20ο Μάθημα

25/11/2020

Για τον υπολογισμό τριπλών ολοκληρωμάτων χρησιμοποιούμε συντεταγμένες :

Καρτεσιανές (x, y, z)

Κυλινδρικές $(x = r\sigma\upsilon\nu\theta,\ y = r\eta\mu\theta,\ z)$

$r \in (0, +\infty)$, $\theta \in [0, 2\pi)$

$(\vec{T}(r, \theta, z) = (r\sigma\upsilon\nu\theta, r\eta\mu\theta, z)$, ορίζουσα $\underline{\underline{r}}$)

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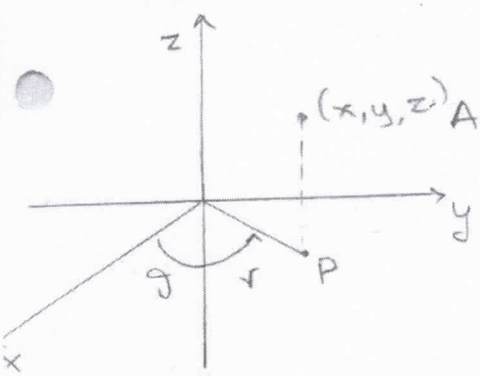


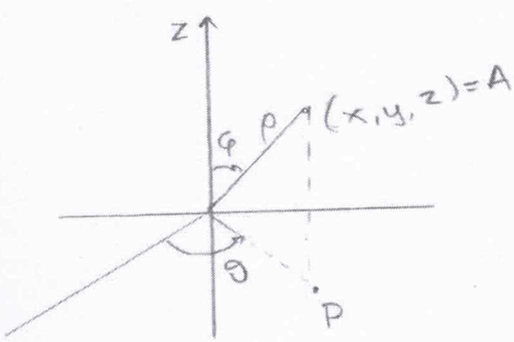
, $r = \sqrt{x^2+y^2}$, $\theta = \sphericalangle(O\vec{x}, O\vec{P})$

Σφαιρικές $(x = \rho\sigma\upsilon\nu\theta\eta\mu\varphi,\ y = \rho\eta\mu\theta\eta\mu\varphi,\ z = \rho\sigma\upsilon\nu\varphi$

$\rho \in (0, +\infty)$, $\theta \in [0, 2\pi)$, $\varphi \in (0, \pi)$

$\vec{T}(\rho, \theta, \varphi) = (\rho\sigma\upsilon\nu\theta\eta\mu\varphi, \rho\eta\mu\theta, \eta\mu\varphi, \rho\sigma\upsilon\nu\varphi)$

(ορίζουσα $\rho^2\eta\mu\varphi$)



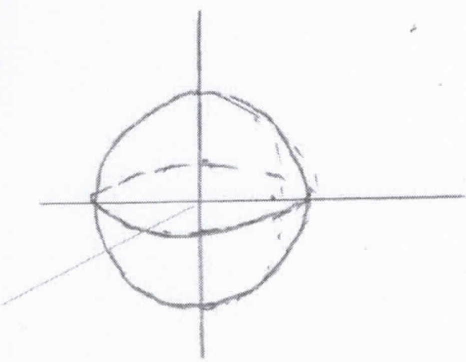
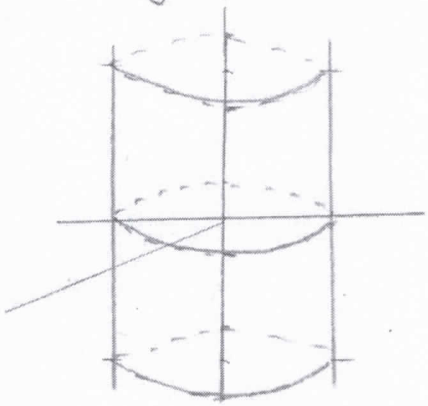
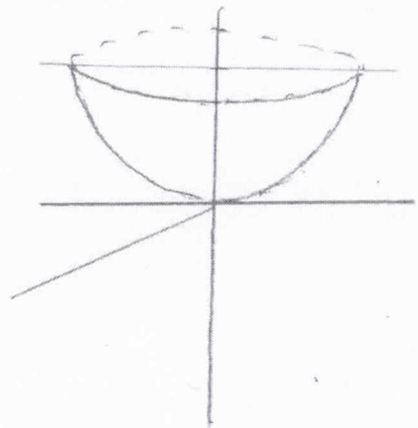
$\rho = \sqrt{x^2+y^2}$

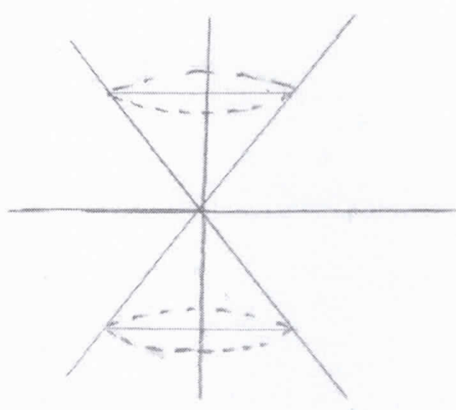
$\theta = \sphericalangle(O\vec{x}, O\vec{P})$

$\varphi = \sphericalangle(O\vec{z}, O\vec{A})$

Οι επιφάνειες που εμφανίζονται να περιβάλλουν στερεό B είναι συνήθως οι εξής:

- $\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2+z^2=a^2\}$ $(a>0)$ Σφαίρα
- $\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2=b^2, z\in\mathbb{R}\}$ $(b>0)$ Κύλινδρος
- $\{(x,y,z)\in\mathbb{R}^3 : z=c(x^2+y^2)\}$ $(c\neq 0)$ Παραβολοειδές
- $\{(x,y,z)\in\mathbb{R}^3 : z^2=c(x^2+y^2)\}$ $(c>0)$ Κώνος (Διπλός)



<u>Και</u> μεταφορές ή και στροφές αυτών

Ασκήσεις

1. $B_a = \{(x,y,z) : (x-a_1)^2 + (y-a_2)^2 + (z-a_3)^2 \leq a^2\}$ $(\alpha>0)$

Σφαίρα κέντρου : $(\alpha_1, \alpha_2, \alpha_3)$ και ακτίνας α

Να υπολογιστεί ο όγκος της σφαίρας αυτής $(V(B_a))$

ΛΥΣΗ

$B((0,0,0), \alpha)$

$V(B_a) = V(B(\vec{0}, \alpha)) = \alpha^3 V(B(\vec{0}, 1))$ $\left(V_d(\lambda D) = \lambda^d V_d(B)_{\lambda\geq 0} \right)$

Αρκεί να υπολογίσουμε τον $V(B_1)$, $B_1 = \{(x,y,z) : x^2+y^2+z^2 \leq 1\}$

i) Καρτεσιανές

$B_1 = \{(x,y,z) : -1 \leq x \leq +1, -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}, -\sqrt{1-x^2-y^2} \leq z \leq \sqrt{1-x^2-y^2}$

$$V(B_1) = \int_{-1}^{+1} \left[\int_{-\sqrt{1-x^2}}^{+\sqrt{1-x^2}} \left(\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} 1\, dz \right) dy \right] dx =$$

$$= \int_{-1}^{+1} \left[\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 2\sqrt{1-x^2-y^2}\, dy \right] dx =$$

(Απ. Λογισμός II) $= \int \sqrt{\beta^2 - y^2}\, dy = \frac{1}{2}\left(\beta^2 \text{τοξημ} \frac{y}{\beta} + y\sqrt{\beta^2-y^2} \right)$

$V(B_1) = \frac{4\pi}{3}$ //

ii) Κυλινδρικές

$x^2+y^2+z^2 = 1$ / $x = r\,\text{συν}\theta$, $y = r\,\text{ημ}\theta$, $z = z$

$r^2 + z^2 = 1 \implies z^2 = 1 - r^2$, $r \in [0,1]$

$$V(B_1) = \int_0^{2\pi} \int_0^1 \left[\int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} r\, dz \right] dr\, d\theta = \frac{4\pi}{3}$$

iii) Σφαιρικές

$x^2+y^2+z^2=1$, $x=\rho\,\sigma\upsilon\nu\theta\,\eta\mu\varphi$, $y=\rho\,\eta\mu\theta\cdot\eta\mu\varphi$, $z=\rho\,\sigma\upsilon\nu\varphi$

$\rho=1$

$$V(B_1)=\int_0^{2\pi}\int_0^{\pi}\left(\int_0^1 \rho^2\,\eta\mu\varphi\, d\rho\right)d\varphi\, d\theta = \frac{4\pi}{3}$$

iv) Ως όγκος εκ περιστροφής

v) Μέθοδος Αρχιμήδη (πχ. Νεγρεπόντης κ.ά, Εύδοξος)

2) $V(\mathcal{E})$, $\mathcal{E}=\left\{(x,y,z): \left(\frac{x}{\alpha}\right)^2+\left(\frac{y}{\beta}\right)^2+\left(\frac{z}{\gamma}\right)^2\leq 1\right\}$, $(\alpha,\beta,\gamma>0)$

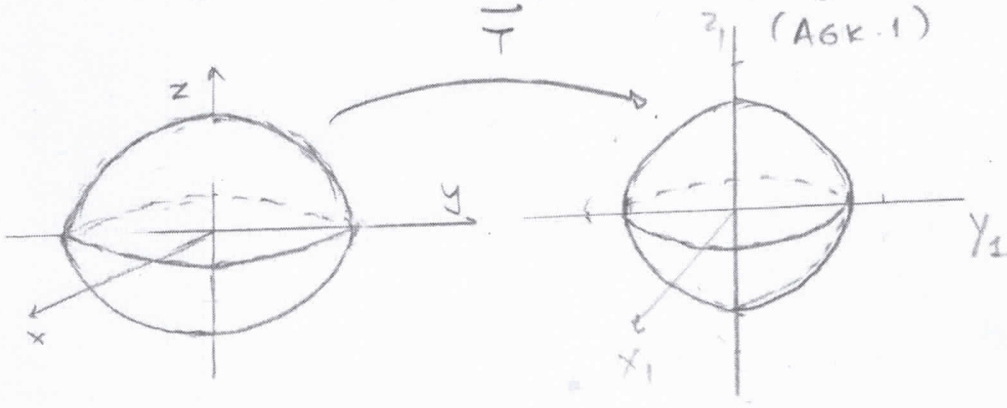
Θέτουμε, $x_1=\frac{x}{\alpha}$, $y_1=\frac{y}{\beta}$, $z_1=\frac{z}{\gamma}$ / $\det J_{\vec{T}}(x_1,y_1,z_1)=\alpha\beta\gamma$

$\vec{T}(x_1,y_1,z_1)=(\alpha x_1, \beta y_1, \gamma z_1)$

$\mathcal{E}^*=\{(x_1,y_1,z_1): x_1^2+y_1^2+z_1^2\leq 1\}$

$$V(\mathcal{E})=\iiint_{\mathcal{E}^*}\alpha\beta\gamma\, dx_1\,dy_1\,dz_1=\alpha\beta\gamma\,\frac{4\pi}{3} \;//$$

$\downarrow$ (Ασκ. 1)



3) $I = \iiint_D z^2 \, dx\, dy\, dz$, $D = \{(x,y,z) : x^2+y^2+z^2 \le 1, z \ge 0\}$

(Απειροστικός $t^2 = \frac{y^2}{1-x^2}$
Λογ. II)
Καρτεσιανές

$\int (1-t^2)^{3/2} dt = t(1-t^2)^{3/2} + \frac{3}{8}\left[\text{τοξημ}\, t - \frac{1}{4}\, \text{ημ}(4\,\text{τοξημ}\, t)\right]$

$I = \frac{2\pi}{15}$

Κυλινδρικές: $x = r\,\text{συν}\theta$, $y = r\,\text{ημ}\theta$, $z = z$ / $r^2+z^2=1$ Σφαίρα

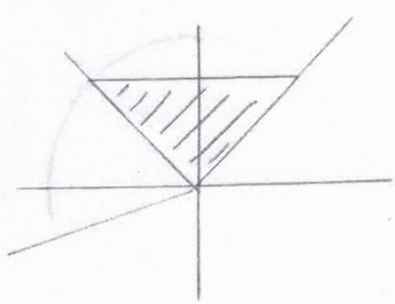
$I = \int_0^{2\pi} \int_0^1 \left(\int_0^{\sqrt{1-r^2}} r \cdot z^2 dz \right) dr\, d\theta = \frac{2\pi}{15}$

Σφαιρικές $z \ge 0$, $\rho\,\text{συν}\varphi \ge 0$ κ' $\varphi \in [0,\pi] \Rightarrow \varphi \in [0, \frac{\pi}{2}]$

$I = \int_0^{2\pi} \int_0^{\pi/2} \left[\int_0^1 (\rho^2 \text{ημ}\varphi)(\rho\,\text{συν}\varphi)^2 d\rho \right] d\varphi\, d\theta = \frac{2\pi}{15}$

4) $B = \{(x,y,z) : z \ge \sqrt{x^2+y^2}, x^2+y^2+z^2 \le 1\}$

$I = \iiint_B (x^2+y^2)\, dV$, $J = \iiint_B \sqrt{z}\, dV$



Λύση

Κυλινδρικές

$x = r\,\text{συν}\theta$
$y = r\,\text{ημ}\theta$
$z = z$

Κώνος $z = \sqrt{x^2+y^2}$, $z = r$

Σφαίρα $z \ge 0$, $z = \sqrt{1-r^2}$

$(0 \le)\, r \le z \le \sqrt{1-r^2}$ $\left(0 \le z \le \sqrt{1-z^2},\right.$ $z^2 \le 1 - z^2$, $2z^2 \le 1, 0 \le z \le \left.\frac{1}{\sqrt{2}}\right)$

$0 \le r \le \frac{1}{\sqrt{2}}$

$$B^* = \left\{ (r, \vartheta, z) : 0 \le \vartheta \le 2\pi,\ 0 \le r \le \frac{1}{\sqrt{2}},\ r \le z \le \sqrt{1-r^2} \right\}$$

$$I = \int_0^{2\pi} \int_0^{\frac{1}{\sqrt{2}}} \left(\int_r^{\sqrt{1-r^2}} r \cdot r^2 \, dz \right) dr \, d\vartheta$$

$$J = \int_0^{2\pi} \int_0^{1/\sqrt{2}} \left(\int_r^{\sqrt{1-r^2}} \sqrt{z} \cdot r \, dz \right) dr \, d\vartheta$$

Σφαιρικές

$x = \rho \,\text{συν}\vartheta\, \text{ημ}\varphi$

$y = \rho \,\text{ημ}\vartheta\, \text{ημ}\varphi$

$z = \rho \,\text{συν}\varphi$

Σφαίρα $\rho = 1$

Κώνος $\varphi = \frac{\pi}{4}$

$$B^* = \{ (\rho, \vartheta, \varphi) : 0 \le \vartheta \le 2\pi,\ 0 \le \varphi \le \pi/4,\ 0 \le \rho \le 1 \}.$$

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5) $V(B)$

$$B = \{ (x, y, z) : x^2 + y^2 + z^2 \le 5,\ \sqrt{3}\, z \ge \sqrt{x^2 + y^2} \}$$

Λύση

Κυλινδρικές

Σφαίρα, $(z \ge 0)$, $z = \sqrt{5 - r^2}$

Κώνος, $z = \frac{1}{\sqrt{3}} r$

$$B^* = \left\{ (r, \vartheta, z) : 0 \le \vartheta \le 2\pi,\ 0 \le r \le \frac{\sqrt{15}}{2},\ \frac{r}{\sqrt{3}} \le z \le \sqrt{5 - r^2} \right\}$$

(όμως. Ασ. Η $0 \le z \le \frac{\sqrt{15}}{2}$)

$$V(B) = \int_0^{2\pi} \int_0^{\frac{\sqrt{15}}{2}} \left(\int_{\frac{r}{\sqrt{3}}}^{\sqrt{5-r^2}} r \, dz \right) dr \, d\vartheta = \frac{5\sqrt{5}}{3} \pi.$$

Σφαιρικές

Σφαίρα, $\rho = \sqrt{5}$

Κωνος, $\varphi = \frac{\pi}{3}$

$$V(B) = \int_0^{2\pi}\int_0^{\pi/3}\left(\int_0^{\sqrt{5}} \rho^2 \eta\mu\varphi \, d\rho\right) d\varphi \, d\theta = \frac{5\sqrt{5}}{3}\pi$$

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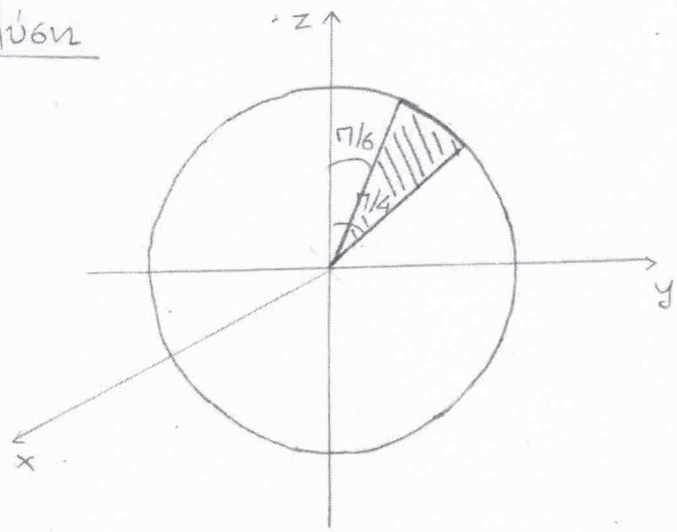
6) $$I = \iiint_K x^2 \sqrt{z} \, dV$$

$$K = \{(x,y,z) : x^2+y^2+z^2 \leq a^2, \ \sqrt{x^2+y^2} \leq z \leq \sqrt{3(x^2+y^2)}\} \quad (\alpha > 0)$$

- Λύση



Κυλινδρικές

2 ολοκληρώματα

Σφαιρικές

Σφαίρα $\rho = a$

Κώνος $z = \sqrt{x^2+y^2}$; $\varphi = \frac{\pi}{4}$

Κώνος $z = \sqrt{3(x^2+y^2)}$, $\varphi = \frac{\pi}{6}$

- $$I = \int_0^{2\pi}\int_{\pi/6}^{\pi/4}\left[\int_0^{a} (\rho^2 \eta\mu\varphi)(\rho\sigma\upsilon\nu\theta\eta\mu\varphi)^2 \sqrt{\rho\sigma\upsilon\nu\varphi} \, d\rho\right] d\varphi \, d\theta$$

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7) $$I = \iiint_B (1-4x^2-9y^2-z^2) \, dV$$

$$B = \{(x,y,z) : 4x^2+9y^2+z^2 \leq 1\}.$$

$x_1 = 2x$

$y_1 = 3y$

$z_1 = z$

$\vec{T}(x_1,y_1,z_1) = \left(\frac{x_1}{2}, \frac{y_1}{3}, z_1\right)$ Ορίζουσα $\frac{1}{6}$.

$$B^* = \{(x_1,y_1,z_1) : x_1^2+y_1^2+z_1^2 \le 1\}$$

$$I = \frac{1}{6} \iiint_{B^*} (1-x_1^2-y_1^2-z_1^2)\, dV$$

$$I = \frac{1}{6}\int_0^{2\pi}\int_0^{\pi}\left[\int_0^1 (\rho^2 \eta\mu\varphi)(1-\rho^2)\,d\rho\right] d\varphi\, d\theta = \frac{4\pi}{45}.$$

8) $$I = \iiint_K e^{(x^2+y^2+z^2)^{3/2}}\, dV$$

$$K = \{(x,y,z) : x^2+y^2+z^2 \le a^2\} \quad (a>0)$$

Λύση

Κυλινδρικές. $\left(\int e^{(\varsigma^2+t^2)^{3/2}} dt\right)$ (Δεν υπολογίζεται !)

Σφαιρικές. $$I = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} e^{\rho^3}(\rho^2 \eta\mu\varphi)\,d\rho\, d\varphi\, d\theta =$$

$$= \frac{4\pi}{3}(e-1)$$

9) $$I = \lim_{\varepsilon\to 0^+} \iiint_{\varepsilon^2 \le x^2+y^2+z^2 \le (1-\varepsilon)^2} \frac{dx\,dy\,dz}{\sqrt{(x^2+y^2+z^2)(1-x^2-y^2-z^2)}}$$

$(0<\varepsilon<1)$

Λυση $$I_\varepsilon = \int_0^{2\pi}\int_0^{\pi}\int_\varepsilon^{1-\varepsilon} \frac{\rho^2 \eta\mu\varphi}{\sqrt{\rho^2(1-\rho^2)}}\,d\rho\, d\varphi\, d\theta =$$

$$= 4\pi \int_\varepsilon^{1-\varepsilon} \frac{\rho}{\sqrt{1-\rho^2}}\,d\rho = 4\pi\left[-\sqrt{1-\rho^2}\,\Big|_\varepsilon^{1-\varepsilon}\right] = 4\pi\left[\sqrt{1-\varepsilon^2} - \sqrt{1-(1-\varepsilon)^2}\right]$$

$$= 4\pi$$

10) Για ποια $\lambda \in \mathbb{R}$ το $I = \lim_{\varepsilon \to 0^+} \iiint_{\varepsilon^2 \leq x^2+y^2+z^2<1} \frac{dx\,dy\,dz}{(x^2+y^2+z^2)^{\lambda}} \in \mathbb{R}$;

Λύση

$$I_\varepsilon = \int_0^{2\pi} \int_0^{\pi} \left(\int_\varepsilon^1 \frac{\rho^2 \eta\mu\varphi}{\rho^{2\lambda}} d\rho \right) d\varphi\, d\theta = 4\pi \int_\varepsilon^1 \rho^{2-2\lambda} d\rho =$$

$$= \begin{cases} 4\pi \left. \dfrac{\rho^{3-2\lambda}}{3-2\lambda} \right|_\varepsilon^1 , & \lambda \neq \frac{3}{2} \\ 4\pi \cdot \ln \rho \Big|_\varepsilon^1 , & \lambda = \frac{3}{2} \end{cases}$$

$$= \begin{cases} 4\pi \cdot \dfrac{1}{3-2\lambda} \left(1 - \varepsilon^{3-2\lambda} \right), & \lambda \neq \frac{3}{2} \\ -4\pi \cdot \ln \varepsilon , & \lambda = \frac{3}{2} \end{cases}$$

$0 < \varepsilon < 1$

Υπάρχει $\lim_{\varepsilon \to 0^+} I_\varepsilon \iff 3-2\lambda > 0$, $\lambda < \frac{3}{2}$

$$I = \frac{4\pi}{3-2\lambda}$$

11) $\iiint_K z^2 \eta\mu(xz) \cdot e^{z^2} dV$, $K = \{(x,y,z) : 0 \leq x \leq 1,\ x \leq y \leq z - x^2,\ -3 \leq z \leq +3 \}$

Λύση

$f(x,y,z) = z^2 \eta\mu(xz)\, e^{z^2}$, $x \geq 0$, $-3 \leq z \leq +3$

$f(x,y,-z) = -f(x,y,z)$ $\quad z \in [-3,3]$ $\quad \underline{\underline{I = 0}}$

12)

$$I_1=\int_{-2}^{+2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_{\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}}(x+z^2)\,dz\,dx\,dy$$

$$I_2=\int_{0}^{1}\int_{x}^{2-x^2}\int_{-3}^{+3}z^2\,ημ(xz)\,dz\,dy\,dx$$

Ένα από τα δύο ολοκληρώματα είναι 0. Ποιο είναι;

Λυση

$J_2=0$ //

Συμπληρωματικές Ασκήσεις

$\mathbb{R}^2$, $K\subseteq\mathbb{R}^2$ και έχει εμβαδόν (πχ. απλό)

$(x,y)\in K$ πυκνότητα μάζας $\delta(x,y)>0$

$m=\iiint_K \delta\,dxdy$ / ΚΒ $(\bar{x},\bar{y})=\frac{1}{m}\left(\iint_K x\delta\,dxdy,\ \iint_K y\delta\,dxdy\right)$

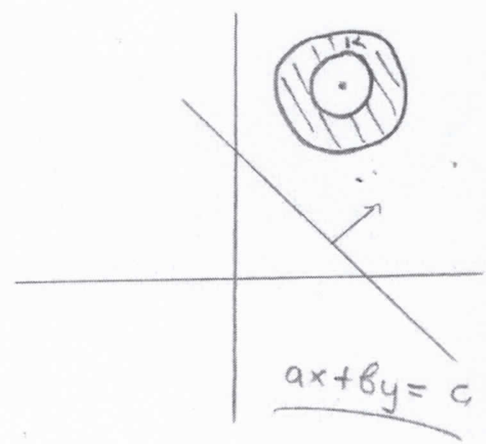
$\mathbb{R}^3$ Ανάλογα. —

Ασκηση

Έστω $K\subseteq\mathbb{R}^2$ και $ax+by\geq c$, $(x,y)\in K$.

$((a,b)\neq(0,0))$

ΝΔΟ, $a\bar{x}+b\bar{y}\geq c$.



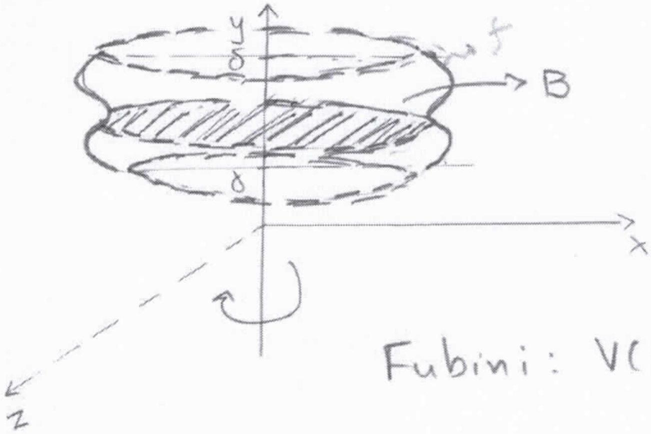
Λύση

$$a\bar{x}+b\bar{y} = a\,\frac{\iint_K x\delta\,dx\,dy}{m} + b\,\frac{\iint_K y\delta\,dx\,dy}{m} =$$

$$= \frac{\iint_K \overbrace{(ax+by)}^{\geq c}\,\delta(x,y)\,dx\,dy}{\iint_K \delta(x,y)\,dx\,dy} \geq \frac{\iint_K c\,\delta(x,y)\,dx\,dy}{\iint_K \delta(x,y)\,dx\,dy} = c$$

Στερεό εκ περιστροφής

1)



$x=f(y)$, $y\in[\gamma,\delta]$

$f\geq 0$

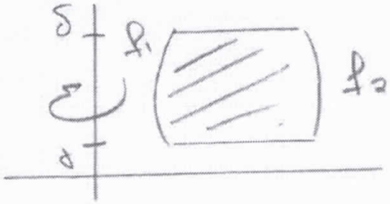
Fubini: $V(B)=\int_\gamma^\delta E(y)\,dy$

$E(y)$: Εμβαδόν κύκλου

$(0,y,0)$, Ακτίνα $f(y)$ / $E(y)=\pi f^2(y)$

$$V(B)=\pi\int_\gamma^\delta f^2(y)\,dy$$

2)

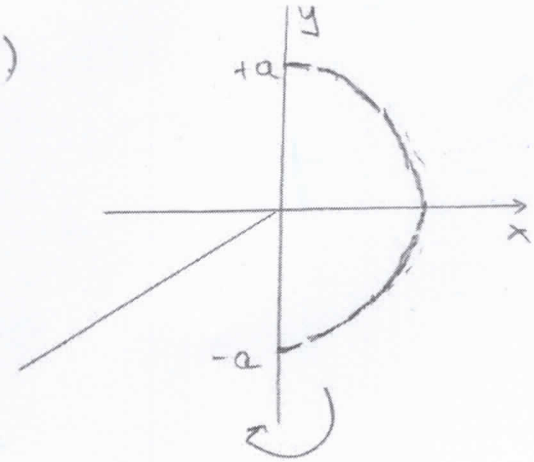


$D=\{(x,y): \gamma\leq y\leq\delta,\ f_1(y)\leq x\leq f_2(y)\}$, $f_2, f_1>0$.

$$V(B)=\pi\int_\gamma^\delta \left(f_2^2(y)-f_1^2(y)\right)dy$$

Εφαρμογές

1)

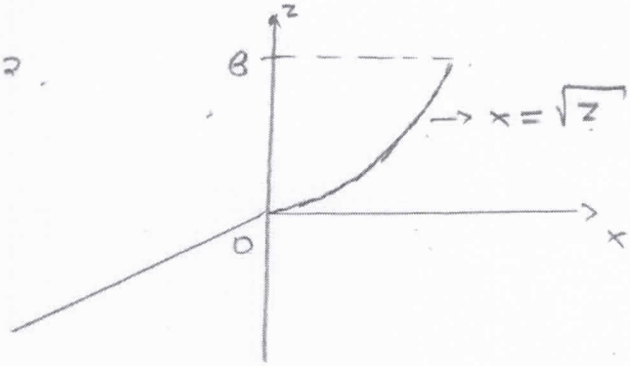


$x=\sqrt{a^2-y^2}=f(y)$, $y\in[-a,+a]$

Όγκος της $B((0,0,0,\alpha)$

$$V(B)=\pi\int_{-a}^{+a}(a^2-y^2)\,dy=\frac{4\pi}{3}a^3$$

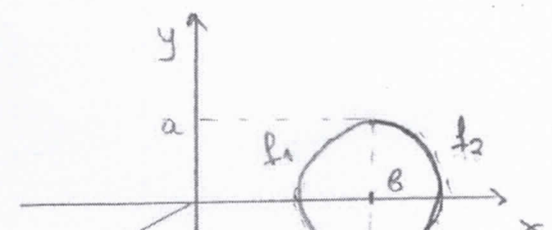
2) $z=x^2$.



$z\in[0,b]$ $(b>0)$

$$V(B)=\pi\int_0^b z\,dz=\frac{\pi b^2}{2}$$

3) Όγκος Ντόνατς / Λουκουμά / Torus



$(0<a<b)$

$f_2(y)=b+\sqrt{a^2-y^2}$

$f_1(y)=b-\sqrt{a^2-y^2}$ $y\in[-\alpha,\alpha]$

$$V(\Lambda)=\pi\int_{-a}^{+a}\left(f_2^2(y)-f_1^2(y)\right)dy=$$

$$=\pi\int_{-a}^{+a}\left(\left(b+\sqrt{a^2-y^2}\right)^2-\left(b-\sqrt{a^2-y^2}\right)^2\right)dy=$$

$$=\pi\int_{-a}^{+a}4\sqrt{a^2-y^2}\,b\,dy=2b\pi^2a^2$$