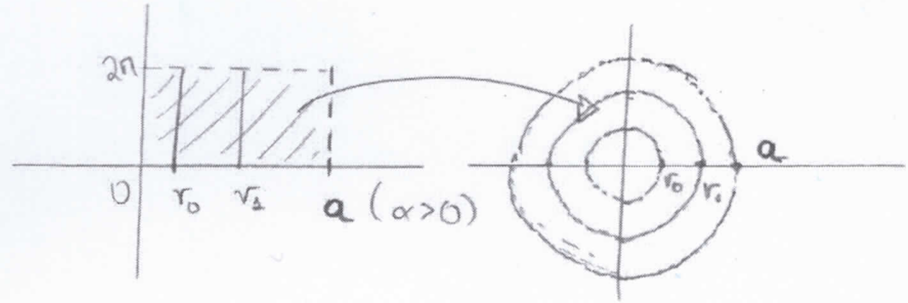


$\Delta x \cdot \Delta y = \left|\det \begin{pmatrix} a & b \\ \gamma & \delta \end{pmatrix}\right| \Delta u \cdot \Delta v$

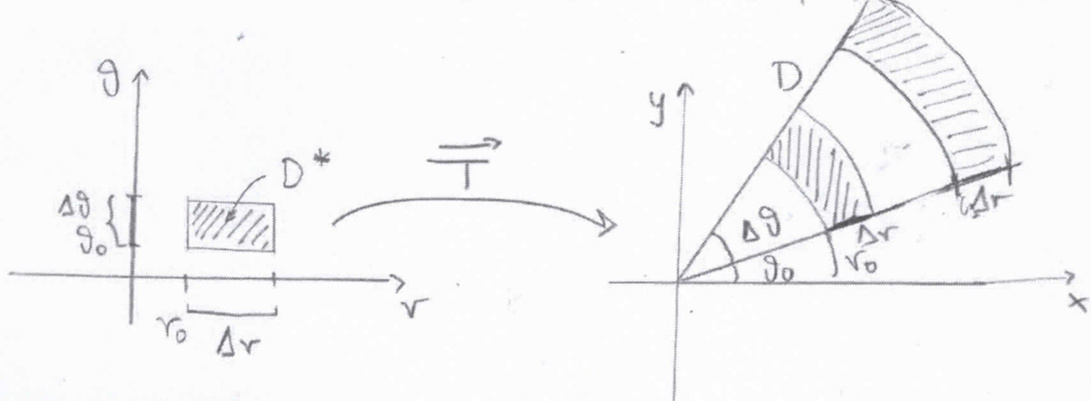
Πολικός Μετασχηματισμός

$\vec{T}(r,\vartheta)=(r\,\text{συν}\vartheta,\ r\,\text{ημ}\vartheta)$, $r\in(0,+\infty)$, $\vartheta\in[0,2\pi)$ (Μη γραμμικός ως προς ϑ)

επί του $\mathbb{R}^2\setminus\{(0,0)\}$, C^1, 1-1.



$\vec{T}(r_0,\vartheta)=(r_0\,\text{συν}\vartheta,\ r_0\,\text{ημ}\vartheta)$



$\Delta r\,\Delta\vartheta$

$\Delta x\cdot\Delta y \cong (r_0\Delta\vartheta)\Delta r = r_0\,\Delta r\,\Delta\vartheta$

(Εξαρτάται από το r)

$\vec{T}(r,\vartheta)=(r\,\text{συν}\vartheta,\ r\,\text{ημ}\vartheta)$

$$J_{\vec{T}}(r,\vartheta)=\begin{pmatrix}\text{συν}\vartheta & -r\,\text{ημ}\vartheta\\ \text{ημ}\vartheta & r\,\text{συν}\vartheta\end{pmatrix},\quad \det J_{\vec{T}}(r,\vartheta)=r\,\text{συν}^2\vartheta+r\,\text{ημ}^2\vartheta$$

$$=\underline{\underline{r}}$$

Θ. Αλλαγής Μεταβλητής για Διπλό Ολοκλήρωμα

$f: \underset{(x,y)}{D} \longrightarrow \mathbb{R}$, D x-απλό / f συνεχής

$\vec{T}: \underset{(u,v)}{D^*} \longrightarrow \underset{(x,y)}{D}$, C^1, 1-1, επί, $\det J_{\vec{T}}(u,v) \neq 0$

D^* u-απλό

Τότε $$\iint_D f(x,y)\,dx\,dy = \iint_{D^*} f(\vec{T}(u,v))\,|\det J_{\vec{T}}(u,v)|\,du\,dv$$

Συμβολισμός $J_{\vec{T}}(u,v) = \dfrac{\partial(x,y)}{\partial(u,v)}$ /

$$\iint_D f(x,y)\,dx\,dy = \iint_{D^*} f(\vec{T}(u,v))\left|\det \frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv.$$

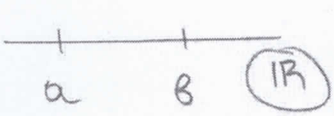
Τι υπολογίζουμε με τη βοήθεια Μονού - Διπλού - Τριπλού Ολοκληρώματος;

$\mathbb{R}^d$, $d=1,2,3,\dots$ / $V_d(K) =: \int_K 1$ όγκος του K (K= "καλό" σύνολο)

(οι συναρτήσεις είναι συνεχείς)

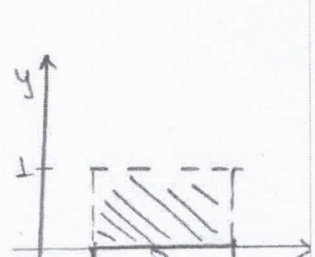
• $d=1$, $f:[a,b]=D_1 \longrightarrow \mathbb{R}$, $f \geq 0$

Μάζα του $[a,b]$ με συνάρτηση πυκνότητας $f(x)$, $x\in[a,b]$



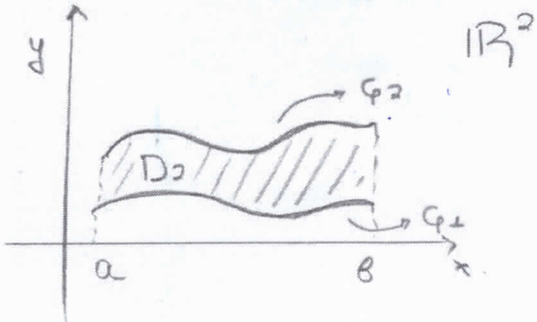
$$m =: \int_{D_1} f(x)\,dx \quad \text{(ΚΒ, Ροπές κ.α.)}$$

$$\overline{V_1(D_1)} =: \int_{D_1} 1 = \int_a^b 1 = 1(b-a) = b-a$$



• $d=2 \quad f: D_2 \longrightarrow \mathbb{R}, \quad D_2 = \{(x,y)\in\mathbb{R}^2 : x\in D_1,\ \varphi_1(x)\le y\le \varphi_2(x)\}$ (x-απλό)

$f \ge 0$

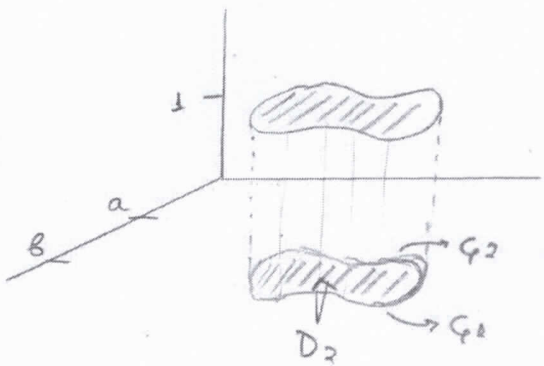


Μάζα του D_2

$m =: \iint_{D_2} f$ (ΚΒ, Ροπές, κα)

$$V_2(D_2) =: \iint_{D_2} 1\, dx\, dy = \int_\alpha^\beta \left(\int_{\varphi_1(x)}^{\varphi_2(x)} 1\, dy \right) dx = \int_\alpha^\beta \left(\varphi_2(x) - \varphi_1(x) \right) dx$$

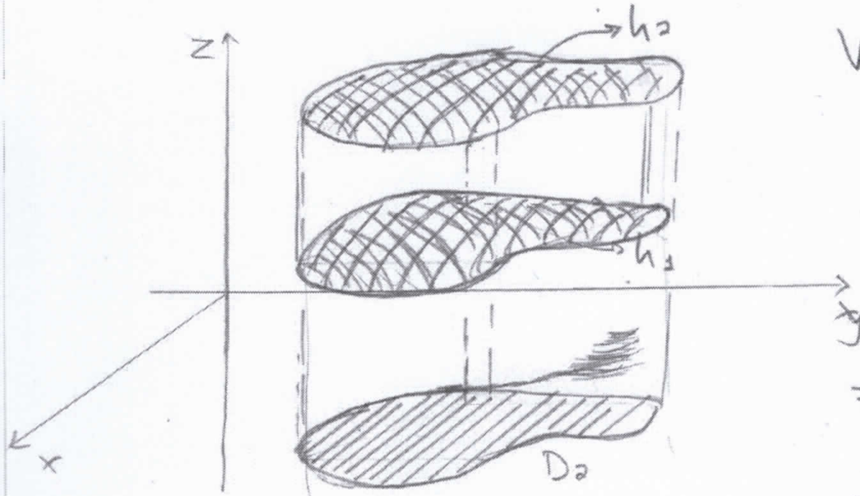
Δηλαδή, το εμβαδόν του D_2 μπορεί να υπολογιστεί με μονό ή διπλό ολοκλήρωμα.



(*) Δηλαδή, το εμβαδόν του D_2 είναι ίσο με τον όγκο στερεού βάσης D_2 και ύψους 1.

• $d=3$, $f: D_3 \longrightarrow \mathbb{R}$, $D_3 = \{(x,y,z) : (x,y)\in D_2,\ h_1(x,y)\le z\le h_2(x,y)\}$

$f\ge 0$, $m =: \iiint_{D_3} f$ (ΚΒ, Ροπές, κα)



$$V_3(D_3) = \iiint_{D_3} 1\, dx\, dy\, dz$$

$$= \iint_{D_2} \left(\int_{h_1(x,y)}^{h_2(x,y)} 1\, dz \right) dy\, dx =$$

$$= \iint_{D_2} \left(h_2(x,y) - h_1(x,y) \right) dy\, dx$$

Ασκήσεις

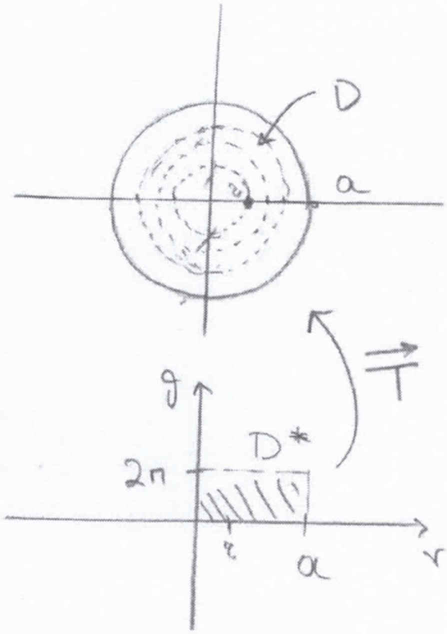
1. i) εμβαδόν $A(D)$, $D=\{(x,y)\cdot x^2+y^2\le a^2\}$ $(\alpha>0)$

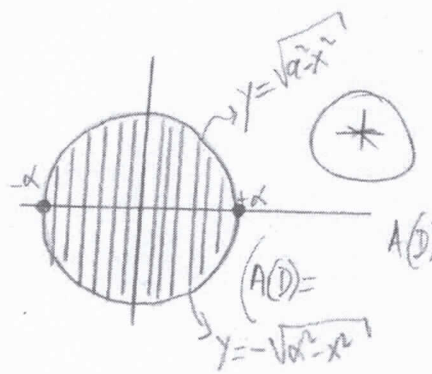
ii) εμβαδόν $A(D)$, $D=\left\{(x,y): \left(\frac{x}{\alpha}\right)^2+\left(\frac{y}{\beta}\right)^2\le 1\right\}$ $(a,\beta>0)$

iii) εμβαδόν $A(K)$, K καρδιοειδές, $K=\{(x,y): x^2+y^2\le a(\sqrt{x^2+y^2}+x)\}$ $\alpha>0$

Λύση i)

$D^*=\{(r,\theta): 0<r\le\alpha,\ 0\le\theta\le 2\pi\}$



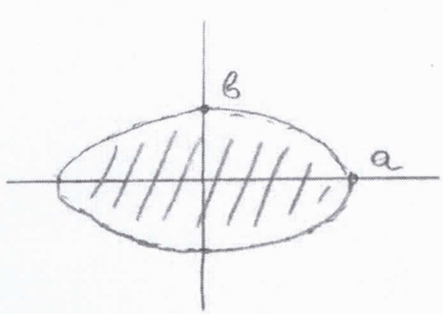


$A(D)=\int_{-\alpha}^{+\alpha}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy\,dx$

$$A(D)=\iint_D 1\,dx\,dy=\iint_{D^*} 1\,|\det J_{\vec{T}}(x,y)|\,dr\,d\theta=\int_0^a\left(\int_0^{2\pi} r\,d\theta\right)dr=$$

$$=\int_0^\alpha (2\pi r)\,dr=2\pi\frac{r^2}{2}\Big|_0^a=\pi a^2$$

ii) $\left(\frac{x}{\alpha}\right)^2+\left(\frac{y}{\beta}\right)^2=1$ $\quad \frac{x}{a}=r\sigma\upsilon\nu\theta,\ \frac{y}{\beta}=r\eta\mu\theta$



$\vec{T}(r,\theta)=(ar\sigma\upsilon\nu\theta,\ \beta r\eta\mu\theta)=(x,y)$

$\det\frac{\partial(x,y)}{\partial(r,\theta)}=a\beta r>0$ / $D^*=\{(r,\theta): 0\le\theta\le 2\pi,\ 0\le r\le 1\}$

$$A(D)=\int_0^1\int_0^{2\pi}(a\beta r)\,d\theta\,dr=a\beta\pi$$

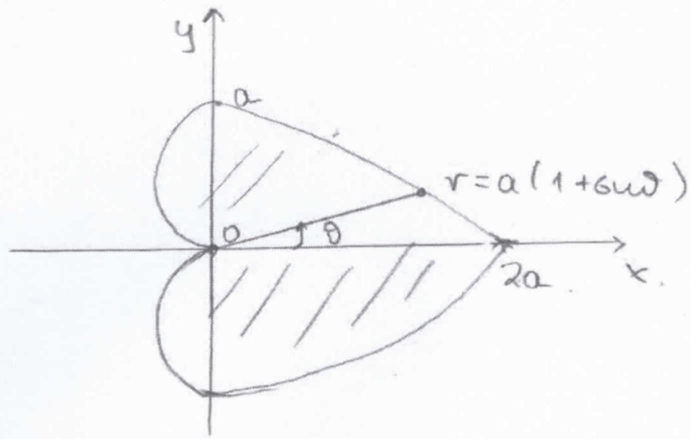
iii) $x^2+y^2=a\left(\sqrt{x^2+y^2}+x\right)$

Παλιό θέμα.

$$\begin{array}{l|l} x=r\,\sigma\upsilon\nu\theta & r^2=a(r+r\,\sigma\upsilon\nu\theta) \\ y=r\,\eta\mu\theta & r=a(1+\sigma\upsilon\nu\theta) \end{array}$$

$$D^*=\{(r,\theta):0\le\theta\le 2\pi,\ 0\le r\le a(1+\sigma\upsilon\nu\theta)\}$$

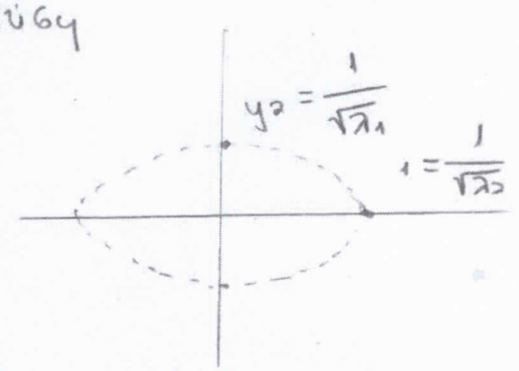
$$A(D)=\iint_D 1\,dx\,dy=\iint_{D^*}1\cdot r\,dr\,d\theta=2\int_0^{\pi}\int_0^{a(1+\sigma\upsilon\nu\theta)} r\,dr\,d\theta=\ldots=\frac{3\pi}{2}a^2$$



2. $I=\iint_D (1-x^2-4y^2)^{3/2}\,dx\,dy$, όπου $D=\{(x,y)\in\mathbb{R}^2: x^2+4y^2\le 1\}$

$$(x,y)\begin{pmatrix}1&0\\0&4\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}\le 1$$

Λύση



$\lambda_1,\lambda_2=$ ιδ. $\begin{pmatrix}1&0\\0&4\end{pmatrix}$

$$\vec{T}\quad \begin{array}{l|l} x=r\,\sigma\upsilon\nu\theta & x=r\,\sigma\upsilon\nu\theta \\ 2y=r\,\eta\mu\theta & y=\frac{r}{2}\eta\mu\theta \end{array}\qquad \det\vec{T}(r,\theta)=\frac{1}{2}$$

$$D^*=\{(r,\theta):0\le\theta\le 2\pi,\ 0\le r\le 1\}$$

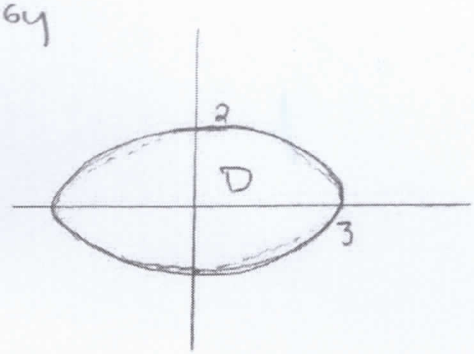
$$(r^2=x^2+4y^2\le 1)$$

$$I=\int_0^{2\pi}\int_0^1 (1-r^2)^{3/2}\left(\frac{1}{2}r\right)dr\,d\theta=2\pi\int_0^1(1-r^2)^{3/2}\frac{r}{2}\,dr=\ldots=\frac{\pi}{5}$$

3) $I = \iint\limits_D x^2 \sqrt{4x^2+9y^2}\, dx\, dy \quad , \quad D = \left\{ (x,y) : \frac{x^2}{9} + \frac{y^2}{4} \le 1 \right\}$

Λύση

<u>Ιούλιος '07</u>



$\frac{x}{3} = r\,\sigma\upsilon\nu\theta$

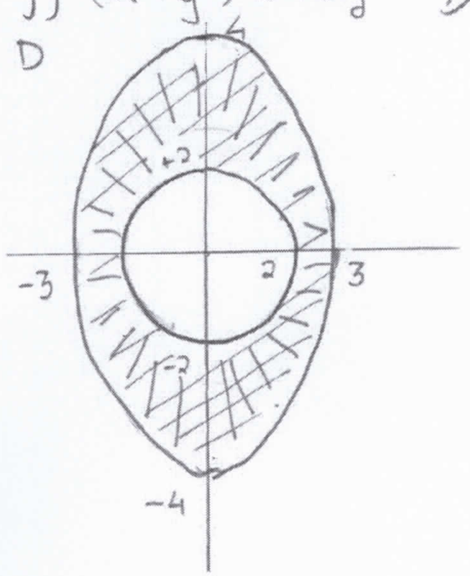
$\frac{y}{2} = r\,\eta\mu\theta \quad \Big/ \quad \left| \det \frac{\partial(x,y)}{\partial(r,\theta)} \right| = 6r$

$$I = \int_0^{2\pi}\int_0^1 (3r\,\sigma\upsilon\nu\theta)^2 \sqrt{4(3r\,\sigma\upsilon\nu\theta)^2 + 9(2r\,\eta\mu\theta)^2} \cdot 6r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 9r^4 \sigma\upsilon\nu^2\theta \cdot 6 \cdot 6\, dr\, d\theta = \int_0^{2\pi}\int_0^1 (36 \cdot 9) r^4 \sigma\upsilon\nu^2\theta\, dr\, d\theta$$

$$= \frac{324}{5}\pi$$

4) $I = \iint\limits_D (x^2+y^2)\, dx\, dy \quad D = \left\{ (x,y) : x^2+y^2 \ge 4,\ \frac{x^2}{9} + \frac{y^2}{16} \le 1 \right\}$



- $D_1 = \{ (x,y) : x^2+y^2 \le 4 \}$,

  $D_1^* = \{ (r,\theta) : 0 \le \theta \le 2\pi \ , 0 \le r \le 2 \}$

  $x = r\,\sigma\upsilon\nu\theta, \ y = r\,\eta\mu\theta$

- $D_2 = \left\{ (x,y) : \left(\frac{x}{3}\right)^2 + \left(\frac{y}{4}\right)^2 \le 1 \right\}$

  $D_2^* = \{ (r,\theta) : 0 \le \theta \le 2\pi \ , 0 \le r \le 1 \}$

  $x = 3r\,\sigma\upsilon\nu\theta, \ y = 4r\,\eta\mu\theta$

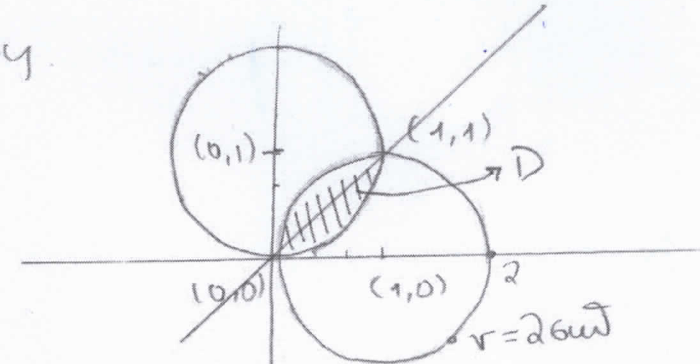
$$I_1 = \int_0^{2\pi}\int_0^2 r^2 \cdot r\, dr\, d\theta \quad (= 8\pi)$$

$$I_2 = \int_0^{2\pi}\int_0^1 \left((3r\,\sigma\upsilon\nu\theta)^2 + (4r\,\eta\mu\theta)^2 \right) 12r\, dr\, d\theta$$

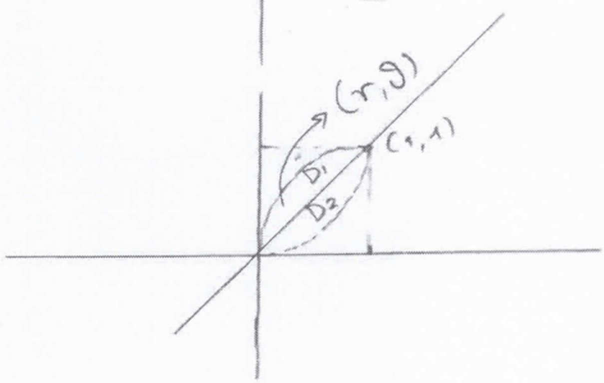
5) $I = \iint\limits_{D} x \, dx\,dy \qquad D = \{(x,y) : (x-1)^2 + y^2 \leq 1 \,,\; x^2 + (y-1)^2 \leq 1\}$

Φεβρουάριος '09

Λύση



$$\left.\begin{aligned} x^2+y^2 &\leq 2x \\ x^2+y^2 &\leq 2y \end{aligned}\right\}$$

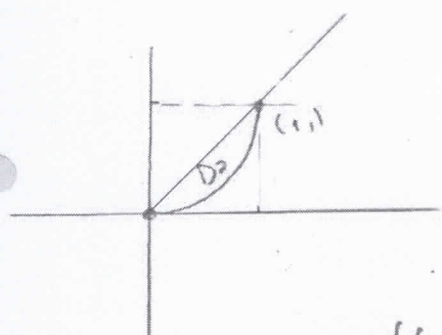


$$I = \iint\limits_{D_1} x + \iint\limits_{D_2} x$$

$D_1 : x^2+y^2 \leq 2x \Big/ \begin{array}{l} x = r\,\text{συν}\theta \\ y = r\,\text{ημ}\theta \end{array}$

$0 \leq r \leq 2\,\text{συν}\theta \,,\; \frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$

$$\iint\limits_{D_1} x\,dx\,dy = \int_{\frac{\pi}{4}}^{\pi/2} \int_0^{2\text{συν}\theta} (r\,\text{συν}\theta)\cdot r\,dr\,d\theta = \dots = \gamma$$



$D_2 : x^2+y^2 \leq 2y \,/\, \begin{array}{l} x = r\,\text{συν}\theta \\ y = r\,\text{ημ}\theta \end{array}$

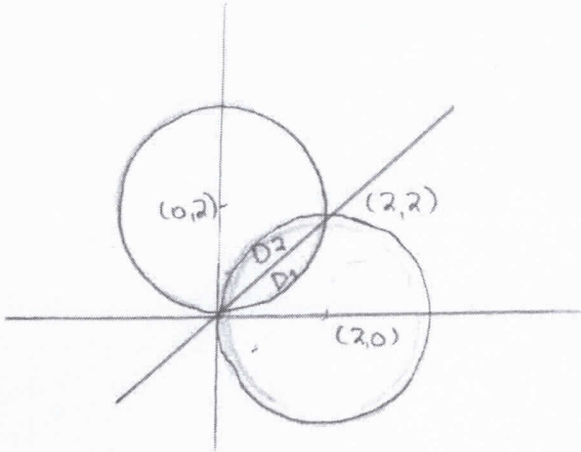
$D_2^* : 0 \leq \theta \leq \frac{\pi}{4} \,,\; 0 \leq r \leq 2\,\text{ημ}\theta$

$$\iint\limits_{D_2} x\,dx\,dy = \int_0^{\pi/4} \int_0^{2\text{ημ}\theta} (r\,\text{συν}\theta)\,r\,dr\,d\theta = \frac{1}{6}$$

6) Απρίλιος '14

$$I = \iint_D xy\,dx\,dy$$

$$D = \{(x,y) : \underbrace{(x-2)^2+y^2 \le 4}_{D_2},\ \underbrace{x^2+(y-2)^2 \le 4}_{D_1}\}$$



$f(x,y) = f(y,x)$

$D = D_1 \cup D_2$ / Συμ. $y = x$

$$\iint_D xy\,dx\,dy = 2\iint_{D_1} xy\,dx\,dy$$

D_1, $x^2+y^2 \le 4y$ / $0 \le r \le 4ημθ$, $0 \le θ \le \frac{π}{4}$

$$\iint_D xy\,dx\,dy = 2\int_0^{π/4}\left(\int_0^{4ημθ}(r^2 ημθ συνθ)\cdot r\,dr\right)dθ =$$

$$= 2\int_0^{π/4}\left(\frac{1}{4}(4ημθ)^4 ημθ συνθ\, dθ\right) = \frac{4^4}{2}\int_0^{π/4} ημ^5θ\, συνθ\, dθ =$$

$$= \frac{4^4}{2}\left[\frac{1}{6} ημ^6θ\right]_{θ=0}^{π/4} = \frac{4^4}{12}\left(\frac{1}{\sqrt{2}}\right)^6$$