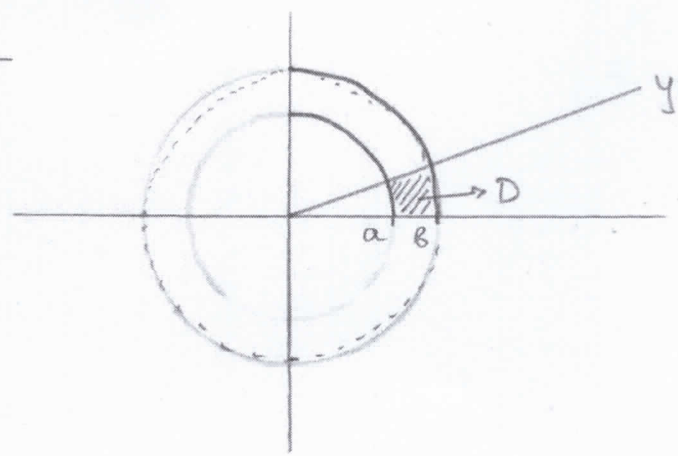
Μάθημα 17 (18/11/2020)

Ασκήσεις (...συνέχεια...)

7) i) $I=\iint\limits_D ημ(x^2+y^2)\,dx\,dy$, D 1ο τεταρτημόριο του x-y φράσσεται από τις $x^2+y^2=a^2$, $x^2+y^2=b^2$ $(0<a<b)$ και $y=0$, $x=\sqrt{3}\,y$

ii) $J=\iint\limits_D \ln(x^2+y^2)\,dx\,dy$, D 1ο τεταρτημόριο του x-y φράσσεται από τις $x^2+y^2=a^2$, $x^2+y^2=b^2$ $(0<a<b)$

Λύση

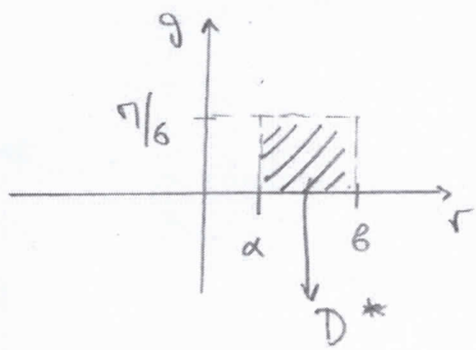


$$D=\left\{(x,y):\ a^2\le x^2+y^2\le b^2,\ 0\le y\le \frac{1}{\sqrt{3}}x\right\}$$

$x=r\,συνθ$

$y=r\cdot ημθ$

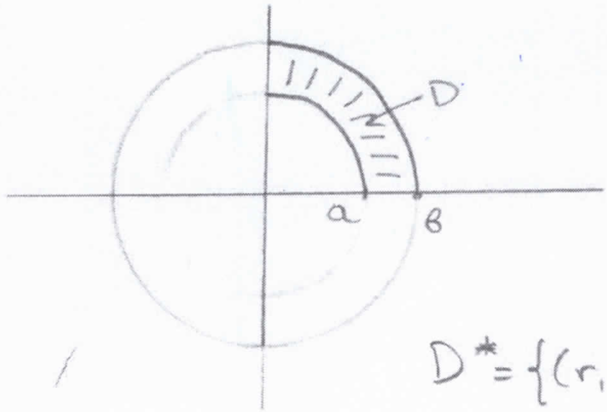
$$D^*=\left\{(r,\theta):\ a\le r\le b,\ 0\le\theta\le\frac{\pi}{6}\right\}.$$



$$\left(\begin{array}{l} r\,ημθ=\frac{1}{\sqrt{3}}\,r\,συνθ \\ εφθ=\frac{1}{\sqrt{3}},\ \theta=\frac{\pi}{6}\end{array}\right)$$

$$I = \int_a^{\beta}\int_0^{\frac{\pi}{6}} \eta\mu(r^2)\cdot r\, d\vartheta\, dr = \frac{\pi}{6}\left[-\frac{\sigma\upsilon\nu r^2}{2}\right]_{\alpha}^{\beta} = \frac{\pi}{12}\left(\sigma\upsilon\nu(\alpha^2) - \sigma\upsilon\nu(\beta^2)\right)$$

ii)



$$D = \left\{(x,y) : a^2 \le x^2 + y^2 \le \beta^2 \right\}$$
$$x, y \ge 0$$
$$\left(\begin{array}{l} x, y \ge 0 \\ \sigma\upsilon\nu\vartheta, \eta\mu\vartheta \ge 0 \end{array} \right)$$

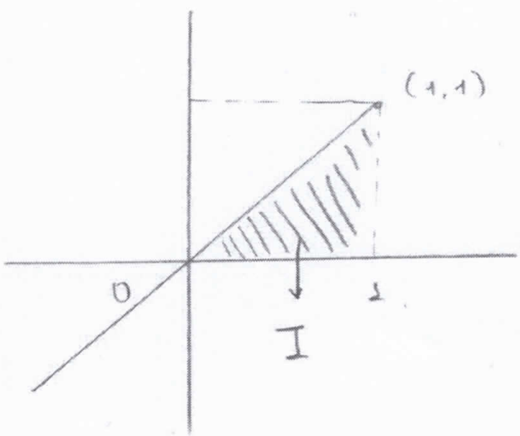
$$D^* = \left\{(r,\vartheta) : a \le r \le \beta,\ 0 \le \vartheta \le \frac{\pi}{2}\right\}$$

$$J = \int_a^{\beta}\int_0^{\frac{\pi}{2}} \ln(r^2)\cdot r\, d\vartheta\, dr = \frac{\pi}{4}\left[\beta^2 \ln\beta^2 - a^2\ln a^2 - (\beta^2 - a^2)\right]$$

8) $I = \iint_D dx\,dy$, $D = \left\{(x,y)\in\mathbb{R}^2 : 0 \le y \le x,\ x \le 1\right\}$ Απρίλιος '14

Να υπολογιστεί το I με καρτεσιανές και πολικές συντεταγμένες.

Λύση



Καρτεσιανές Συντεταγμένες

$$I = \int_0^1 \left(\int_0^x dy\right) dx = \int_0^1 x\, dx$$
$$= \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}.$$

Πολικές Συντεταγμένες

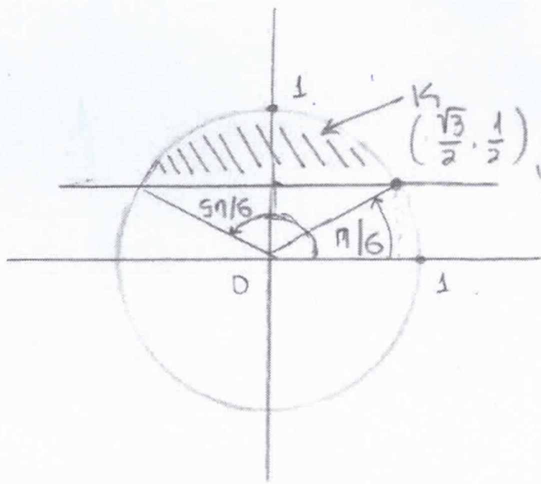
$$D^* = \left\{(r,\vartheta) : 0 \le \vartheta \le \frac{\pi}{4},\ 0 < r \le \frac{1}{\sigma\upsilon\nu\vartheta}\right\}$$

$$x = 1,\ r\,\sigma\upsilon\nu\vartheta = 1,\ r = \frac{1}{\sigma\upsilon\nu\vartheta}$$

$$J = \int_0^{\frac{\pi}{4}}\left(\int_0^{\frac{1}{\sigma\upsilon\nu\vartheta}} r\, dr\right) d\vartheta = \frac{1}{2}\int_0^{\frac{\pi}{4}} \frac{1}{\sigma\upsilon\nu^2\vartheta}\, d\vartheta = \frac{1}{2}\left(\varepsilon\varphi\vartheta\right)\Big|_0^{\frac{\pi}{4}} = \frac{1}{2}(1-0) = \frac{1}{2}$$

9) $I = \iint\limits_K \dfrac{y^3}{(x^2+y^2)^{3/2}}$, $K = \{(x,y) : x^2+y^2 \le 1,\ 1 \le 2y \le 2\}$.

Λύση



$y = \frac{1}{2}$ $\quad r\,\eta\mu\vartheta = \frac{1}{2}$

$y \ge \frac{1}{2}$ $\quad r \ge \dfrac{1}{2\eta\mu\vartheta}$

$K^* = \left\{ (r,\vartheta) : \dfrac{\pi}{6} \le \vartheta \le \dfrac{5\pi}{6},\ \dfrac{1}{2\eta\mu\vartheta} \le r \le 1 \right\}$

$$I = \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \int_{\frac{1}{2\eta\mu\vartheta}}^{1} \frac{(r\,\eta\mu\vartheta)^3}{(r^2)^{3/2}} \cdot r\,dr\,d\vartheta = \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{1}{2}\left(1 - \frac{1}{4\eta\mu^2\vartheta}\right)\eta\mu^3\vartheta\,d\vartheta = \ldots$$

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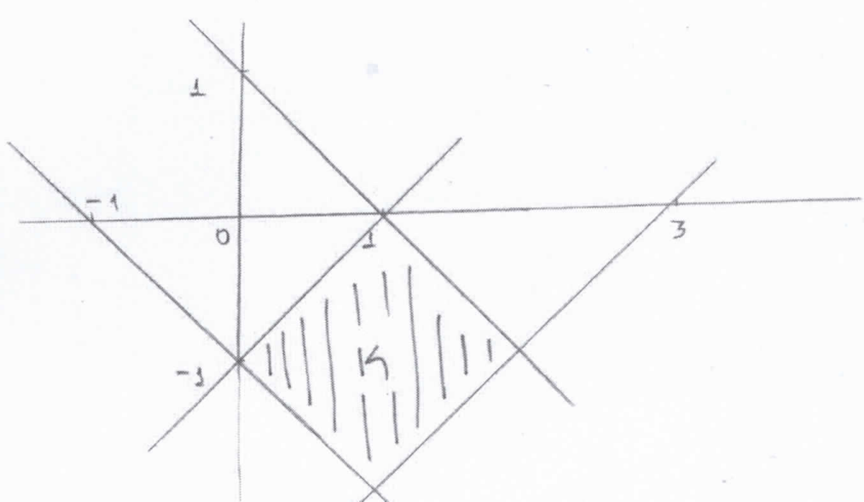
10) i) $I = \iint\limits_K \dfrac{(x+y)^4}{(x-y)^{5/2}}\,dx\,dy$, $K = \{(x,y) : -1 \le x+y \le 1,\ 1 \le x-y \le 3\}$

ii) $J = \iint\limits_K \dfrac{(x+2y)^3}{(2x+5y^2+2xy)^{3/2}}\,dx\,dy$, $K = \{(x,y) : 2x^2+5y^2+2xy \le 1$

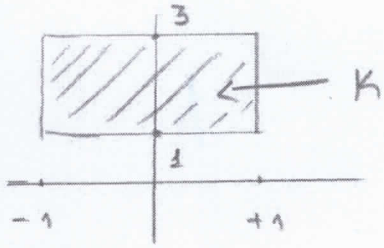
$\frac{1}{2} \le x+2y \le 1\}$.

Λύση i)



$u = x+y$, $v = x-y$ $\Big|$ $x = \frac{u+v}{2}$, $y = \frac{u-v}{2}$ $\quad \vec{T}(u,v) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \left(\frac{u+v}{2}, \frac{u-v}{2}\right)$

$\det J_{\vec{T}}(u,v) = \det \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} = -\frac{1}{2} < 0$ (αριστερόστροφο)



$K^* = \{(u,v) : -1 \le u \le 1,\ 1 \le v \le 3\}$

$$I = \iint_K \frac{(x+y)^4}{(x-y)^{5/2}}\,dx\,dy = \int_{-1}^{+1}\left(\int_1^3 \frac{u^4}{v^{5/2}} \cdot \left|-\frac{1}{2}\right| dv\right) du =$$

$$= \frac{1}{2}\left(\int_{-1}^{1} u^4 \left(\int_{+1}^{3} \frac{1}{v^{5/2}}\,dv\right) du = \ldots\right.$$

ii) $K = \{(x,y) \cdot 2x^2 + 5y^2 + 2xy \le 1,\ \frac{1}{2} \le x + 2y \le 1\}$

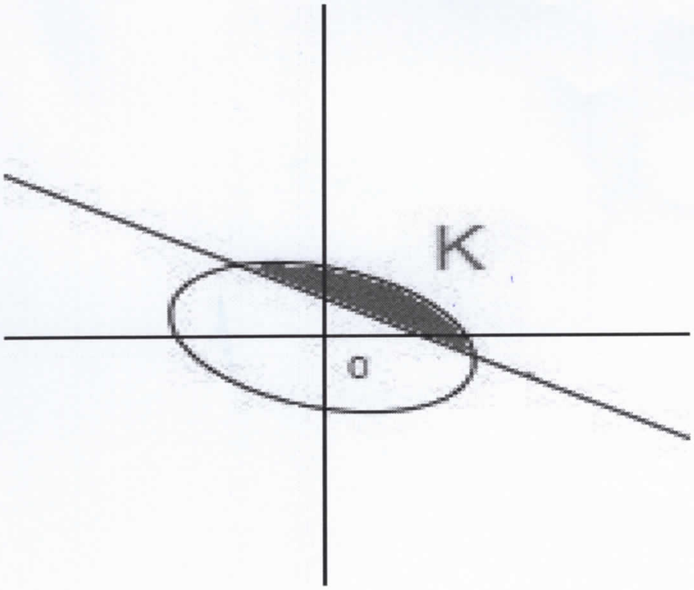
$u = x+2y$, $v = x-y$ $\Big|$ $u^2+v^2 = 2x^2+5y^2+2xy$ $\Big|$ $x = \frac{u+2v}{3}$, $y = \frac{u-v}{3}$

$\vec{T}(u,v) = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$ $\quad \det J_{\vec{T}}(u,v) = -\frac{1}{9} - \frac{2}{9} =$

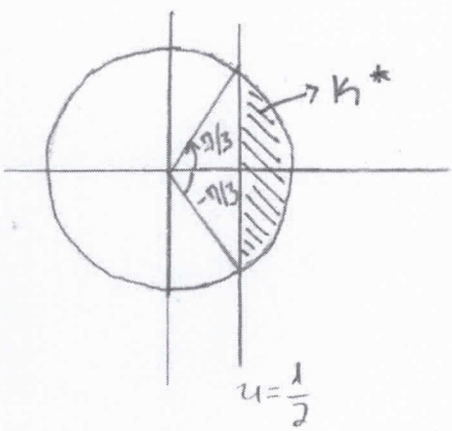
$= -\frac{3}{9} = -\frac{1}{3} < 0$

(αριστερόστροφο)

$$\left\{\begin{array}{l} 2x^2 + 5y + 2xy = 1 \ (*) \quad (x,y)\begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 1 \\ \text{Ιδιοτιμές του πίνακα } \begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix} : \lambda_1, \lambda_2 > 0 \Rightarrow \\ \Rightarrow \text{Η εξίσωση } (*) \text{ παριστάνει έλλειψη} \end{array}\right\}$$



$$K^* = \left\{ (u,v) : u^2+v^2 \le 1, \ \frac{1}{2} \le u \le 1 \right\}$$



$$J = \frac{1}{3} \iint \frac{u^3}{(u^2+v^2)^{3/2}} \, du\, dv$$

Πολικός Μετασχηματισμός

$u = r\,\sigma\upsilon\nu\theta$
$v = r\,\eta\mu\theta$
$\left(u = \frac{1}{2}, \ r = \frac{1}{2\sigma\upsilon\nu\theta} \right)$

$$K^{**} = \left\{ (r,\theta) : -\frac{\pi}{3} \le \theta \le \frac{\pi}{3}, \ \frac{1}{2\sigma\upsilon\nu\theta} \le r \le 1 \right\}$$

Άρα $$J = \frac{1}{3} \int_{-\frac{\pi}{3}}^{+\frac{\pi}{3}} \left(\int_{\frac{1}{2\sigma\upsilon\nu\theta}}^{1} \frac{(r\,\sigma\upsilon\nu\theta)^3}{(r^2)^{3/2}} \, dr \right) d\theta = \frac{1}{6} \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \sigma\upsilon\nu^3\theta \left(1 - \frac{1}{4\sigma\upsilon\nu^2\theta} \right) d\theta$$

$= \dots$

Σημείωση Ασκ 10

i) $\vec{T}(u,v) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$

$\left(\frac{1}{2}, \frac{1}{2}\right)^{\text{του } (u,v)} \longrightarrow \left(\frac{1}{4}, 0\right)^{\text{του } (x,y)}$

$\left(\frac{1}{2}, -\frac{1}{2}\right) \longrightarrow \left(0, \frac{1}{4}\right)$

Δηλαδή, η ορθογώνια βάση του (u,v), $\frac{1}{2}(1,1)$, $\frac{1}{2}(1,-1)$
$\xrightarrow{\vec{T}} \frac{1}{4}(1,0)$, $\frac{1}{4}(0,1)$ ορθογώνια βάση

ii) $\vec{T}(u,v) = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$

Η βάση $(1,1)$, $(2,-1)$, $(u,v) \longrightarrow$ όχι ορθογώνια βάση

$\quad\quad\quad\;\downarrow \quad\quad\quad \downarrow$

$\quad\quad\quad(1,0) \quad\quad (0,1) \longrightarrow$ ορθογώνια (βλ. Γραμ. Αλγ. Ι)

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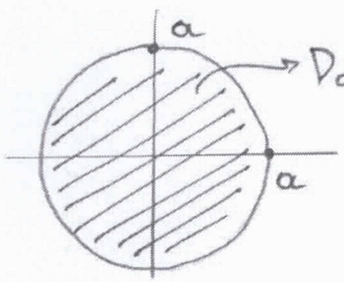
11) $I_a = \iint\limits_{D_\alpha} e^{-(x^2+y^2)}\,dx\,dy$, $D_\alpha = \{(x,y) : x^2+y^2 \le \alpha^2\}$ $(\alpha > 0)$

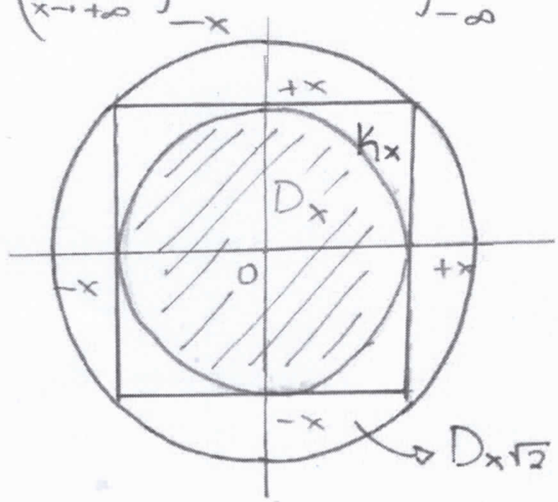


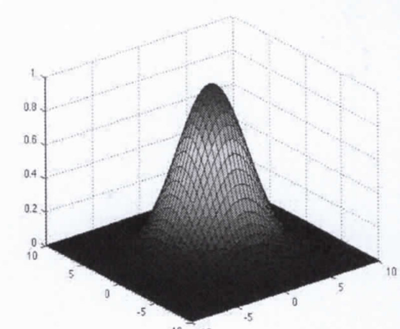
Λύση

$x = r\,\text{συν}\theta$ $\quad I_a = \int_0^{2\pi}\left(\int_0^a e^{-r^2} r\,dr\right) d\theta = \pi\left(1 - e^{-\alpha^2}\right)$

$y = r\,\text{ημ}\theta$

12) $\int_{-\infty}^{+\infty} e^{-t^2} dt = \sqrt{\pi}$ $\left(\lim\limits_{x \to +\infty} \int_{-x}^{+x} e^{-t^2} dt = \int_{-\infty}^{+\infty} e^{-t^2} dt\right)$.





Λύση $\quad x > 0$

$K_x = [-x, x] \times [-x, x]$

$\iint\limits_{K_x} e^{-(t_1^2 + t_2^2)} dt_1\,dt_2 =$

$= \int_{-x}^{+x}\left(\int_{-x}^{+x} e^{-t_1^2 - t_2^2} dt_2\right) dt_1 = \left(\int_{-x}^{+x} e^{-t_1^2} dt_1\right)\left(\int_{-x}^{+x} e^{-t_2^2} dt_2\right) =$

$= \left(\int_{-x}^{+x} e^{-t^2} dt\right)^2$

$$D_x = \left\{ (t_1, t_2) : \sqrt{t_1^2 + t_2^2} \le x \right\}$$

$$D_{x\sqrt{2}} = \left\{ (t_1, t_2) : \sqrt{t_1^2 + t_2^2} \le x\sqrt{2} \right\}$$

$$\begin{cases} e^{-(t_1^2 + t_2^2)} > 0 \\ D_x \subseteq K_x \subseteq D_{x\sqrt{2}} \end{cases}$$

$$\iint\limits_{D_x} e^{-(t_1^2+t_2^2)} \le \iint\limits_{K_x} e^{-(t_1^2+t_2^2)} \le \iint\limits_{D_{x\sqrt{2}}} e^{-(t_1^2+t_2^2)}$$

Άρα $$\underset{\text{Ασκ. } 11}{\pi\left(1 - e^{-x^2}\right)} \le \iint\limits_{K_x} e^{-(t_1^2+t_2^2)} = \left(\int_{-x}^{+x} e^{-t^2} dt \right)^2 \le \underset{\text{Άσκηση } 11}{\pi\left(1 - e^{-2x^2}\right)}$$

$$\Longrightarrow \lim_{x \to +\infty} \left(\int_{-\infty}^{+\infty} e^{-t^2} dt \right)^2 = \pi$$ Άρα $$\int_{-\infty}^{+\infty} e^{-t^2} dt = \sqrt{\pi} \quad \left(e^{-t^2} > 0\right)$$